Department of Mathematics
MTL 106 (Introduction to Probability Theory and Stochastic Processes)
Tutorial Sheet No. 10
Answer for selected Problems

1

$$\frac{d}{dt}p_{0n}(t) = -\lambda\, p_{0n}(t) + \lambda\, p_{1n}(t),$$

$$\frac{d}{dt}p_{1n}(t) = -(\lambda + \mu)\, p_{1n}(t) + \lambda\, p_{2n}(t) + \mu\, p_{0n}(t),$$

$$\frac{d}{dt}p_{in}(t) = -(\lambda + 2\mu)\, p_{in}(t) + \lambda\, p_{i+1,n}(t) + 2\mu\, p_{i-1,n}(t), \qquad i \geq 2.$$

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$$\mu = \frac{5}{(0.1)^{1/5}} \approx 7.93 \text{ customers per minute.}$$

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$$W_q \approx 6.35 \text{ minutes.}$$

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$$L_q = \frac{\rho^2}{1-\rho} = \frac{(0.625)^2}{1-0.625} \approx 1.042.$$

6 (a) Mean number in the shop:

$$L = \frac{\rho}{1-\rho} = \frac{5}{\mu - 5}.$$

(b) Mean number waiting:

$$L_q = \frac{\rho^2}{1-\rho} = \frac{25}{\mu(\mu-5)}.$$

(c) Probability an arrival need not wait:

$$P_0 = 1 - \rho = 1 - \frac{5}{\mu}.$$

(d) Probability no seat is available (4 seats):

$$\Pr(N \geq 5) = \rho^5 = \left(\frac{5}{\mu}\right)^5.$$

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$$\lambda = 4, \quad K = 6, \quad \rho = \frac{\lambda}{\mu} = \frac{4}{\mu}$$

$$P_n = \frac{(1-\rho)\rho^n}{1-\rho^7}, \quad n = 0, 1, \ldots, 6$$

$$\lambda_{\text{eff}} = 4\left(1 - \frac{(1-\rho)\rho^6}{1-\rho^7}\right)$$

$$L = \frac{\rho\left(1 - 7\rho^6 + 6\rho^7\right)}{(1-\rho)(1-\rho^7)}$$

$$W = \frac{L}{\lambda_{\text{eff}}} = \frac{\dfrac{\rho\left(1-7\rho^6+6\rho^7\right)}{(1-\rho)(1-\rho^7)}}{4\left(1-\dfrac{(1-\rho)\rho^6}{1-\rho^7}\right)}$$

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$$\pi_n = \pi_0 \prod_{k=0}^{n-1} \frac{\lambda_k}{\mu_{k+1}} = \pi_0 \left(\frac{\lambda}{\mu}\right)^n \frac{M!}{(M-n)!}, \qquad n = 0, 1, \ldots, M.$$

$$\pi_0 = \left[\sum_{n=0}^{M} \frac{M!}{(M-n)!} \left(\frac{\lambda}{\mu}\right)^n\right]^{-1}.$$

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$$P_{\text{reject}} = P_4 = \frac{(1-\rho)\rho^4}{1-\rho^5} = \frac{16}{211} \approx 0.0758.$$

$$L = \frac{\rho\left[1-5\rho^4+4\rho^5\right]}{(1-\rho)(1-\rho^5)} = \frac{262}{211} \approx 1.242.$$

10 (a) Mean number waiting:

$$L_q = \frac{P_0(\lambda/\mu)^c\rho}{c!(1-\rho)^2} = \frac{8}{9} \approx 0.889.$$

(b) Total time in clinic:

$$W = \frac{L_q}{\lambda} + \frac{1}{\mu} = \frac{13}{27}\text{ hr} \approx 28.9\text{ min}.$$

(c) Average idle time of each doctor:

$$1-\rho = \frac{1}{3} = 33.33\%.$$

11 Probability that at least one customer is in system:

$$P(N \geq 1) = 1 - \pi(0,0) = 1 - \frac{1}{G}$$

Mean number of customers in system:

$$E[N] = \sum_{i=0}^{c}\sum_{j=0}^{c-i}(i+j)\,\pi(i,j) = \frac{1}{G}\sum_{i=0}^{c}\sum_{j=0}^{c-i}(i+j)\frac{\rho_1^i}{i!}\frac{\rho_2^j}{j!}$$

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$$\lambda = 1 \text{ per minute}, \quad \mu = \frac{1}{6} \text{ per minute}, \quad c = 3, \quad K = 7$$

$$\rho = \frac{\lambda}{c\mu} = \frac{1}{3\cdot(1/6)} = 2$$

Steady-state probabilities for $M/M/3/7$:

$$P_n = \begin{cases} \dfrac{1}{G}\dfrac{(\lambda/\mu)^n}{n!}, & 0 \leq n \leq 3, \\ \dfrac{1}{G}\dfrac{(\lambda/\mu)^n}{3!\,3^{n-3}}, & 3 < n \leq 7 \end{cases}$$

Normalization constant:

$$G = \sum_{n=0}^{3} \frac{(\lambda/\mu)^n}{n!} + \sum_{n=4}^{7} \frac{(\lambda/\mu)^n}{3!\, 3^{\,n-3}}$$

Mean number in system:

$$L = \sum_{n=0}^{7} n P_n$$

Effective arrival rate:

$$\lambda_{\text{eff}} = \lambda(1 - P_7)$$

Mean time in system (Little's law):

$$W = \frac{L}{\lambda_{\text{eff}}}$$

Expected number of blocked customers per hour:

$$\text{Blocked rate} = 60 \cdot \lambda P_7 = 60 P_7$$

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