Department of Mathematics
MTL 601 (Probability and Statistics)
Tutorial Sheet No. 1

1. Determine the sample space for each of the following random experiments.
    1. A student is selected at random from a probability and statistics lecture class, and the student's total marks are determined.
    2. A coin is tossed three times, and the sequence of heads and tails is observed.
2. Items coming off a production line are marked defective (D) or non-defective (N). Items are observed and their condition is noted. This is continued until two consecutive defectives are produced or four items have been checked, which ever occurs first. Describe the sample space for this experiment.
3. One urn contains three red balls, two white balls, and one blue ball. A second urn contains one red ball, two white balls, and three blue balls:
    1. One ball is selected at random from each urn. Describe the sample space.
    2. If the balls in two urns are mixed in a single urn and then a sample of three is drawn, find the probability that all three colors are represented when sampling is drawn (i) with replacement (ii) without replacement.
4. A fair coin is continuously flipped. What is the probability that the first five flips are (i) H, T, H, T, T (ii) T, H, H, T, H.
5. In a certain colony, 75% of the families own a car, 65% own a computer and 55% own both a car and a computer. If a family is randomly chosen, what is the probability that this family owns a car or a computer but not both?
6. A fair die is tossed once. Let A be the event that face 1, 3, or 5 comes up, B be the event that it is 2, 4, or 6, and C be the event that it is 1 or 6. Show that A and C are independent. Find $P(A, B,$ or C occurs).
7. An urn contains four tickets marked with numbers $112, 121, 211, 222$, and one ticket is drawn at random. Let A_i $(i = 1, 2, 3)$ be the event that ith digit of the number of the ticket drawn is 1. Discuss the independence of the events A_1, A_2 and A_3.
8. Let A and B are two independent events. Show that A^c and B^c are also independent events.
9. What can you say about the event A if it is independent of itself?. If the events A and B are disjoint and independent, what can you say of them?
10. If A and B are independent and $A \subseteq B$ show that either $P(A) = 0$ or $P(B) = 1$.
11. Let $A = (a, b)$ and $B = (c, d)$ be disjoint open intervals of $\mathbf{R}$. Let $C_n = A$ if n is odd and $C_n = B$ if n is even. Find $\limsup C_n$ and $\liminf C_n$. Does $\lim_{n\to\infty} C_n$ exist ?
12. Prove that $\limsup\ (A_n \cup B_n) = \limsup A_n\ \cup \limsup B_n$.
13. Show that $\sigma\{A, B\} = \sigma\{A \cap B, A \cap B^c, A^c \cap B, A^c \cap B^c\}$ on space Ω, A, B being any subsets of Ω.
14. Show that $\Im \cap A$ is a $\sigma-$ field on $A\ \subset \Omega$ if $\Im$ is a $\sigma-$field on Ω.
15. Show that for any class C of subsets of Ω and $A \subset \Omega$ the minimal $\sigma-$field $\sigma_A(C \cap A)$ generated by class $C \cap A$ on A is $\sigma(C) \cap A$ where $\sigma(C)$ is minimal $\sigma-$field generated by C on Ω.
16. Let $A_1, A_2, \ldots, A_n$ be events on a probability space $(\Omega, \Im, P)$.

    (i) For $n = 3$, if $P(A_1 \cap A_2 \cap A_3) = P(A_1^c \cap A_2^c \cap A_3^c)$, then each equals $\frac{1}{2}\left[1 - \sum_{i=1}^{3} P(A_i) + \sum_{i>j=1}^{3}\sum^{3} P(A_i \cap A_j)\right]$.

(ii) $P(\cap_{i=1}^{n} A_i) \geq \sum_{i=1}^{n} P(A_i) - (n-1)$.

17. Let $\Omega = \{s_1, s_2, s_3, s_4\}$ and $P(s_1) = \frac{1}{6}$, $P(s_2) = \frac{1}{5}$, $P(s_3) = \frac{1}{3}$ and $P(s_4) = \frac{3}{10}$. Define:

$$A_n = \begin{cases} \{s_1, s_2\}, & \text{if n is odd} \\ \{s_2, s_4\}, & \text{if n is even} \end{cases}$$

Find $P(\liminf A_n)$, $P(\limsup A_n)$, $\liminf P(A_n)$ and $\limsup P(A_n)$.

