Department of Mathematics
Tutorial Sheet No. 7
MAL 509 (Probability Theory)

1. Let $\{X_n\}$ be a sequence of random variables defined on the probability space $([0,\infty), \beta_1, P),\ P(\{0\}) = 0$.

   (a) Define $X_n(s) = \frac{1}{s}\left(1 - \frac{1}{n}\right),\ s \in [0,\infty)$. Then show that $X_n \overset{a.s}{\to} X$, where $X(s) = \frac{1}{s},\ s \in [0,\infty)$.

   (b) Define $X_n(s) = \frac{1}{ns},\ s \in [0,\infty)$. Then show that $X_n \overset{a.s}{\to} 0$.

   (c) Let $P(I) = \displaystyle\int_I e^{-x}dx$. Define

   $$X_n(s) = \begin{cases} 0, & \text{if } s \text{ is rational} \\ (-1)^n, & \text{if } s \text{ is irrational} \end{cases}$$

   Then show that $\{X_n\}$ diverges almost surely.

2. Let $S = [0,1]$ and P be uniform distribution.

   (a) Define
   $$X_n(s) = e^{-n^2(ns-1)},\ 0 \le s \le 1;\ n = 1, 2, \ldots$$
   Show that $X_n \overset{a.s}{\to} 0$, but $X_n \overset{m.s}{\nrightarrow} 0$.

   (b) Define
   $$X_n(s) = \begin{cases} 3^n, & \text{if } 0 \le s \le 1 \\ 0, & \text{otherwise} \end{cases}$$
   Show that $X_n \overset{p}{\to} 0$, but $X_n \overset{m.s}{\nrightarrow} 0$.

3. Consider a sequence of random variables $\{X_n\}$ with $P(X_n = 0) = 1 - \frac{1}{n^\alpha},\ P(X_n = \pm n) = \frac{1}{2n^\alpha}$. Determine the value of α for which the sequence obeys WLLN.

4. Consider a sequence of random variables $\{X_n\}$ with $E(X_n) = m$ and

   $$Cov(X_i, X_j) = \begin{cases} \sigma^2, & i = j \\ a\sigma^2, & i = j \pm 1,\ \text{where } |a| < 1,\ \sigma^2 > 0 \text{ are given contants} \\ 0, \text{otherwise} & \end{cases}$$

   Show that WLLN holds for $\{X_n\}$.

5. Use CLT to show that $\displaystyle\lim_{n\to\infty} e^{-n} \sum_{i=0}^{n} \frac{n^i}{i!} = 0.5$

6. Consider a sequence of independent random variables $\{X_n\}$ such that

   $$P(X_n = 0) = 1 - \frac{2}{n^3},\ P(X_n = \pm n) = \frac{1}{n^3},\ n > 1$$

   Does the sequence $\{X_n\}$ obey CLT ?

7. If $X_n \overset{p}{\to} 0$, then find the median of $X_n \ \to\ 0$ as $n \to \infty$.

8. Consider a sequence $\{X_n\}$ of identically distributed random variables with the property that $nP(|X_i| > n \to 0$ as $n \to \infty$. Show that $\frac{1}{n} \max_{1 \le i \le n} X_i \overset{p}{\to} 0$

9. Suppose $|X_n - X| \le Y_n$, almost surely for some random variable X, then show that of $E(Y_n) \to 0$, then $E(X_n) \to E(X)$ and $X_n \overset{p}{\to} X$.

10. Show that $X_n \overset{2}{\to} X \ \Rightarrow\ E(X_n) \to E(X),\ E(X_n^2) \to E(X^2)$ as $n \to \infty$.

11. Let $\{X_i\}$ be a sequence of independent random variables, such that each X_i has mean 0 and variance 1. Show that

$$\sqrt{n}\frac{X_1 + X_2 + \ldots + X_n}{X_1^2 + X_2^2 + \ldots + X_n^2} \xrightarrow{l} Z \sim N(0,1)$$

12. Does WLLN hold for the following sequences

   (a) $P(X_k = \pm 2^k) = \frac{1}{2}$

   (b) $P(X_k = \pm \frac{1}{k}) = \frac{1}{2}$

13. For what values of α, does the strong law of large numbers hold for the sequence $\{X_n\}$, where $P(X_k = \pm k^\alpha) = \frac{1}{2},\ k = 1, 2, \ldots$.

14. Let $\{X_n\}$ be a sequence of independent random variables with the following probability distribution. In each case, does the Lindeberg Condition for CLT hold ?

   (a) $P(X_k = \pm \frac{1}{2^n}) = \frac{1}{2}$

   (b) $P(X_k = \pm \frac{1}{2^{n+1}}) = \frac{1}{2^{n+3}},\ P(X_n = 0) = 1 - \frac{1}{2^{n+2}}$

15. From an urn containing 10 identical balls numbered 0 through 9, n balls are drawn with replacement,

   (a) What does the law of large number tell you about the appearance of 0's in n drawings.

   (b) How many drawings must be made in order that with probability atleast 0.95, the relative frequency of occurence of 0's will be between 0.09 and 0.11 ?

   (c) Use CLT to find the probability that among n numbers thus chosen, the number 5 will appear between $\frac{n-35n}{10}$ and $\frac{n+35n}{10}$ times, if (i) $n = 25$ (ii) $n = 100$.