Department of Mathematics
MTL 601 (Probability and Statistics)
Tutorial Sheet No. 5

1. (Jensens's Inequality) If g is a convex function and $E(X)$ exists, then show that $g(E(X)) \leq E(g(X))$. Hence show that $E(X) \leq (E(|X|^r))^{1/r}$.

2. Let $g(X) \geq 0, \forall x \in [0, \infty)$ be a non-decreasing even function. Show that for any random variable X such that $E(g(X))$ exists $P(|X| \geq \epsilon) \leq \frac{E(g(X))}{g(\epsilon)}$.

3. Let $\{X_n\}$ be a sequence of random variables defined on the probability space $([0, \infty), \beta_1, P),\ P(\{0\}) = 0$.

   (a) Define $X_n(s) = \frac{1}{s}\left(1 - \frac{1}{n}\right),\ s \in [0, \infty)$. Then show that $X_n \overset{a.s}{\to} X$, where $X(s) = \frac{1}{s},\ s \in [0, \infty)$.

   (b) Define $X_n(s) = \frac{1}{ns},\ s \in [0, \infty)$. Then show that $X_n \overset{a.s}{\to} 0$.

   (c) Let $P(I) = \displaystyle\int_I e^{-x} dx$. Define

   $$X_n(s) = \begin{cases} 0, & \text{if } s \text{ is rational} \\ (-1)^n, & \text{if } s \text{ is irrational} \end{cases}$$

   Then show that $\{X_n\}$ diverges almost surely.

4. Consider a sequence of random variables $\{X_n\}$ with $E(X_n) = m$ and

   $$Cov(X_i, X_j) = \begin{cases} \sigma^2, & i = j \\ a\sigma^2, & i = j \pm 1, \text{ where } |a| < 1,\ \sigma^2 > 0 \text{ are given contants} \\ 0, & \text{otherwise} \end{cases}$$

   Show that WLLN holds for $\{X_n\}$.

5. Consider a sequence of independent random variables $\{X_n\}$ such that

   $$P(X_n = 0) = 1 - \frac{2}{n^3},\ P(X_n = \pm n) = \frac{1}{n^3},\ n > 1$$

   Does the sequence $\{X_n\}$ obey CLT ?

6. Consider a sequence $\{X_n\}$ of identically distributed random variables with the property that $nP(|X_i| > n) \to 0$ as $n \to \infty$. Show that $\frac{1}{n} \max_{1 \leq i \leq n} X_i \overset{p}{\to} 0$

7. Suppose $|X_n - X| \leq Y_n$, almost surely for some random variable X, then show that of $E(Y_n) \to 0$, then $E(X_n) \to E(X)$ and $X_n \overset{p}{\to} X$.

8. Show that $X_n \overset{2}{\to} X \ \Rightarrow\ E(X_n) \to E(X),\ E(X_n^2) \to E(X^2)$ as $n \to \infty$.

9. Let $\{X_i\}$ be a sequence of independent random variables, such that each X_i has mean 0 and variance 1. Show that

   $$\sqrt{n}\frac{X_1 + X_2 + \cdots + X_n}{X_1^2 + X_2^2 + \cdots + X_n^2} \overset{l}{\to} Z \sim N(0, 1)$$

10. Does WLLN hold for the following sequences

    (a) $P(X_k = \pm 2^k) = \frac{1}{2}$

    (b) $P(X_k = \pm \frac{1}{k}) = \frac{1}{2}$

11. For what values of α, does the strong law of large numbers hold for the sequence $\{X_n\}$, where $P(X_k = \pm k^\alpha) = \frac{1}{2},\ k = 1, 2, \ldots$.

12. Let $\{X_n\}$ be a sequence of independent random variables with the following probability distribution. In each case, does the Lindeberg Condition for CLT hold ?

