

Department of Mathematics
MTL 5005 (Introduction to Probability)
Tutorial Sheet No. 1

1. Determine the sample space for each of the following random experiments.
 1. A student is selected at random from a probability and statistics lecture class, and the student's total marks are determined.
 2. A coin is tossed three times, and the sequence of heads and tails is observed.
2. Items coming off a production line are marked defective (D) or non-defective (N). Items are observed and their condition is noted. This is continued until two consecutive defectives are produced or four items have been checked, whichever occurs first. Describe the sample space for this experiment.
3. One urn contains three red balls, two white balls, and one blue ball. A second urn contains one red ball, two white balls, and three blue balls:
 1. One ball is selected at random from each urn. Describe the sample space.
 2. If the balls in two urns are mixed in a single urn and then a sample of three is drawn, find the probability that all three colors are represented when sampling is drawn (i) with replacement (ii) without replacement.
4. A fair coin is continuously flipped. What is the probability that the first five flips are (i) H, T, H, T, T (ii) T, H, H, T, H.
5. In a certain colony, 75% of the families own a car, 65% own a computer and 55% own both a car and a computer. If a family is randomly chosen, what is the probability that this family owns a car or a computer but not both?
6. A fair die is tossed once. Let A be the event that face 1, 3, or 5 comes up, B be the event that it is 2, 4, or 6, and C be the event that it is 1 or 6. Show that A and C are independent. Find $P(A, B, \text{ or } C \text{ occurs})$.
7. Let $A = (a, b)$ and $B = (c, d)$ be disjoint open intervals of $\mathbf{R}$. Let $C_n = A$ if n is odd and $C_n = B$ if n is even. Find $\limsup C_n$ and $\liminf C_n$. Does $\lim_{n \rightarrow \infty} C_n$ exist?
8. Prove that $\limsup (A_n \cup B_n) = \limsup A_n \cup \limsup B_n$.
9. Show that $\sigma\{A, B\} = \sigma\{A \cap B, A \cap B^c, A^c \cap B, A^c \cap B^c\}$ on space Ω , A, B being any subsets of Ω .
10. Show that $\mathfrak{F} \cap A$ is a σ -field on $A \subset \Omega$ if $\mathfrak{F}$ is a σ -field on Ω .
11. Show that for any class C of subsets of Ω and $A \subset \Omega$ the minimal σ -field $\sigma_A(C \cap A)$ generated by class $C \cap A$ on A is $\sigma(C) \cap A$ where $\sigma(C)$ is minimal σ -field generated by C on Ω .
12. Let $A_1, A_2, \dots, A_n$ be events on a probability space $(\Omega, \mathfrak{F}, P)$.

(i) For $n = 3$, if $P(A_1 \cap A_2 \cap A_3) = P(A_1^c \cap A_2^c \cap A_3^c)$, then each equals $\frac{1}{2} \left[1 - \sum_{i=1}^3 P(A_i) + \sum_{i>j=1}^3 \sum_{j=1}^3 P(A_i \cap A_j) \right]$.

(ii) $P(\cap_{i=1}^n A_i) \geq \sum_{i=1}^n P(A_i) - (n-1)$.

13. Let $\Omega = \{s_1, s_2, s_3, s_4\}$ and $P(s_1) = \frac{1}{6}$, $P(s_2) = \frac{1}{5}$, $P(s_3) = \frac{1}{3}$ and $P(s_4) = \frac{3}{10}$. Define:

$$A_n = \begin{cases} \{s_1, s_2\}, & \text{if } n \text{ is odd} \\ \{s_2, s_4\}, & \text{if } n \text{ is even} \end{cases}$$

Find $P(\liminf A_n)$, $P(\limsup A_n)$, $\liminf P(A_n)$, $\limsup P(A_n)$.

