Department of Mathematics
MTL 601 (Probability and Statistics)
Tutorial Sheet No. 4

1. Let
$$F(x,y) = \begin{cases} 0, & x<0, y<0 \text{ or } x+y<1 \\ 1, & \text{otherwise} \end{cases}$$
Show that F is not a distribution function in R^2.

2. A point (u, v) is chosen as follows. First a u is chosen at random in the interval $(0, 1)$, then a point v is chosen at random on the interval $(0, u)$. Find the joint p.d.f. of (X, Y), defined by $X(u, v) = u$, $Y(u, v) = v$. Also find the conditional PDF of X, given $Y = y$ and that of Y, given $X = x$.

3. Suppose two cards are drawn from a deck of 52 cards. Let X = number of aces obtained and Y = number of queens obtained. Discuss whether or not random variables X, Y are independent.

4. Let the joint CDF function of X and Y is
$$F(x,y) = \begin{cases} (1-e^{-2x})(1-e^{-2y}) & x>0, y>0 \\ 0 & \text{otherwise} \end{cases}.$$
What is the joint PDF and Y, and the $P(1 < X < 4, 1 < Y < 3)$?

5. Let the joint PMF function of X and Y is
$$p(x,y) = \begin{cases} \frac{1}{33}(2x+y) & x=1,2,3, y=1,2 \\ 0 & \text{otherwise} \end{cases}.$$
What are the marginals of X and Y ?

6. For what value of c is the function
$$p(x,y) = \begin{cases} c(x+2y) & x=1,2, y=1,2 \\ 0 & \text{otherwise} \end{cases}.$$
a joint PMF?

7. Let the joint PDF function of X and Y is
$$f(x,y) = \begin{cases} 2e^{-(2x+y)} & 0 \le x, y < \infty \\ 0 & \text{otherwise} \end{cases}.$$
what is $P(X \ge Y \ge 3)$?

8. Let the joint PDF function of X and Y is
$$f(x,y) = \begin{cases} \frac{2}{9}x & 0<x<y<3 \\ 0 & \text{otherwise} \end{cases}.$$
What is the marginal PDF of Y?

9. Let the joint PDF function of X and Y is
$$f(x,y) = \begin{cases} \frac{2}{9}(3-x-y) & 0<x; y<3; 0<x+y<3 \\ 0 & \text{otherwise} \end{cases}.$$
What is the conditional probability $P(X < 1 | Y < 1)$?

10. A girl and boy decide to meet between 5:00 and 6:00 p.m. Assume they all arrive at separate times that are uniformly distributed at random within this time span. Each will wait no longer than 10 minutes for the other, after which they will both depart. What is the likelihood that they actually leave the house?

11. Let X and Y be two independent uniformly distributed r.v.s on $[0,1]$. Calculate, $P(Y \geq \frac{1}{2} | Y \geq 1-3X)$.

12. Verify that the normal distribution, geometric distribution, and Poisson distribution have reproductive property, but the uniform distribution and exponential distributions do not.

13. Suppose a two dimensional random variable has a joint PDF:

$$f_{X,Y}(x,y) = \begin{cases} kx(x-y), & 0 < x < 2, -x < y < x \\ 0, & \text{otherwise} \end{cases}$$

  (a) Evaluate the constant k

  (b) Compute $P(X+Y<1),\ P(XY<1)$

  (c) Compute $P(Y > -1/2 | X = 1)$

14. Let X_1, X_2 be i.i.d. random variables each having PMF $P(X = \pm 1) = \frac{1}{2}$. Define $X_3 = X_1 X_2$. Show that random variables X_1, X_2, X_3 are pairwise independent but not mutually independent.

15. Suppose X, Y are independent random variables each having binomial distribution with parameters n and $p,\ (0<p<1)$. FInd the joint pmf of $(X+Y, X-Y)$.

16. Suppose X, Y are independent Poisson random variables, show that the conditional distribution of X given $Z = X+Y$, is binomial.

17. Let X, Y be i.i.d each having standard normal distribution. Show that $U = \sqrt{X^2+Y^2}$ and $V = \frac{X}{Y}$ are independent.

18. Let (X, Y) be uniformly distributed on the region $\{(x,y) : 0 < x < y < 1\}$.

  (a) Find $V(2X+3Y-4),\ Cov(X+Y, X-Y),\ E(X^3+XY^2-X^2Y)$.

  (b) Find regression of Y on X and of X on Y.

  (c) Find MGF of (X, Y) and use this to compute correlation coefficient between X, Y.

19. Let $X_1, X_2 \dots, X_{m+n}$ be i.i.d. random variables each having a finite second order moment. Find the correlation coefficient between S_n and $S_{m+n} - S_n,\ n > m$, where $S_k = \sum_{i=1}^{k} X_i,\ k = 1, 2, \dots, m+n$.