    (a) $P(X_k = \pm\frac{1}{2^n}) = \frac{1}{2}$

    (b) $P(X_k = \pm\frac{1}{2^{n+1}}) = \frac{1}{2^{n+3}},\ P(X_n = 0) = 1 - \frac{1}{2^{n+2}}$

13. Suppose that the number of customers who visit SBI, IIT Delhi on a Saturday is a r.v. with $\mu = 75$ and $\sigma = 5$. Find the lower bound for the probability that there will be more than 50 but fewer than 100 customers in the bank?

14. From an urn containing 10 identical balls numbered 0 through 9, n balls are drawn with replacement,

    (a) What does the law of large number tell you about the appearance of 0's in n drawings.

    (b) How many drawings must be made in order that with probability atleast 0.95, the relative frequency of occurrence of 0's will be between 0.09 and 0.11 ?

15. Does the r.v. X exist for which
$$P\left[\mu - 2\sigma \le X \le \mu + 2\sigma\right] = 0.6.$$

16. Use CLT to show that
$$\lim_{n\to\infty} e^{-nt}\sum_{k=0}^{n}\frac{(nt)^k}{k!} = 1 = \begin{cases} 1, & 0 < t < 1 \\ 0.5, & t = 1 \\ 0, & t > 1 \end{cases}.$$

17. Let $X_1, X_2, \ldots, X_{50}$ be a random sample of size 50 from a distribution with density
$$f(x) = \begin{cases} \frac{1}{\Gamma(\alpha)\theta^{\alpha}x^{\alpha-1}e^{-x/\theta}} & 0 < x < \infty \\ 0 & \text{otherwise.} \end{cases}$$
    Find out the mean and variance of the sample mean $\bar{X}$?

18. A random sample from a uniform distribution on the range $[1, 12]$ would be $X_1, X_2, \ldots, X_9$. How likely is it that the next-smallest number will be higher than or equal to 4?

19. Assume that Z is the sample range and $X_1, X_2, \ldots, X_n$ is a random sample from the uniform distribution on (0, 1). How likely is it that Z will be less than or equal to 0.5?

20. Suppose that a 1996-size random sample from a distribution with PDF is $X_1, X_2, \ldots, X_{1996}$
$$f(x) = \frac{e^{-\lambda}\lambda^x}{x!}.$$
    What is the distribution of $1996\bar{X}$?

21. A random sample of size n drawn from a Bernoulli distribution with success probability $p = \frac{1}{2}$ is defined as $X_1, X_2, \ldots, X_n$. What is the sample mean X's limiting distribution?

22. A random sample of size n from a normal distribution with a mean of mu and a variance of 1 will be considered. What is the sample size n if the statistic $W = \sum_{i=1}^{n}(X_i - \bar{X})^2$ has a 75th percentile of 28.24?

23. Let $X_1, X_2, \ldots, X_n$ be a random sample from a uniform distribution on the interval from 0 to 5. What is the limiting moment generating function of $\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}$ as $n \to \infty$?

24. A continuous distribution has been used to draw five observations at random and independently. What is the chance that the sixth observation won't be as good as the first five combined?

25. A normal population with a mean of 100 and a variation of 4 is to be selected at random from by taking a sample of size 16. What is the sample mean's 90th percentile of the distribution?

26. A certain medicine has been shown to produce allergic reactions in 1% of the population. If this medicine is dispensed to 500 people, what is the probability that at most 10 of them show any allergy symptoms?

27. Let $X \sim B(n, p)$. Use the CLT to find n such that

$$P[X > n/2] \geq 1 - \alpha.$$

Calculate the value of n when $\alpha = 0.90$ and $p = 0.45$.

28. If X is a nonnegative r.v. with mean 2, what can be said about $E(X^3)$ and $E(lnX)$.

29. Construct a sequence of independent random variables $\{X_n\}$ with $P(X_n = 0) = 1 - \frac{1}{n^\alpha}$, $P(X_n = \pm n) = \frac{1}{2n^\alpha}$. Determine the values of α for which the sequence $\{X_n\}$ obeys WLLN.

