

Department of Mathematics
MTL 601 (Probability and Statistics)
Tutorial Sheet No. 3

1. Let the joint CDF function of X and Y is

$$F(x, y) = \begin{cases} (1 - e^{-2x})(1 - e^{-2y}) & x > 0, y > 0 \\ 0 & \text{otherwise} \end{cases}.$$

What is the joint PDF and Y , and the $P(1 < X < 4, 1 < Y < 3)$?

2. Let the joint PMF function of X and Y is

$$p(x, y) = \begin{cases} \frac{1}{33}(2x + y) & x = 1, 2, 3, y = 1, 2 \\ 0 & \text{otherwise} \end{cases}.$$

What are the marginals of X and Y ?

3. Let the joint PDF function of X and Y is

$$f(x, y) = \begin{cases} \frac{2}{9}(3 - x - y) & 0 < x; y < 3; 0 < x + y < 3 \\ 0 & \text{otherwise} \end{cases}.$$

What is the conditional probability $P(X < 1 | Y < 1)$?

4. Suppose a two dimensional random variable has a joint PDF:

$$f_{X,Y}(x, y) = \begin{cases} kx(x - y), & 0 < x < 2, -x < y < x \\ 0, & \text{otherwise} \end{cases}$$

- (a) Evaluate the constant k
- (b) Compute $P(X + Y < 1)$, $P(XY < 1)$
- (c) Compute $P(Y > -1/2 | X = 1)$

5. Let X, Y be discrete random variables with respective PMFs given by

$$p_X(x_1) = p_1, p_X(x_2) = 1 - p_1, p_Y(y_1) = p_2, p_Y(y_2) = 1 - p_2$$

Show that X, Y are independent iff X, Y are uncorrelated.

6. Using the concept of conditional expectation, prove that:

$$\text{Var}(X) = \text{Var}(E(X|Y)) + E(\text{Var}(X|Y))$$

7. Suppose X, Y are independent random variables each having binomial distribution with parameters n and p , ($0 < p < 1$). Find the joint pmf of $(X + Y, X - Y)$.

8. Let X, Y be i.i.d random variables with common pdf

$$f(x) = \begin{cases} e^{-x}, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases}$$

Find the pdf of random variables $\min\{X, Y\}$, $\max\{X, Y\}$. Let $U = X + Y$ and $V = X - Y$. Find the conditional pdf of V , given $U = u$ for some fixed $u > 0$.

9. Let the joint PMF function of X and Y is

$$p(x, y) = \begin{cases} \binom{7}{y} x^y (1-x)^{7-y} & y = 0, 1, 2, \dots, 7 \\ 0 & \text{otherwise} \end{cases}.$$

Find the conditional expectation of Y given $X = x$?

10. Let the joint PDF function of X and Y is

$$f(x, y) = \begin{cases} \frac{3}{4}x & 0 < y < x < 1 \\ 0 & \text{otherwise} \end{cases}.$$

Find the conditional expectation of Y given $X = 0.5$?

11. Let the joint PDF function of X and Y is

$$f(x, y) = \begin{cases} \frac{6}{7}x & 1 \leq x + y \leq 2, x \geq 0, y \geq 0 \\ 0 & \text{otherwise} \end{cases}.$$

12. Let X, Y be i.i.d each having standard normal distribution. Show that $U = \sqrt{X^2 + Y^2}$ and $V = \frac{X}{Y}$ are independent.

13. Let the joint PMF function of X and Y is

$$p(x, y) = \begin{cases} \binom{7}{y} x^y (1-x)^{7-y} & y = 0, 1, 2, \dots, 7 \\ 0 & \text{otherwise} \end{cases}.$$

Find the conditional expectation of Y given $X = x$?

14. Let the joint PDF of X and Y is

$$f(x, y) = \begin{cases} 4x & 0 < x < \sqrt{y} < 1 \\ 0 & \text{otherwise} \end{cases}.$$

Find the conditional variance of Y for $X = x$?

15. Does WLLN hold for the following sequences

(a) $P(X_k = \pm 2^k) = \frac{1}{2}$

(b) $P(X_k = \pm \frac{1}{k}) = \frac{1}{2}$.

16. Let $X_1, X_2, \dots, X_n$ be a random sample from a uniform distribution on the interval from 0 to 5. What is the limiting moment generating function of $\frac{\bar{X} - \mu}{\sigma/\sqrt{n}}$ as $n \rightarrow \infty$?

17. Let $X_1, X_2, \dots$ be a sequence of iid r.v. with mean 1 and variance 1600, and assume that these variables are non-negative. Let $Y = \sum_{k=1}^{100} X_k$. Use the central limit theorem to approximate the probability $P(Y \geq 900)$.

18. Let X be a r.v. having a Gamma distribution with parameters n and 1. How large must n be in order to guarantee that

$$P\left(\left|\frac{X}{n} - 1\right| > 0.01\right) < 0.01?$$

19. Let $\{X_n, n = 1, 2, 3, \dots\}$ be a sequence of iid r.v. with $E(X_1) = Var(X_1) = \lambda \in (0, \infty)$ and $P(X_1 > 0) = 1$. Show that for $n \rightarrow \infty$,

$$\sqrt{n} \frac{Y_n - \lambda}{\sqrt{\lambda}} \xrightarrow{d} N(0, 1)$$

where

$$Y_n = \frac{1}{n} \sum_{k=1}^n X_k.$$

20. Let Y be a Gamma distributed r.v. with PDF

$$f(y) = \begin{cases} \frac{1}{\Gamma(p)} e^{-y} y^{p-1} & y > 0 \\ 0 & \text{otherwise} \end{cases}.$$

Find the limiting distribution of $\frac{Y - E(Y)}{\sqrt{Var(Y)}}$ as $p \rightarrow \infty$.

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