18. Let $\Im_i$ be a $\sigma-$field on Ω_i, $i = 1, 2$. Let $f : \Omega_1 \to \Omega_2$ be such that $A \in \Im_2 \Rightarrow f^{-1}(A) \in \Im_1$. If P is a probability measure on $(\Omega_1, \Im_1)$, then show that $Q(A) = P(f^{-1}(A))$, $A \in \Im_2$ is a probability measure on $(\Omega_2, \Im_2)$.

19. Prove that
(a) $\sum_{i=1}^{n} P(A_i) - \sum_{j=1}^{n} \sum_{i<j} P(A_i \cap A_j) \leq P(\cup_{i=1}^{n} A_i) \leq \sum_{i=1}^{n} P(A_i)$
(b) $P(\cap_{i=1}^{n} A_j) \geq 1 - \sum_{j=1}^{n} P(\bar{A}_j)$

20. In a series of independent tosses of a coin, let A_n be the event that a head occurs in the n^{th} toss. If p is the probability of A_n for all n, obtain the probability that infinitely many A_n's occur. Obtain the probability that infinitely many B_n's occur, where B_n is the event that head occurs for the first time at the n^{th} toss.

21. Let $(\Omega, \Im, P)$ be a probability space. Let $A, B, C \in S$ with $P(B)$ and $P(C) > 0$. If B and C are independent, show that

$$P\{A/B\} = P\{A/B \cap C\}P(C) + P\{A/B \cap C^c\}P(C^c).$$

Conversely, if this relation holds, $P\{A/B \cap C\} \neq P\{A/B\}$, and $P(B) > 0$, then B and C are independent.

22. Five percent of patients with a particular ailment are chosen to get a new therapy that is thought to boost recovery rates from 30 to 50%. After the course of treatment is complete, one of these patients is chosen at random and assessed for recovery. What is the probability that the patient got the new treatment?

23. From the rural jail, four recordings lead. A prisoner broke out of the facility. In the event that road 1 is chosen, there is a 1/8 chance of success, in the event that road 2 is chosen, there is a 1/6 chance of success; in the event that road 3 is chosen, there is a 1/4 chance of success; and in the event that road 4 is chosen, there is a 9/10 chance of success.

    1. How likely is it that the prisoner will be able to escape?
    2. What is the probability that the prisoner will use roads 4 and 1 to escape if they are successful?

24. The probability that an airplane accident which is due to structure failure is identified correctly is 0.85, and the probability that an airplane accident which is not due to structure failure is identified as due to structure failure is 0.15. If 30% of all airplane accidents are due to structure failure, find the probability that an airplane accident is due to structure failure given that it has been identified to be caused by structure failure.

25. The numbers $1, 2, 3, \ldots, n$ are put in that order at random. Calculate the probability that the digits $1, 2, \ldots, k$ $(k < n)$ appear next to each other in that order.

26. A secretary has to send n letters. She writes addresses on n envelopes and absent mindedly places letters one in each envelope. Find the probability that at least one letter reaches the correct destination.

27. In a town with $(n+1)$ residents, one person spreads a rumour to another, who then tells it to a third person, etc. The target of the rumour is selected at random among the n people accessible at each step. Calculate the probability that the rumour will be spread r times without being returned to the source.

28. Until the first time the same outcome occurs three times in a row (three heads or three tails), a biased coin with a chance of success (head) of p, $0 < p < 1$ is tossed. Calculate the probability that the seventh throw will bring the game to a close.

29. Show that the probability that exactly one of the events A or B occurs is equal to $P(A)+P(B)-2P(A\cap B)$.

30. The coefficients a, b and c of the quadratic equation $ax^2+bx+c=0$ are determined by rolling a fair die three times in a row. What is the probability that both roots of the equation are real?. What is the probability that both roots of the equation are complex?

31. Suppose that n independent trials, each of which results in any of the outcomes 0, 1 and 2, with respective probabilities 0.3, 0.5 and 0.2, are performed. Find the probability that both outcome 1 and outcome 2 occur at least once.

32. A pair of dice is rolled until a sum of 7 or an even number appears. Find the probability that 7 appears first.

33. In $2n$ tosses of a fair coin, what is the probability that more tails occur than the heads?

34. A biased coin (with probability of obtaining a 'Head' equal to $p > 0$) is tossed repeatedly and independently until the first head is observed. Compute the probability that the first head appears at an even numbered toss.

35. An Integrated MTech student has to take 5 courses a semester for 10 semesters. In each course he/she has a probability 0.3 of getting an 'A' grade. Assuming the grades to be independent in each course, what is the probability that he/she will have all 'A' grades in at least three semesters.