30. The bulbs manufactured by a plant have life lengths which follow exponential distribution with mean 100 hours. A series of 50 independent life tests is performed. Each test consist of
1. choosing 4 bulbs at random from the plant
2. putting them into operation simultaneously and independently of each other
3. the life test ends as soon as any one of the four bulbs fails and time taken to complete the test is noted

Let $Y_1, Y_2, \ldots, Y_{50}$ respectively be the times of 50 life tests.

(a) What is the pdf of any Y_i

(b) What is the pdf of T, the total time of 50 tests

(c) Use central limit theorem to compute approximately $P(T > 1000)$

31. Suppose that X_i , $i = 1, 2, \ldots, 30$ are independent r.v. each having a Poisson distribution with parameter 0.01. Let $S = X_1 + X_2 + \cdots + X_{30}$.
(a) Using central limit theorem evaluate $P(S \geq 3)$.
(b) Compare the answer in (a) with exact value of this probability.

32. Let $X_1, X_2, \ldots$ be a sequence of iid r.v. with mean 1 and variance 1600, and assume that these variables are non-negative. Let $Y = \sum_{k=1}^{100} X_k$. Use the central limit theorem to approximate the probability $P(Y \geq 900)$.

33. Let $X_1, X_2, \ldots, X_{100}$ be a random sample of size 100 from a distribution with PMF

$$p(x) = \begin{cases} \frac{e^{-\lambda}\lambda^x}{x!} & x = 0, 1, 2, \ldots \\ 0 & \text{else} \end{cases} n.$$

Find the probability of $\overline{X}$ being greater that or equal to 0.5?

34. Let X and Y be two r.v.s with joint PDF

$$f(x, y) = \begin{cases} 12x^2 & 0 < y < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the expected value of the r.v. $Z = X^3Y^3 + X^3 - X^2 - Y^2$?

35. A machine requires 4 of the 6 independent parts to function. Let each component's lifetime be represented by $X_1, X_2, \ldots, X_6$. Assume they are all distributed exponentially with theta as the parameter. What is the machine lifetime PDF?

36. The weight of the male and female students at a particular large university is roughly normally distributed, with means and standard deviations of 180 and 20, and 130 and 15, respectively. What is the likelihood that, if a man and woman are chosen at random, the sum of their weights will be less than 280?

37. Let X be a r.v. having the PDF

$$f(x) = \begin{cases} 3(1-x)^2 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}.$$

What is the probability that the larger of 4 independent observations of X will exceed $\frac{1}{3}$?

38. A fair dice is rolled as many times as needed until the sum of all the results obtained is greater than 200. What is the probability that at least 50 tosses are needed?

39. A coffee exporter a particular brand reports of that the amount of impurities in 1 grams of coffee is a r.v. with mean 4 grams and standard deviation 5.4 grams. In a sample of 50 grams of coffee from this exporter, what is the probability that the sample mean is greater than 6.7 grams?

40. Let X be a r.v. having a Gamma distribution with parameters n and 1. How large must n be in order to guarantee that

$$P\left(\left|\frac{X}{n} - 1\right| > 0.01\right) < 0.01?$$

41. How many tosses of a fair coin are needed so that the probability that the average number of heads obtained differs at most 0.01 from 0.5 is at least 0.90?

42. Let X be a nonnegative r.v.. Prove:

$$E(X) \le [E(X^2)]^{\frac{1}{2}} \le [E(X^3)]^{\frac{1}{3}} \le \ldots.$$

Verify the above result for $X \sim \ Geo(p)$.

43. Examine the nature of convergence of $\{X_n,\ n = 1, 2, 3, \ldots\}$ defined below for the different values of k for $n = 1, 2, \ldots$:

$$\begin{aligned} P(X_n = n^k) &= \frac{1}{n} \\ P(X_n = 0) &= 1 - \frac{2}{n} \\ P(X_n = -n^k) &= \frac{1}{n}. \end{aligned}$$

44. Show that the convergence in the rth mean does not imply almost sure convergence for the sequence $\{X_n,\ n = 1, 2, 3, \ldots\}$ defined below:

$$\begin{aligned} P(X_n = 0) &= 1 - \frac{1}{n} \\ P(X_n = -n^{1/2r}) &= \frac{1}{n}. \end{aligned}$$

45. For each $n \ge 1$, let X_n be an uniformly distributed r.v. over set $\{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}$, i.e.,

$$P\left(X_n = \frac{k}{n} = \frac{1}{n+1}, k = 0, 1, \ldots, n\right).$$

Let U be a r.v. with uniform distribution in the interval $[0, 1]$. Show that $X_n \xrightarrow{d} U$.

