Department of Mathematics
MTL 601 (Probability and Statistics)
Tutorial Sheet No. 2

1. For $f : \Omega \to \Omega'$, show that $A \subset f^{-1}(f(A))$, if $A \subset \Omega$ and $f(f^{-1}(B)) \subset B$, if $B \subset \Omega'$.

2. Let $f : R \to R$ where $f(x) = \cos(x)$. Describe the $\sigma-$field induced by f and verify that it is a sub $\sigma-$field on R.

3. If A_1, A_2 are measurable sets and $f : \Omega \to R$ is defined by

$$f(w) = \begin{cases} -1, & w \in A_1 \\ 1, & w \in A_1^c \cap A_2 \\ 0, & w \in A_1^c \cap A_2^c \end{cases}$$

   Examine whether f is a measurable function over (a) $\sigma(A_1)$ (b) $\sigma(A_2)$ (c) $\sigma(\{A_1, A_2\})$.

4. Consider measurable space (Ω, F), where $\Omega = \{a, b, c, d\}$ and $F = \{\phi, \Omega, \{a, b\}, \{c, d\}\}$. Examine if the function $X : \Omega \to R$ defined by $X(a) = X(b) = 1,\ X(c) = 1,\ X(d) = 2$ is $F-$measurable. Find the minimal $\sigma-$field w.r.t which X is measurable.

5. Let f, g be Borel functions on (R, β). Define for $A \in \beta$:

$$h(w) = \begin{cases} f(w), & w \in A \\ g(w), & w \in A^c \end{cases}$$

   Show that h is a Borel function.

6. Give an example to show that $|f|$ is measurable function does not imply that f is measurable function.

7. If X is a random variable on a probability space (Ω, F, P), then show that $a + bX,\ |X|$,

$$X^+ = \begin{cases} X, & X \geq 0 \\ 0, & X < 0 \end{cases}$$

   are random variables on this probability space.

8. In a bombing attack, there is 50% chance that a bomb can strike the target. Two hits are required to destroy the target completely. How many bombs must be dropped to give a 99% chance or better of completely destroying the target?

9. Five-digit codes are chosen at random from the collection $\{0, 1, 2, \ldots, 9\}$ with replacement. Let X be the r.v. that indicates how many zeros there are in the selected codes. Find the PMF of r.v. X?

10. There are 10 coins in an urn, 4 of which are fake. One coin at a time is taken out of the urn until all counterfeit coins have been discovered. Let X be the r.v. that indicates how many coins must be eliminated in order to locate the first fake coin. Find the PMF of X?

11. For what values of α and p does the following function represent a PMF $p_X(x) = \alpha p^x,\quad x = 0, 1, 2, \ldots$.

12. A r.v. X has the following PMF

$X = x$	0	1	2	3	4	5	6	7	8
$P(X = x)$	k	$3k$	$5k$	$7k$	$9k$	$11k$	$13k$	$15k$	$17k$

   1. Determine the value of k.
   2. Find $P(X < 4),\ P(X \geq 5),\ P(0 < X < 4)$.
   3. Find the CDF of X.

4. Find the smallest value of x for which $P(X \leq x) = 1/2$.

13. Let X be a r.v. with PMF

$$p_X(x) = \frac{4c}{5^x}, x = 1, 2, \ldots$$

for some constant c. What is the value of c? Compute the probability that X is even?

14. A random number is chosen from the interval $[0, 1]$ by a random mechanism. What is the probability that (i) its first decimal will be 3 (ii) its second decimal will be 3 (iii) its first two decimal will be 3's?

15. Suppose X, the random number of eggs laid by a bird is a random variable having pmf:

$$p_X(x) = \frac{e^{-\lambda}\lambda^x}{x!}, \; x = 0, 1, 2, \ldots$$

where $\lambda > 0$ is a known constant. The probability of an egg developing is p. Find the pmf of random variable Y, the number of eggs developing. Also find the pmf of $|Y - 4|$.

16. Two balls are chosen at random from an urn of five balls with labels numbered 1 through 5 without replacement. The r.v. X should indicate the total of the numbers on the two balls. What is the the PDF of X?

17. The sum of the two six-sided dice's numbers is determined after rolling them. Let X, the r.v. that represents the total of the numbers rolled, be the variable. What is the the PDF of X?