36. In a certain colony, 75% of the families own a car, 65% own a computer and 55% own both a car and a computer. If a family is randomly chosen, what is the probability that this family owns a car or a computer but not both?

37. Four tennis players A, B, C, D have the probabilities of winning a tournament as $P(A) = 0.35, P(B) = 0.15, P(C) = 0.3, P(D) = 0.2$. Before the tournament, the player B is injured and withdraws. Find the new probabilities of winning the tournament for A, C and D.

38. Two tokens are taken at random without replacement from an urn containing 10 tokens numbered 1 to 10. What is the probability that the larger of the two numbers obtained is 3?

39. In a room, there are four 18 year old males, six 18 year old females, six 19 year old males and x 19 years old females. What is the value of x if we want age and gender to be independent when a student is chosen at random.

40. Suppose there are two full bowls of cookies. Bowl 1 has 10 chocolate chip and 30 plain cookies, while bowl 2 has 20 of each. Our friend Raj picks a bowl at random, and then picks a cookie at random. We may assume there is no reason to believe Raj treats one bowl differently from another, likewise for the cookies. The cookie turns out to be a plain one. How probable is it that Fred picked it out of Bowl 1?

41. In a city with one hundred taxis, 1 is green and 99 are blue. A witness observes a hit-and-run case by a taxi at night and recalls that the taxi was green, so the police arrest the green taxi driver who was on duty that night. The driver proclaims his innocence and hires you to defend him in court. You hire a scientist to test the witness' ability to distinguish green and blue taxis in night. The data suggests that the witness sees green cars as green 97% of the time and blue cars as green 5% of the time. Write a mathematical speech for the jury to give them reason to believe innocence of your client's guilt.

42. Box I contains 3 red and 2 blue marbles while Box II contains 2 red and 8 blue marbles. A fair coin is tossed. If the coin turns up heads, a marble is chosen from Box I; if it turns up tails, a marble is chosen from Box II. Find the probability that a red marble is chosen.

43. 70% of the light aircraft that disappear while in flight in a certain country are subsequently discovered. Of the aircraft that are discovered, 60% have an emergency locator, whereas 90% of the aircraft not discovered do not have such a locator. Suppose that a light aircraft has disappeared. If it has an emergency locator, what is the probability that it will be discovered?

44. There are two identical boxes containing, respectively, four white and three red balls; three white and seven red balls. A box is chosen randomly, and a ball is drawn from it. Find the probability that the ball is white. If the ball is white, what is the probability that it is from the first box?

45. Customers are used to evaluate preliminary product designs. In the past, 95% of highly successful products received good reviews, 60% of moderately successful products received good reviews, and 10% of poor products received good reviews. In addition, 40% of products have been highly successful, 35% have been moderately successful, and 25% have been poor products. If a new design attains a good review, what is the probability that it will be a highly successful product? If a product does not attain a good review, what is the probability that it will be a highly successful product?

46. Consider that box A has 6 red chips and 3 blue chips, while box B has 4 red chips and 5 blue chips. From box A, one chip is randomly selected, and it is then put in box B. The next chip is then randomly selected from those currently in box B. Given that the red chip selected from box B, what is the likelihood that a blue chip was moved from box A to box B?

47. A system engineer is evaluating the dependability of a rocket with three stages. The first stage's engine must lift the rocket off the ground in order for the mission to be successful, and the second stage's engine must then take the rocket into orbit. The rocket's third stage engine is then employed to finish the mission. The likelihood of the mission being successfully completed determines how reliable the rocket is. The odds of the stages 1, 2, and 3 engines running efficiently are 0.99, 0.97, and 0.98, respectively. What is the rocket's reliability?

48. Identical and fraternal twins can both exist. Since identical twins are produced from the same egg, they are also the same sex. On the other side, there is a 50/50 probability that fraternal twins will be of the same sex. Between all sets of twins, there is a 1/3 chance that they are fraternal and a 2/3 chance that they are identical. What is the probability that the following set of twins, if they are the same sex, are identical?

49. What is the probability that a 7 will be rolled before an 8 if two fair dice are rolled?

50. In an urn, there are six red balls and three blue balls. A ball is chosen randomly from the urn and replaced with a ball of the opposite color. Then, another ball is chosen from the urn. What is the probability that the first ball chosen was red, given that the second ball chosen was also red?

