Department of Mathematics
MTL 601 (Probability and Statistics)
Tutorial Sheet No. 3

1. A discrete random variable X has a uniform probability distribution on the set $\{-k, -(k-1), \ldots, -1, 0, 1, 2, \ldots, r\}$. Find the probability distribution of (a) $|X|$ and (b) $(X+1)^2$.

2. Let X be a random variable with Poisson distribution with parameter λ. Show that the characteristic function of X is $\varphi_X(t) = exp\left[\lambda(e^{it} - 1)\right]$. Hence, compute $E(X^2)$, $Var(X)$ and $E(X^3)$.

3. Let X be a random variable with $N(0, \sigma^2)$. Find the moment generating function for the random variable X. Deduce the moments of order n about zero for the random variable X from the above result.

4. Show that the PGF's of the geometric, negative binomial and Poisson distribution exists and hence calculate them.

5. Prove that, the r.v. X has exponential distribution and satisfies a memoryless property or Markov property which is given as
$$P(X > x + s/X > s) = P(X > x) \quad x, s \in \mathbb{R}^+. \tag{1}$$

6. Suppose that diameters of a shaft s manufactured by a certain machine are normal r.v. with mean 10 and s.d. 0.1. If for a given application the shaft must meet the requirement that its diameter falls between 9.9 and 10.2 cm. What proportion of shafts made by this machine will meet the requirement?

7. A machine automatically packs a chemical fertilizer in polythene packets. It is observed that 10% of the packets weigh less than 2.42 kg while 15% of the packets weigh more than 2.50 kg. Assuming that the weight of the packet is normal distributed, find the mean and variance of the packet.

8. Let $Y \sim N(\mu, \sigma^2)$ where $\mu \in \mathbb{R}$ and $\sigma^2 < \infty$. Let X be another r.v. such that $X = e^Y$. Find the distribution function of X. Also, verify that $E(\log(X)) = \mu$ and $Var(\log(X)) = \sigma^2$.

9. Demonstrate that any continuous nonnegative exponential r.v. is an exponential r.v..

10. Give α and β positive real numbers with the relationship $\alpha < \beta$. What is the probability that the distance between two randomly chosen points along a straight line segment of length β is at least α?

11. Let X have a beta distribution i.e. its pdf is
$$f_X(x) = \tfrac{1}{\beta(a,b)} x^{a-1}(1-x)^{b-1} \ , \ \ 0 < x < 1$$
and Y given $X = x$ has binomial distribution with parameters (n, x). Find the pdf of Y.

12. One of the inputs to a certain program is a random variable whose value is a nonnegative real number, call it Y. The pdf of Y is given by $f_Y(y) = ye^{-y}, \ y > 0$. Conditioned on $Y = y$, the execution time X of the program is an exponentially distributed random variable with parameter y.
    (a) Compute the pdf of the program execution time X.
    (b) Find the probability that input value is atleast 200 given that the program execution time is 99 hours.

13. Let X and Y be an independent random variables with pmfs $b(m,p)$ and $b(n,p)$ respectively. Prove that $P(X = x \mid X + Y = k)$ is hypergeometric distribution.

14. Let X and Y be i.i.d. random variables and let $P(X = k) = p_k > 0$, $k = 0, 1, \ldots$. Prove that, if
$$P(X = t/X + Y = t) = P(X = t-1/X + Y = t) = \frac{1}{t+1}, \quad t \geq 0$$
then X and Y are modified geometric random variables.

15. A certain industrial process yields a large number of steel cylinders whose lengths are distributed normal with mean 3.25 inches and standard deviation 0.05 inches. If two such cylinders are chosen at random and placed end to end what is the probability that their combined length is less than 6.60 inches?

16. A player takes out, simultaneously and randomly, two balls from a box that contains 8 white balls, 5 black balls and 3 blue balls. Suppose that the player wins Rs. 2 for each black ball selected and looses Rs. 1 for each white ball selected. Let X be a r.v. that denotes the players fortune. Find the PMF of the r.v. X. Also find $P(X^2 > 3)$.