18. A die is tossed two times. Define X =sum of scores and Y =absolute difference of scores on the two tosses. Identify the probability space and verify that X, Y are random variables on this probability space. Write down the events: $\{X = 3\}$, $\{Y \leq 1\}$, $\{X > 5\}$, $-1 < X \leq 4.5$ and determine their probabilities. Also find the p.m.f. of X and the distribution of Y. Using these determine $P(X \leq 7|X > 4)$ and $P(Y > 2.5|Y \leq 4)$.

19. A discrete random variable X has a uniform probability distribution on the set $\{-k, -(k-1), \ldots, -1, 0, 1, 2, \ldots, r\}$. Find the probability distribution of (a) $|X|$ and (b) $(X + 1)^2$.

20. Let the PDF of X be

$$f(x) = \begin{cases} c(8x - 5x^2), & 0 < x < 2 \\ 0, & \text{otherwise} \end{cases}.$$

   1. What is the value of c?
   2. What is the distribution of X?
   3. Obtain $P\left(\frac{1}{4} < X < \frac{3}{4}\right)$.

21. The life length of a certain equipment in hours is a continuous random variable with pdf

$$f_X(x) = \begin{cases} xe^{-kx}, & x > 0 \\ 0, & \text{elsewhere} \end{cases}$$

Determine the constant k.

(i) Given that the equipment has been operative for the last 10 hours, what is the probability that it will not fail during next 15 hours.

(ii) Ten pieces of equipment are tested independently for 50 hours and Y= the number of pieces that fail during the testing time. Find pmf of the random variable Y.

(iii) If n pieces of the equipment are hooked up in series and operate independently, find the pdf of the life of series system. Find n so that the reliability of the series system for 20 hours is 80%.

22. A bombing plane flies directly above a railroad track. Assume that if a larger(small) bomb falls within 40(15) feet of the track, the track will be sufficiently damaged so that traffic will be disrupted. Let X denote the perpendicular distance from the track that a bomb falls. Assume that

$$f_X(x) = \begin{cases} \frac{100-x}{5000}, & if\ x \in 0 < x < 100 \\ 0, & \text{otherwise} \end{cases}.$$

1. Find the probability that a larger bomb will disrupt traffic.
2. If the plane can carry three large (eight small) bombs and uses all three(eight), what is the probability that traffic will be disrupted?

23. An urn contains n cards numbered $1, 2, \ldots, n$. Let X be the least number on the card obtained when m cards are drawn without replacement from the urn. Find the probability distribution of r.v. X. Compute $P(X \geq 3/2)$.

24. Let

$$F_X(x) = \begin{cases} 0, & x < 0 \\ 1 - 2e^{-x} + e^{-2x}, & x \geq 0 \end{cases}.$$

Is F_X a distribution function? What type of r.v. is X? Find the PMF/PDF of X?

25. A continuous type r.v. X has the PDF

$$f(x) = \begin{cases} \frac{4x^2}{7} & 0 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}.$$

Compute the probability that X is greater than its 75^{th} percentile?

26. Let the CDF of X be

$$F(x) = \begin{cases} 0 & -\infty < x < 0 \\ 1 - \sum_{k=0}^{4} \frac{x^k e^{-x}}{k!} & 0 \leq x < \infty \end{cases}.$$

Find $P(X > 0)$?

27. Let X be a r.v. with CDF

$$F(x) = \begin{cases} 0 & -\infty < x < 0 \\ 1 - e^{-2x} & 0 \leq x < \infty \end{cases}.$$

Find $P(0 \leq e^X \leq 5)$?

28. The r.v. X has PDF

$$f(x) = \begin{cases} (k+1)x^3 & 0 < x < 2 \\ 0 & \text{otherwise} \end{cases},$$

where k is a constant. Compute the probability of X between the first and third quartiles?

29. Let X be a continuous type r.v. having CDF $F(x)$. What is the CDF of $Y = max(0, X)$?

30. What is the PDF of the r.v. X if its CDF is given by

$$F(x) = \begin{cases} 0 & x < 3 \\ 0.4 & 3 \leq x < 4 \\ 0.6 & 4 \leq x < 5 \\ 1 & x \geq 5 \end{cases}.$$

31. Let $\Omega = [0,1]$. Define $X : \to R$ by

$$X(w) = \begin{cases} w, & 0 \le w \le \frac{1}{2} \\ w - \frac{1}{2}, & \frac{1}{2} < w \le 1 \end{cases}$$

For any interval $I \subseteq [0,1]$, define $P(I) = \int_I 2x dx$. Identify the probability space and verify that X is a random variable on this space. Determine the distribution function of X and use this to find $P(X > \frac{1}{2}),\ P(\frac{1}{4} < X < \frac{1}{2}),\ P(X < \frac{1}{2} | X > \frac{1}{4})$.

