Department of Mathematics
Tutorial Sheet No. 4
MAL 509 (Probability Theory)

1. Verify the following as a consequence of definition of expectation:

   (a) If X is a bounded random variable, then $E(X)$ exists.

   (b) If $X \in (a, b)$ with probability 1, then $a < E(X) < b$.

   (c) If $X$is symmetrical abt a point μ, then $E(X) = \mu$.

   (d) If $X \in \{1, 2, \ldots, n, \ldots\}$ with probability 1, then $E(X) = \sum_{n=1}^{\infty} P(X \geq n)$.

2. Let A, B, C be events in a sample space with $P(A) = 0.1,\ P(B) = 0.4,\ P(C) = 0.3,\ P(A \cap B) = 0.1, A \cap C = B \cap C = \phi$. Find mean and variance of a random variable $X = I_A + I_B - I_C$, where $I_{[.]}$ is the indicator function of the set [.].

3. Show that $E(|X|) = \int_{-\infty}^{0} F_X(x)dx + \int_{0}^{\infty} (1 - F_X(x))dx$.

4. Show that the matrix $\sum = (\mu_{i+j})_{k\times k}$ is non negative definite where μ_p is the central moment of order p of the random variable X.

5. The r^{th} factorial moment of a random variable X defined by $\mu'_{(r)} = E[X(X-1)\ldots(X_r+1)]$. If X is a discrete random variable with p.m.f. $p_X(i) > 0,\ i = 0, 1, \ldots, r$, and 0 elsewhere, show that $\mu'_{(k)} = 0,\ k = r+1, r+2, \ldots$.

6. From a point on the circumfrence of the circle of radius r, a chord is drawn in a random direction. Show that the expected value of length og the chord is $\frac{4r}{\pi}$ and its variance is $2r^2\left(1 - \frac{8}{\pi^2}\right)$.

7. For the random variable X with p.m.f. $p_X(x) = \frac{e^{-\lambda}\lambda^x}{x!},\ x = 0, 1, 2, \ldots$, show that $\mu_2 = \mu_3 = \lambda,\ \mu_4 = \lambda + 3\lambda^2$.

8. Show that for a random variable with p.d.f. $f_X(x) = \frac{k\alpha^k}{(x+\alpha)^{k+1}},\ x > 0$, the α^{th} absolute moment exists for X, for $\alpha < k$.

9. For a random variable with p.d.f. $f_X(x) = \frac{1}{\alpha!}e^{-x}x^{\alpha},\ x > 0$, where α is a positive integer, show that $P(0 < X < 2(\alpha + 1)) > \frac{\alpha}{\alpha+1}$ by using Chebyshev inequality.

10. For the random variable with p.m.f. $p_X(x) = \binom{r+x-1}{x} p^r q^x,\ x = 0, 1, 2, \ldots$, where $q = 1 - p$, find the m.g.f. and hence the mean and variance of X.

11. Find the mean, variance of the random variable X having the distribution function

$$F_X(x) = \begin{cases} 0, & x < 0 \\ p + (1-p)(1 - e^{-\lambda x}), & o \leq x < T \\ 1, & x \geq T \end{cases}$$

12. Fir the Laplace distribution, i.e.e the random variable having p.d.f. given by: $f_X(x) = \frac{1}{2\lambda} exp\left(\frac{|X-\mu|}{\lambda}\right),\ x \in R,\ \lambda > 0,\ 0 < \mu < \infty$, find m.g.f. and its mean and variance.

13. Let X be a random variable having m.g.f $M(t),\ t > 0$. Then show that for $t > 0,\ P(tX > s^2 + \log(M(t)) < e^{s^2}$.

14. (Jensens's Inequality) If g is a convex function and $E(X)$ exists, then show that $g(E(X)) < E(g(X))$. Hence show that $E(X) \leq (E(|X|))^{1/r}$.

15. Let $g(X) \geq 0, \forall x \in [0, \infty)$ be a non-decreasing even function. Show that for any random variable X such that $E(g(X))$ exists $P(|X| \geq \epsilon) \leq \frac{E(g(X))}{g(\epsilon)}$.

16. X has the power series distribution if it has a p.m.f. $P(X = j) = \frac{a_j \theta^j}{f(\theta)},\ x = 0, 1, 2, \ldots,\ \theta > 0,\ a_j \geq 0$ and where $f(\theta) = \sum_{j=0}^{\infty} a_j \theta^j$. Find the m.g.f. of X. Use it to find the factorial moments of X.

17. Show that the sequence of moments determine the probability distribution of the random variable uniquely if it is a bounded random variable.

18. Show that absolute moment of no order exists for the random variable having p.d.f $f_X(x) = \frac{1}{2|x|(\ln|x|)^2}$ for $|x| > e$.