Department of Mathematics
MTL 601 (Probability and Statistics)
Tutorial Sheet No. 7

1. Suppose $X \sim B(n, p)$ show that the sample proportion X/n is an unbiased estimator of p.

2. Prove that MLE is consistent for the estimation of $p > 0$ in the Gamma family
$$f(x) = \frac{e^{-x}x^{r-1}}{\Gamma(r)}, \quad 0 < x < \infty.$$

3. Let $X_1, X_2, \ldots, X_n$ be a random sample from Poisson distribution with parameter λ. Show that $\alpha\overline{X} + (1-\alpha)s^2, 0 \leq \alpha \leq 1$, is a class of unbiased estimators for λ. Find the UMVUE for λ. Also, find an unbiased estimator for $e^{-\lambda}$.

4. Let $X_1, X_2, \ldots, X_n$ be a random sample from Bernoulli distribution $B(1, p)$. Find an unbiased estimators for p^2 if it exists.

5. Suppose that 200 independent observations $X_1, X_2, \ldots, X_{200}$ are obtained from random variable X. We are told that $\sum_{i=1}^{200} x_i = 300$ and that $\sum_{i=1}^{200} x_i^2 = 3754$. Using these values, obtain unbiased estimates for $E(X)$ and $Var(X)$.

6. If X_1, X_2 and X_3 are three independent random variables having the Poisson distribution with the parameter λ, show that
$$\widehat{\lambda_1} = \frac{X_1 + 2X_2 + 3X_3}{6}$$
is an unbiased estimator of λ. Also, compare the efficiency of $\widehat{\lambda_1}$ with that of the alternate estimator
$$\widehat{\lambda_2} = \frac{X_1 + X_2 + X_3}{3}.$$

7. Let X be Cauchy-distributed random variable with PDF
$$f(x;\theta) = \frac{1}{\pi}\frac{1}{(1 + (x-\theta)^2)}, \quad -\infty < x < \infty, -\infty < \theta < \infty.$$
Find the Cramer–Rao lower bound for the estimation of the location parameter θ.

   Consider the normal distribution $N(0, \theta)$. With a random sample $X_1, X_2, \ldots, X_n$ we want to estimate the standard deviation $\sqrt{\theta}$. Find the constant c so that $Y = c\sum_{i=1}^{n} |X_i|$ is an unbiased estimator of $\sqrt{\theta}$ and determine its efficiency.

8. If $X_1, X_2, \ldots, X_n$ is a random sample from $N(\mu, 1)$. Find a lower bound for the variance of an estimator of μ^2. Determine an unbiased minimum variance estimator of μ^2, and then compute its efficiency.Please suggest whether the sentence 'If $X_1, X_2, \ldots, X_n$...' conveys the intended meaning.

9. Prove that $\overline{X}$ the mean of a random sample of size n from a distribution that is $N(\mu, \sigma^2)$, $-\infty < \mu < \infty$ is an efficient estimator of μ for every known $\sigma^2 > 0$.

10. Assuming population to be $N(\mu, \sigma^2)$, show that sample variance is a consistent estimator for population variance σ^2.

11. Let $X_1, X_2, \ldots, X_n$ be a random sample from uniform distribution on an interval $(0, \theta)$. Show that $\left(\prod_{i=1}^{n} X_i\right)^{1/n}$ is consistent estimator of θe^{-1}.

12. The number of births in randomly chosen hours of a day is as follows.

Number of births	4	0	6	5	2	1	2	0	4	3

Use this data to estimate the proportion of hours that had 2 or fewer births.

13. Consider the number of students attended Probability and Statistics lecture classes for 42 lectures. Let $X_1, X_2, \ldots, X_{30}$ denote the number students attended in randomly chosen 30 lecture classes. Suppose the observed data is as follows

100	90	85	95	88	82	92	84	88	87	82	88	82	91	92	91
82	90	82	87	92	70	84	79	88	81	82	78	81	82	90	79

(a) Using method of moments, find the estimators for the population mean and population variance.

(b) Assume that, X_i's are i.i.d random variables each having discrete uniform distribution with interval 70 to 90, find ML estimators for the population mean and population variance.

14. Using method of moments, find the estimators of the parameters for the following population distributions (i) $N(\mu, \sigma^2)$ (ii) $Exp(\lambda)$ (iii) $P(\lambda)$.

15. Let $X_1, X_2, \ldots, X_n$ be i.i.d from the uniform distribution $U(a, b)$, $-\infty < a < b < \infty$. Prove that, using the method of moments, the estimators of a and b are, respectively,

$$\hat{a} = \overline{X} - \sqrt{\frac{3(n-1)}{n} S^2} \quad \text{and} \quad \hat{b} = \overline{X} + \sqrt{\frac{3(n-1)}{n} S^2}.$$

16. Let $X_1, X_2, \ldots, X_n$ be i.i.d from the binomial distribution $B(n, p)$, with unknown parameters n and p. Prove that, using the method of moments, the estimators of n and p are, respectively,

$$\hat{n} = \frac{\overline{X}}{1 - \frac{n-1}{n}\frac{S^2}{\overline{X}}} \quad \text{and} \quad \hat{p} = 1 - \frac{n-1}{n}\frac{S^2}{\overline{X}}.$$

17. Let $X_1, X_2, \ldots, X_n$ be i.i.d from the gamma distribution $G(r, \lambda)$, with unknown parameters r and λ. Prove that, using the method of moments, the estimators of r and λ are, respectively,

$$\hat{r} = \frac{\overline{X}^2}{\frac{1}{n}\sum_{i=1}^{n} X_i^2 - \overline{X}^2} \quad \text{and} \quad \hat{\lambda} = \frac{\frac{1}{n}\sum_{i=1}^{n} X_i^2 - \overline{X}^2}{\overline{X}}.$$

18. Consider a queueing system in which the arrival of customers follows Poisson process. Let X be the distribution of service time, which has gamma distribution; i.e., the PDF of X is given by

$$f(x; \lambda, r) = \begin{cases} \frac{\lambda(\lambda x)^{r-1} e^{-\lambda x}}{\Gamma(r)}, & 0 < x < \infty \\ 0, & \text{otherwise} \end{cases}.$$

Suppose that r is known. Let $X_1, X_2, \ldots, X_n$ be a random sample on X. Obtain the ML estimate of λ based on this sample. Further, assume that r is also unknown. Determine the ML estimators of λ and r.