32. Let $f(x)$ be a PDF of the continuous type r.v. X. Show that for every $-\infty < \mu < \infty$ and $\sigma > 0$, the function $\frac{1}{\sigma} f(\frac{x-\mu}{\sigma})$ is also a PDF of some continuous type r.v., say Z.

33. A point is chosen at random inside a square D with vertices at (0,0),(1,0),(0,1) and (1,1) in R^2. Define $X(u,v) = u + v,\ (u,v) \in D$. For any quadrilateral or triangle E in D, define $P(E)$ =area of E. Identify the probability space and verify that X is a random variable on this space. Find the distribution function of X.

34. An urn contains n cards numbered $1, 2, \ldots, n$. Let X be the least number on the m cards drawn randomly without replacement from the urn. Find probability distribution of random variable X. Compute $P(X \ge \frac{3}{2})$.

35. A point is chosen at random inside a circle or radius r. Let X = distance of the point from the center of the circle. Find the pdf of the random variable.

36. A point is chosen at random on the circumference of a circle of radius r. Find the pdf of the abscissa of the chosen point.

37. Let $F(x),\ x \ge 0$ be a distribution for a non-negative random variable and

$$G(x) = \begin{cases} 0, & x \le 0 \\ 1 - exp[-1 + \alpha(1 - F(x))], & x > 0 \end{cases}$$

where $\alpha > 0$ is a constant. Show that $G(x)$ is a distribution function and compute $P(X = 0 | X \le 1)$ assuming $G(x)$ is the distribution function of the random variable X.

38. Let X be a continuous random variable taking values in the interval $[0,1]$. If $P(x < X \le y)$, for all x, y, $0 \le x < y \le 1$ depends only on $(y - x)$, then show that X has uniform probability distribution on the interval $[0,1]$.

39. Let X be a random variable with distribution function

$$F_X(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{6} + \frac{x}{3}, & 0 \le x < 1 \\ 1, & x \ge 1 \end{cases}$$

Show that $F_X(x) = \alpha F_1(x) + (1-\alpha) F_2(x)$ for some α where F_1 and F_2 are respectively distribution functions of a discrete and a continuous random variable. Find α. Evaluate conditional probability $P\left(\frac{1}{2} \le X \le 1 | X > \frac{1}{4}\right)$.

40. Let X be a random variable such that $P(X > \frac{1}{2}) = \frac{7}{8}$ and its pdf is:

$$f_X(x) = \begin{cases} ax, & 0 \le x < 1 \\ b - x, & 1 \le x < 2 \\ 0, & \text{otherwise} \end{cases}$$

Determine a, b and find the distribution function of X.

41. Let X be a continuous random variable having pdf

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}},\ -\infty < x < \infty$$

Find pdf of:

(i) $Y = a + bX$

(ii) $Y = \ln X$

(iii) $Z = X^2$

(iv) $Y = \begin{cases} X^{1/2}, & X > 0 \\ -|X|^{1/2}, & X \leq 0 \end{cases}$

42. If X has p.d.f

$$f_X(x) = \begin{cases} 6x(1-x), & 0 < x < 1 \\ 0, & \text{elsewhere} \end{cases}$$

find the pdf of

(i) $Y = \frac{X}{1+X}$

(ii) $Z = \begin{cases} 2X, & X < 4 \\ \frac{1}{2}, & \frac{1}{4} \leq X < \frac{3}{4} \\ \frac{2}{3}X, & X \geq \frac{3}{4} \end{cases}$

43. Let

$$g(x) = \begin{cases} x, & |x| \geq b \\ 0, & |x| < b \end{cases}$$

Find probability distribution of g(X) if X has pdf $f(x)$.

44. Verify the following as a consequence of definition of expectation:

(a) If X is a bounded random variable, then $E(X)$ exists.

(b) If $X \in (a, b)$ with probability 1, then $a < E(X) < b$.

(c) If X is symmetrical abt a point μ, then $E(X) = \mu$.

(d) If $X \in \{1, 2, \dots, n, \dots\}$ with probability 1, then $E(X) = \sum_{n=1}^{\infty} P(X \geq n)$.