46. Let $\{X_n,\ n=1,2,3,\ldots\}$ be a sequence of iid r.v. with $E(X_1)=Var(X_1)=\lambda\in(0,\infty)$ and $P(X_1>0)=1$. Show that for $n\to\infty$,

$$\sqrt{n}\frac{Y_n-\lambda}{\sqrt{\lambda}}\xrightarrow{d}N(0,1)$$

where

$$Y_n=\frac{1}{n}\sum_{k-1}^{n}X_k.$$

47. Let $\{X_n,\ n=1,2,3,\ldots\}$ be a sequence of iid r.v. with $P(X_n=1)=P(X_n=-1)=\frac{1}{2}$. Show that

$$\frac{1}{n}\sum_{j=1}^{n}X_j$$

converges in probability to 0.

48. Let $X,X_1,X_2,\ldots$ be a sequence of r.v. defined on a same probability space (Ω,S,P). Prove that

   (a) If $X_n\xrightarrow{a.s.}X$ and $Y_n\xrightarrow{a.s.}Y$, then $X_n+Y_n\xrightarrow{a.s.}X+Y$.
   (b) If $X_n\xrightarrow{a.s.}X$ and $Y_n\xrightarrow{a.s.}Y$, then $X_nY_n\xrightarrow{a.s.}XY$.
   (c) If $X_n\xrightarrow{P}X$ iff $X_n-X\xrightarrow{P}0$.
   (d) If $X_n\xrightarrow{P}X$ and $Y_n\xrightarrow{P}Y$, then $P(X=Y)=1$.
   (e) If $X_n\xrightarrow{P}X$ then $X_n-X_m\xrightarrow{P}0$.
   (f) If $X_n\xrightarrow{P}X$ and $Y_n\xrightarrow{P}Y$, then $X_n+Y_n\xrightarrow{P}X+Y$.
   (g) If $X_n\xrightarrow{P}X$ and $Y_n\xrightarrow{P}Y$, then $X_nY_n\xrightarrow{P}XY$.

49. A fair coin is flipped 100 consecutive times. Let X be the number of heads obtained. Use Chebyschev's inequality to find a lower bound for the probability that $\frac{X}{100}$ differs from $\frac{1}{2}$ by less than 0.1.

50. Let $X_1,X_2,\ldots,X_n$ be n independent Poisson distributed r.v. with means $1,2,\ldots,n$, respectively. Find an x in terms of t such that

$$P\left(\frac{S_n-\frac{n^2}{2}}{n}\le t\right)\approx\Phi(x)$$

for sufficiently large n, where Φ is the CDF of $N(0,1)$.

51. Suppose that $X_i; i=1,2,\ldots,450$, are independent r.v. each having a distribution $N(0,1)$. Evaluate $P(X_1^2+X_2^2+\ldots+X_{450}^2>495)$ approximately.

52. Let Y be a Gamma distributed r.v. with PDF

$$f(y)=\begin{cases}\frac{1}{\Gamma(p)}e^{-y}y^{p-1} & y>0\\ 0 & \text{otherwise}\end{cases}.$$

Find the limiting distribution of $\frac{Y-E(Y)}{\sqrt{Var(Y)}}$ as $p\to\infty$.

53. Suppose that the MGF $(M_x(t))$ of a r.v. X exists. Use Markov's inequality to show that for any $a\in R$ the following holds:

- $P(X\ge a)\le exp(-ta)M_x(t),\forall t>0$.
- $P(X\le a)\le exp(-ta)M_x(t),\forall t<0$.