19. Prove that for the family of uniform distribution on the interval $[0, \theta]$, $max(X_1, X_2, \ldots, X_n)$ is the MLE for θ.

20. Suppose that the random sample arises from a distribution with PDF

$$f(x;\theta) = \begin{cases} \theta x^{\theta-1}, & 0 < x < 1, \quad 0 < \theta < \infty \\ 0 & \text{otherwise} \end{cases}.$$

Show that $\hat{\theta} = -\dfrac{n}{\ln \Pi_{i=1}^{n} X_i}$ is the MLE of θ. Further, prove that in a limiting sense, $\hat{\theta}$ is the minimum variance unbiased estimator of θ and thus θ is asymptotically efficient.

21. Find the maximum likelihood estimator based on a sample of size n from the two-sided exponential distribution with PDF

$$f(x;\theta) = \frac{1}{2}e^{-|x-\theta|}, \qquad -\infty < x < \infty.$$

Is the estimator unbiased?

22. If $X_1, X_2, \ldots, X_n$ is a random sample from $N(\theta, 1)$. Find a lower bound for the variance of an estimator of θ^2. Determine an unbiased minimum variance estimator of θ^2 and then compute its efficiency.

23. Let $X_1, X_2, \ldots, X_n$ be random sample from a distribution with pdf

$$f(x;\theta) = \begin{cases} \theta^x(1-\theta), & x = 0, 1, 2, \ldots; \quad 0 \leq \theta \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the MLE $\hat{\theta}$ of θ.

(b) Show that $\sum_{i=1}^{n} X_i$ is a complete sufficient statistics for θ.

(c) Determine the unbiased maximum variance estimator of θ.

24. Prove that method of moment estimator is consistent for the estimation of $r > 0$ in the gamma family

$$f(x;r) = \begin{cases} \frac{e^{-x}x^{r-1}}{\Gamma(r)} & 0 < x < \infty \\ 0, & \text{otherwise} \end{cases}.$$

25. Let $X_1, X_2, \ldots, X_n$ be a random sample from the normal distribution with both mean and variance equal to an unknown parameter θ, $\theta > 0$.

    1. Is there a sufficient statistics?
    2. What is the MLE for θ?
    3. What is the Cramer–Rao lower bound?

26. Let $X_1, X_2, \ldots, X_n$ be random sample from a Poisson distribution with mean λ. Find the minimum variance unbiased estimator of λ^2.

27. (Lehmann–Scheffe Theorem): An unbiased estimator that is a complete sufficient statistics is the unique UMVUE. Using Lehmann–Scheffe theorem, prove that $\overline{X}$ is the unique UMVUE for θ of a Bernoulli distribution.

28. Let $X_1, X_2, \ldots, X_n$ be a random sample from the Poisson distribution with parameter λ. Show that $\sum_{i=1}^{n} X_i$ is a minimal sufficient statistics for λ.

29. Let $X_1, X_2, \ldots, X_n$ be a random sample from the normal distribution $N(0, \theta)$. Show that $\sum_{i=1}^{n} X_i^2$ is a minimal sufficient statistics for θ.

30. If $X_1, X_2, \dots, X_n$ is a random sample from $N(\theta, 1)$. Find a lower bound for the variance of an estimator of θ^2. Determine the minimum variance unbiased estimator of θ^2, and then compute its efficiency.

31. Let $X_1, X_2, \dots, X_n$ be a random sample from the modified geometric distribution with PMF

$$p(x; q) = (1-q)^{x-1} q, \quad x = 1, 2, \dots.$$

Prove that maximum likelihood estimator of q is

$$\hat{q} = \frac{n}{\sum_{i=1}^{n} x_i} = \frac{1}{\overline{x}}.$$

32. Let $\overline{X}$ be the mean of a random sample of size n from $N(\mu, 25)$. Find the smallest sample size n such that $(\overline{X} - 1, \overline{X} + 1)$ is a 0.95 level confidence interval for μ.

33. The contents of seven similar containers of sulfuric acid are 9.8, 10.2, 10.4, 9.8, 10.0, 10.2, and 9.6 liters. Find a 95% confidence interval for the mean of all such containers, assuming an appropriate normal distribution.

34. Please suggest whether the sentence 'If the standard deviation...' conveys the intended meaning. If the standard deviation of the lifetimes of light bulbs is estimated to be 100 hours. What should be the sample size in order to be 99% confident that the error in the estimated average lifetime will not exceed 20 hours? Repeat the exercise for 95 and 99.73% confidence level.

35. A company has 500 cables. Forty cables were selected at random with a mean breaking strength of 2400 pounds and a standard deviation of 150 pounds.

    (a) What are the 95% confidence limits for estimating the mean breaking strength of remaining 460 cables?

    (b) With what degree of confidence could we say that the mean breaking strength of remaining 460 cables is 2400 ± 35 pounds.