45. Let X and Y be two r.v.s such that their MGFs exist. Then, prove the following:

1. If $M_X(t) = M_Y(t),\ \forall\ t$, then X and Y have same distribution.
2. If $\Psi_X(t) = \Psi_Y(t),\ \forall\ t$, then X and Y have same distribution.

46. Show that $E(|X|) = \int_{-\infty}^{0} F_X(x)dx + \int_{0}^{\infty} (1 - F_X(x))dx$.

47. Let X be a r.v. with mean μ and variance σ^2. Show that $E[(aX - b)^2]$, as a function of b, is minimized when $b = a\mu$.

48. Let X be a continuous type r.v. with PDF

$$f_X(x) = \begin{cases} a + bx^2, & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

If $E(X) = \frac{3}{5}$, find the value of a and b.

49. Let A, B, C be events in a sample space with $P(A) = 0.1,\ P(B) = 0.4,\ P(C) = 0.3,\ P(A \cap B) = 0.1, A \cap C = B \cap C = \phi$. Find mean and variance of a random variable $X = I_A + I_B - I_C$, where $I_{[.]}$ is the indicator function of the set [.].

50. Show that the matrix $\sum = (\mu_{i+j})_{k\times k}$ is non negative definite where μ_p is the central moment of order p of the random variable X.

51. From a point on the circumfrence of the circle of radius r, a chord is drawn in a random direction. Show that the expected value of length of the chord is $\frac{4r}{\pi}$ and its variance is $2r^2\left(1-\frac{8}{\pi^2}\right)$.

52. For the random variable X with p.m.f. $p_X(x) = \frac{e^{-\lambda}\lambda^x}{x!},\ x = 0, 1, 2, \ldots$, show that $\mu_2 = \mu_3 = \lambda,\ \mu_4 = \lambda + 3\lambda^2$.

53. Show that for a random variable with p.d.f. $f_X(x) = \frac{k\alpha^k}{(x+\alpha)^{k+1}},\ x > 0$, the α^{th} absolute moment exists for X, for $\alpha < k$.

54. For the random variable with p.m.f. $p_X(x) = \binom{r+x-1}{x} p^r q^x,\ x = 0, 1, 2, \ldots$, where $q = 1-p$, find the m.g.f. and hence the mean and variance of X.

55. Find the mean, variance of the random variable X having the distribution function

$$F_X(x) = \begin{cases} 0, & x < 0 \\ p + (1-p)(1-e^{-\lambda x}), & 0 \le x < T \\ 1, & x \ge T \end{cases}$$

56. For the Laplace distribution, i.e.e the random variable having p.d.f. given by: $f_X(x) = \frac{1}{2\lambda} exp\left(\frac{|X-\mu|}{\lambda}\right),\ x \in R,\ \lambda > 0,\ 0 < \mu < \infty$, find m.g.f. and its mean and variance.

57. Let X be a random variable having m.g.f $M(t),\ t > 0$. Then show that for $t > 0,\ P(tX > s^2 + \log(M(t)) < e^{-s^2}$.

58. Show that the sequence of moments determine the probability distribution of the random variable uniquely if it is a bounded random variable.

59. Show that absolute moment of no order exists for the random variable having p.d.f $f_X(x) = \frac{1}{2|x|(\ln|x|)^2}$ for $|x| > e$.

60. Let X be a continuous r.v. with PDF

$$f(x) = \begin{cases} \alpha + \beta x^3, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

If $E(X) = 2/5$, find the value of α and β.

61. Consider a r. v. X with $E(X) = 1$ and $E(X^2) = 1$. Find $E\left[(X - E(X))^4\right]$ if it exists.

62. Let X be a discrete r.v. with MGF $M_X(t) = \alpha + \beta e^{2t} + \gamma e^{4t},\quad E(X) = 3,\ Var(X) = 2$. (a) Find α, β and γ? (b) Find the PMF of X? (c) Find $E(2^X)$?

63. The MGF of a r.v. X is given by $M_X(t) = exp(\mu(e^t - 1))$. (a) What is the distribution of X? (b) Find $P(\mu - 2\sigma < X < \mu + 2\sigma)$, given $\mu = 4$.

64. Let X be a r. v. with characteristic function given by $\varphi_X(t) = \frac{1}{7}\left(2 + e^{-it} + e^{it} + 3e^{2it}\right)$. Determine (a) $P\left(-1 \le X \le \frac{1}{2}\right)$. (b) $E(X)$.