17. An airline knows that 5 percent of the people making reservation on a certain flight will not show up. Consequently, their policy is to sell 52 tickets for a flight that can hold only 50 passengers. Assume that passengers come to airport are independent with each other. What is the probability that there will be a seat available for every passenger who shows up?

18. The number of weekly breakdown of a computer is a r.v. having a Poisson distribution with $\alpha = .03$. (a) What is the probability that computer will have even number of breakdown during the given week?. (b) If the computer is run for 10 consecutive weeks what is the probability that (i) at least two weeks have no breakdown (ii) 10th is the first week to have a breakdown?.

19. The number of times that an individual contracts viral infection in a given year is a Poisson r.v. with parameter $\lambda = 5$. Using a new health scheme, 75% of the population reduces the Poisson parameter λ to 1. If an individual does not get viral infection for a year, what is the probability that he/she followed the new health scheme?

20. A student arrives to the bus stop at 6:00 AM sharp, knowing that the bus will arrive in any moment, uniformly distributed between 6:00 AM and 6:20 AM. What is the probability that the student must wait more than five minutes? if at 6:10 AM the bus has not arrived yet, what is the probability that the student has to wait at least five more minutes?

21. Suppose that the life length of two electronic devices say D_1 and D_2 have normal distributions $N(40, 36)$ and $N(45, 9)$ respectively. If a device is to be used for 45 hours, which device would be preferred? If it is to be used for 42 hours which one should be preferred?

22. Let X be a discrete r.v. with MGF $M_X(t) = \alpha + \beta e^{2t} + \gamma e^{4t}$, $E(X) = 3$, $Var(X) = 2$. (a) Find α, β and γ? (b) Find the PMF of X? (c) Find $E(2^X)$?

23. The MGF of a r.v. X is given by $M_X(t) = exp(\mu(e^t - 1))$. (a) What is the distribution of X? (b) Find $P(\mu - 2\sigma < X < \mu + 2\sigma)$, given $\mu = 4$.

24. Let X be a r. v. with characteristic function given by $\varphi_X(t) = \frac{1}{7}\left(2 + e^{-it} + e^{it} + 3e^{2it}\right)$. Determine (a) $P\left(-1 \leq X \leq \frac{1}{2}\right)$. (b) $E(X)$.

25. If $X \sim U(0, 1)$. What is the PDF of $Y = -lnX$?

26. What is the chance that the r.v. will be larger than its square, or $P(X > X^2)$, if the r.v. X has a uniform distribution over the range $[0, a]$?

27. Let $X \sim U(-7, 7)$. What is the probability that the quadratic equation $114t^2 + 25tX + 3X = 0$ has complex solutions?

28. Let $X \sim Exp(\lambda)$. What is the probability density function of the r.v. $Y = X\sqrt{X}$?

29. Let a, b, c, d be any four real values, all of which should be positive. What is the PDF of the r.v. $Y = (b - a)X + a$ if $X \sim Beta(c, d)$?

30. Suppose that the life length of an item is exponentially distributed with parameter 0.5. Assume that ten such items are installed successively so that the ith item is installed immediately after the $(i-1)$th item has failed. Let T_i be the time to failure of the ith item $i = 1, 2, \ldots, 10$ and is always measured from the time of installation. Let S denote the total time of functioning of the 10 items. Assuming that $T_i's$ are independent, evaluate $P(S \geq 15.5)$.

31. A certain industrial process yields a large number of steel cylinders whose lengths are distributed normal with mean 3.25 inches and standard deviation 0.05 inches. If two such cylinders are chosen at random and placed end to end what is the probability that their combined length is less than 6.60 inches?

32. A small industrial unit has 30 machines whose lifetimes are independent exponentially distributed with mean 100 months. If all the machines are under use at a time, find the probability that even after 200 months there are at least five machines working.

33. Suppose that 30 electronic devices say $D_1, D_2, \ldots, D_{30}$ are used in the following manner. As soon as D_1 fails, D_2 becomes operative. When D_2 fails, D_3 becomes operative, etc. Assume that the time to failure of D_i is an exponentially distributed r.v. with parameter $= 0.1\ (\mathrm{h})^{-1}$. Let T be the total time of operation of the 30 devices. What is the probability that T exceeds $350\,\mathrm{h}$?