20. Let X, Y be discrete random variables with respective PMFs given by

$$p_X(x_1) = p_1,\ p_X(x_2) = 1-p_1,\ p_Y(y_1) = p_2,\ p_Y(y_2) = 1-p_2$$

  Show that X, Y are independent iff X, Y are uncorrelated.

21. Using the concept of conditional expectation, prove that:

$$Var(X) = Var(E(X|Y)) + E(Var(X|Y))$$

22. Let $X_1, X_2, \dots, X_n$ be the sequence of i.i.d. random variables and N be discrete random variable taking positive integer values. Suppose $X's$ and N are independent. Define random sum: $S_N = X_1 + X_2 + \ldots + X_N$.

  (a) Show that $E(S_N) = E(N)E(X_1)$ and $Var(S_N) = (E(X_1))^2 Var(N) + E(N)Var(X_1)$.

  (b) Find MGF of S_N in terms of mgf's of X_i and N.

23. Let X, Y be i.i.d random variables with common pdf

$$f(x) = \begin{cases} e^{-x}, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases}$$

Find the pdf of random variables $\min\{X,Y\}$, $\max\{X,Y\}$. Let $U = X + Y$ and $V = X - Y$. Find the conditional pdf of V, given $U = u$ for some fixed $u > 0$.

24. Let X and Y be independent RVs defined on the space $(\Omega, \mathcal{S}, P)$. Let X be uniformly distributed on $(-a, a), a > 0$, and Y be a RV of the continuous type with density f, where f is continuous and positive on $\mathcal{R}$. Let F be the CDF of Y. If $u_0 \in (-a, a)$ is a fixed number, show that

$$f_{Y|X+Y}(y|u_0) = \begin{cases} \frac{f(y)}{F(u_0+a)-F(u_0-a)} & \text{if } u_0 - a < y < u_0 + a \\ 0 & \text{otherwise} \end{cases}$$

where $f_{Y|X+Y}(y|u_0)$ is the conditional density function of Y, given $X + Y = u_0$.

25. One of the inputs to a certain program is a random variable whose value is a nonnegative real number, call it Y. The pdf of Y is given by $f_Y(y) = ye^{-y},\ y > 0$. Conditioned on $Y = y$, the execution time X of the program is an exponentially distributed random variable with parameter y.
   (a) Compute the pdf of the program execution time X.
   (b) Find the probability that input value is atleast 200 given that the program execution time is 99 hours.

26. Let X and Y be independent nonnegative RVs of the continuous type with PDFs f and g, respectively. Let $f(x) = e^{-x}$ if $x > 0$, and$= 0$ if $x \leq 0$, and let g be arbitrary. Show that the MGF of Y, which is assumed to exist, has the property that the CDF of X/Y is $1 - M(-t)$.

27. Let (X, Y) be iid RVs with common standard normal density

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad -\infty < x < \infty.$$

Let $U = X + Y$ and $V = X^2 + Y^2$. Find the MGF of the random variable (U, V). Also, find the correlation coefficient between U and V. Are U and V independent?

28. Let the joint PMF function of X and Y is

$$p(x,y) = \begin{cases} \binom{7}{y} x^y (1-x)^{7-y} & y = 0, 1, 2, \ldots, 7 \\ 0 & \text{otherwise} \end{cases}.$$

Find the conditional expectation of Y given $X = x$?

29. Let the joint PDF function of X and Y is

$$f(x,y) = \begin{cases} \frac{3}{4}x & 0 < y < x < 1 \\ 0 & \text{otherwise} \end{cases}.$$

Find the conditional expectation of Y given $X = 0.5$?

30. Let the joint PDF function of X and Y is

$$f(x,y) = \begin{cases} \frac{6}{7}x & 1 \leq x + y \leq 2, x \geq 0, y \geq 0 \\ 0 & \text{otherwise} \end{cases}.$$

Find the marginal PDF of Y and the conditional variance of X given $Y = \frac{1}{2}$?

31. Let the joint PDF of X and Y is

$$f(x,y) = \begin{cases} 4x & 0 < x < \sqrt{y} < 1 \\ 0 & \text{otherwise} \end{cases}.$$

Find the conditional variance of Y for $X = x$?

32. The joint pdf of (X, Y) is given by:

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{4}(1 + xy(x^2 - y^2)), & |x| \le 1,\ |y| \le 1 \\ 0, & \text{otherwise} \end{cases}.$$

Compute

(a) $E(Y|X),\ E(XY^2 + Y|X),\ Cov(X^2, Y^2)$

(b) $MGF(X,Y),\ MGF(X+Y)$.