14. Let $\mathfrak{F}_i$ be a σ -field on Ω_i , $i = 1, 2$. Let $f : \Omega_1 \rightarrow \Omega_2$ be such that $A \in \mathfrak{F}_2 \Rightarrow f^{-1}(A) \in \mathfrak{F}_1$. If P is a probability measure on $(\Omega_1, \mathfrak{F}_1)$, then show that $Q(A) = P(f^{-1}(A))$, $A \in \mathfrak{F}_2$ is a probability measure on $(\Omega_2, \mathfrak{F}_2)$.
15. Prove that
 (a) $\sum_{i=1}^n P(A_i) - \sum_{j=1}^n \sum_{i < j} P(A_i \cap A_j) \leq P(\cup_{i=1}^n A_i) \leq \sum_{i=1}^n P(A_i)$
 (b) $P(\cap_{i=1}^n A_i) \geq 1 - \sum_{j=1}^n P(\bar{A}_j)$
16. The base and altitude of a right triangle are obtained by picking points randomly from $[0, a]$ and $[0, b]$, respectively. Show that the probability that the area of the triangle so formed will be less than $ab/4$ is $(1 + \ln 2)/2$.
17. Consider a randomly chosen group of $n(\leq 365)$ persons. What is the probability that at least two of them have the same birthdays?
18. In a movie theater that can accommodate $n + k$ persons, n persons are to be seated. What is the probability that $r \leq n$ given seats are occupied.
19. A company manufacturing cornflakes puts a card numbered 1 or 2 or 3, ..., or n at random on each package, all numbers being equally likely to be drawn. If $m(\geq n)$ packages are purchased, show that the probability of being able to assemble atleast one complete set of cards from the packages is:

$$1 - \binom{n}{1} \left(1 - \frac{1}{n}\right)^m + \binom{n}{2} \left(1 - \frac{2}{n}\right)^m - \dots + (-1)^n \binom{n}{n-1} \left(1 - \frac{n-1}{n}\right)^m$$

20. An urn contains 5 white and 4 red balls. Four balls are transferred to a second urn. A ball is then drawn from this urn and it happens to be black. Find the probability of drawing a white ball from among the remaining three.
21. Let a sample of size 4 be drawn with(without) replacement from an urn containing 12 balls of which 5 are white. If the sample contains 3 white balls then find the probability the ball drawn on the third draw was white.
22. There are n urns, each containing α white and β black balls. One ball is taken from urn 1 to urn 2 and then one is taken from urn 2 to urn 3 and so on. Finally a ball is chosen from urn n . If the first ball transferred was a white, what is the probability that the last chosen ball is white? what happens when $n \rightarrow \infty$?
23. A certain test is 95% accurate on those who have cancer and 95% accurate on those who do not have this type. If $\frac{1}{4}\%$ of the people under test have this type of cancer. What is the probability that a person has this type of cancer when his test says he does ?
24. In a series of independent tosses of a coin, let A_n be the event that a head occurs in the n^{th} toss. If p is the probability of A_n for all n , obtain the probability that infinitely many A_n 's occur. Obtain the probability that infinitely many B_n 's occur, where B_n is the event that head occurs for the first time at the n^{th} toss.
25. What can you say about the event A if it is independent of itself?. If the events A and B are disjoint and independent, what can you say of them?
26. If A and B are independent and $A \subseteq B$ show that either $P(A) = 0$ or $P(B) = 1$.
27. Let $(\Omega, \mathfrak{F}, P)$ be a probability space. Let $A, B, C \in \mathcal{S}$ with $P(B)$ and $P(C) > 0$. If B and C are independent, show that

$$P\{A/B\} = P\{A/B \cap C\}P(C) + P\{A/B \cap C^c\}P(C^c).$$

Conversely, if this relation holds, $P\{A/B \cap C\} \neq P\{A/B\}$, and $P(B) > 0$, then B and C are independent.

28. A lot of five identical batteries is life tested. The probability assignment is assumed to be

$$P(A) = \int_A \frac{1}{\lambda} e^{-x/\lambda} dx$$

for any event $A \subseteq [0, \infty)$, where $\lambda > 0$ is a known constant. Thus the probability that a battery fails after time t is given by

$$P(t, \infty) = \int_t^\infty \frac{1}{\lambda} e^{-x/\lambda} dx, \quad t \geq 0.$$

If the times to failure of the batteries are independent, what is the probability that at least one battery will operating after t_0 hours?

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