51. Five children are born into a household; it is assumed that each birth is unrelated to the others and that there is a 0.5 chance that each child will be a girl. Given that the family has at least one male, what is the probability that they also have at least one girl?

52. A diagnostic test is said to be 90% accurate for a particular disease, which means that if a person has the disease, there is a 90% likelihood that the test will find it. Similar to this, there is a 90% likelihood that the test will come back negative if a person does not have the condition. Only 1% of people are affected by the illness. What is the probability that a person who was randomly selected from the population and given the test results actually has the disease?

53. There are 10 cartons of milk at a tiny grocery shop, and two of them are sour. What is the probability of choosing a carton of sour milk if you randomly choose the sixth milk carton sold that day?

54. Suppose P and Q are independent events such that the probability that at least one of them occurs is 1/3 and the probability that P occurs but Q does not occur is 1/9. What is the probability of Q?

55. In a cookie jar, there are three red marbles and one white marble, and in a shoebox, there is one red marble and one white marble. Without replacement, three marbles are drawn at random from the cookie jar and put in the shoebox. The next step is the random, replacement-free selection of 2 marbles from the shoebox. How likely is it that both of the marbles you choose from the shoebox will be red?

56. N black balls and N white balls are in an urn. Without replacement, three balls are randomly selected from the urn. What does n mean if there is a $1/12$ chance that all three balls will be white?

57. Suppose, I have five envelopes hidden in a box with the numbers 3, 4, 5, 6, and 7. I choose an envelope, and if it contains a prime number, I receive the square of that amount in money. If not, I choose another envelope and receive the sum of the squares from the two envelopes I choose (in rupees). What is the probability that amount will be 25?

58. There are 10 balls with the numbers 1 through 10 in an urn. Without replacement, the urn yields five balls at random. The identical two odd-numbered balls must be drawn on the odd-numbered draws for A to occur. How likely is the occurrence of event A?

59. The base and altitude of a right triangle are obtained by picking points randomly from $[0, a]$ and $[0, b]$, respectively. Show that the probability that the area of the triangle so formed will be less than $ab/4$ is $(1 + ln2)/2$.

60. Consider a randomly chosen group of $n(\leq 365)$ persons. What is the probability that at least two of them have the same birthdays?

61. In a movie theater that can accommodate $n + k$ persons, n persons are to be seated. What is the probability that $r \leq n$ given seats are occupied.

62. A company manufacturing cornflakes puts a card numbered 1 or 2 or 3, . . . , or n at random on each package, all numbers being equally likely to be drawn. If $m(\geq n)$ packages are purchased, show that the probability of being able to assemble atleast one complete set of cards from the packages is:

$$1 - \tbinom{n}{1}\left(1 - \frac{1}{n}\right)^m + \tbinom{n}{2}\left(1 - \frac{2}{n}\right)^m - \cdots + (-1)^n \tbinom{n}{n-1}\left(1 - \frac{n-1}{n}\right)^m$$

63. An urn contains 5 white and 4 black balls. Four balls are transferred to a second urn. A ball is then drawn from this urn and it happens to be black. Find the probability of drawing a white ball from among the remaining three.

64. Let a sample of size 4 be drawn with(without) replacement from an urn containing 12 balls of which 5 are white. If the sample contains 3 white balls then find the probability the ball drawn on the third draw was white.

65. There are n urns, each containing α white and β black balls. One ball is taken from urn 1 to urn 2 and then one is taken from urn 2 to urn 3 and so on. Finally a ball is chosen from urn n. If the first ball transferred was a white, what is the probability that the last chosen ball is white? what happens when $n \to \infty$?

66. A lot of five identical batteries is life tested. The probability assignment is assumed to be

$$P(A) = \int_A \frac{1}{\lambda} e^{-x/\lambda} dx$$

for any event $A \subseteq [0, \infty)$, where $\lambda > 0$ is a known constant. Thus the probability that a battery fails after time t is given by

$$P(t, \infty) = \int_t^\infty \frac{1}{\lambda} e^{-x/\lambda} dx, \qquad t \geq 0.$$

If the times to failure of the batteries are independent, what is the probability that at least one battery will operating after t_0 hours?

67. The first generation of a particle is the number of offsprings of a given particle. The next generation is formed by the offsprings of these members. If the probability that a particle has k offsprings (split into k parts) is p_k where $p_0 = 0.4, p_1 = 0.3, p_2 = 0.3$, find the probability that there is no particle in the second generation. Assume that the particles act independently and identically irrespective of the generation.