Department of Mathematics
MTL 106 (Introduction to Probability Theory and Stochastic Processes)
Tutorial Sheet No. 6

1. (Jensens's Inequality) If g is a convex function and $E(X)$ exists, then show that $g(E(X)) \leq E(g(X))$. Hence show that $E(X) \leq (E(|X|^r))^{1/r}$ for $r \geq 1$.

2. Let $g(X) \geq 0, \forall \; x \in [0, \infty)$ be a non-decreasing even function. Show that for any random variable X such that $E(g(X))$ exists and $P(|X| \geq \epsilon) \leq \frac{E(g(X))}{g(\epsilon)}$.

3. Let $\{X_n\}$ be a sequence of random variables defined on the probability space $([0, \infty), \beta_1, P), \; P(\{0\}) = 0$.

   (a) Define $X_n(s) = \frac{1}{s}\left(1 - \frac{1}{n}\right), \; s \in [0, \infty)$. Then show that $X_n \xrightarrow{a.s} X$, where $X(s) = \frac{1}{s}, \; s \in [0, \infty)$.

   (b) Define $X_n(s) = \frac{1}{ns}, \; s \in [0, \infty)$. Then show that $X_n \xrightarrow{a.s} 0$.

   (c) Let $P(I) = \displaystyle\int_I e^{-x} dx$. Define

   $$X_n(s) = \begin{cases} 0, & \text{if } s \text{ is rational} \\ (-1)^n, & \text{if } s \text{ is irrational} \end{cases}$$

   Then show that $\{X_n\}$ diverges almost surely.

4. Consider a sequence of random variables $\{X_n\}$ with $E(X_n) = m$ and

   $$Cov(X_i, X_j) = \begin{cases} \sigma^2, & i = j \\ a\sigma^2, & i = j \pm 1, \text{ where } |a| < 1, \; \sigma^2 > 0 \text{ are given contants} \\ 0, & \text{otherwise} \end{cases}$$

   Show that WLLN holds for $\{X_n\}$.

5. Consider a sequence $\{X_n\}$ of identically distributed random variables with the property that $nP(|X_i| > n) \to 0$ as $n \to \infty$. Show that $\frac{1}{n} \max_{1 \leq i \leq n} X_i \xrightarrow{p} 0$

6. Suppose $|X_n - X| \leq Y_n$, almost surely for some random variable X, then show that if $E(Y_n) \to 0$, then $E(X_n) \to E(X)$ and $X_n \xrightarrow{p} X$.

7. Show that $X_n \xrightarrow{2} X \;\Rightarrow\; E(X_n) \to E(X), \; E(X_n^2) \to E(X^2)$ as $n \to \infty$.

8. Let $\{X_i\}$ be a sequence of independent random variables, such that each X_i has mean 0 and variance 1. Show that

   $$\sqrt{n}\frac{X_1 + X_2 + \cdots + X_n}{X_1^2 + X_2^2 + \cdots + X_n^2} \xrightarrow{l} Z \sim N(0, 1).$$

9. Does WLLN hold for the following sequences

   (a) $P(X_k = \pm 2^k) = \frac{1}{2}$

   (b) $P(X_k = \pm \frac{1}{k}) = \frac{1}{2}$.

10. For what values of α, does the strong law of large numbers hold for the sequence $\{X_n\}$, where $P(X_k = \pm k^\alpha) = \frac{1}{2}, \; k = 1, 2, \ldots$.

11. Suppose that the number of customers who visit SBI, IIT Delhi on a Saturday is a r.v. with $\mu = 75$ and $\sigma = 5$. Find the lower bound for the probability that there will be more than 50 but fewer than 100 customers in the bank?

12. From an urn containing 10 identical balls numbered 0 through 9, n balls are drawn with replacement,

    (a) What does the law of large number tell you about the appearance of 0's in n drawings.

(b) How many drawings must be made in order that with probability atleast 0.95, the relative frequency of occurrence of 0's will be between 0.09 and 0.11 ?

13. Does the r.v. X exist for which
$$P\left[\mu - 2\sigma \le X \le \mu + 2\sigma\right] = 0.6.$$

14. Use CLT to show that
$$\lim_{n\to\infty} e^{-nt} \sum_{k=0}^{n} \frac{(nt)^k}{k!} = 1 = \begin{cases} 1, & 0 < t < 1 \\ 0.5, & t = 1 \\ 0, & t > 1 \end{cases} .$$

15. Let $X_1, X_2, \ldots, X_n$ be a random sample from a uniform distribution on the interval from 0 to 5. What is the limiting moment generating function of $\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}$ as $n \to \infty$?

16. Let $X \sim B(n,p)$. Use the CLT to find n such that
$$P[X > n/2] \ge 1 - \alpha.$$
Calculate the value of n when $\alpha = 0.90$ and $p = 0.45$.

17. Suppose that X_i , $i = 1, 2, \ldots, 30$ are independent r.v. each having a Poisson distribution with parameter 0.01. Let $S = X_1 + X_2 + \cdots + X_{30}$.
(a) Using central limit theorem evaluate $P(S \ge 3)$.
(b) Compare the answer in (a) with exact value of this probability.

18. Let $X_1, X_2, \ldots$ be a sequence of iid r.v. with mean 1 and variance 1600, and assume that these variables are non-negative. Let $Y = \sum_{k=1}^{100} X_k$. Use the central limit theorem to approximate the probability $P(Y \ge 900)$.

19. Let X be a r.v. having a Gamma distribution with parameters n and 1. How large must n be in order to guarantee that
$$P\left(\left|\frac{X}{n} - 1\right| > 0.01\right) < 0.01?$$

20. How many tosses of a fair coin are needed so that the probability that the average number of heads obtained differs at most 0.01 from 0.5 is at least 0.90?

21. Examine the nature of convergence of $\{X_n,\ n = 1, 2, 3, \ldots\}$ defined below for the different values of k for $n = 1, 2, \ldots$:
$$P(X_n = n^k) = \frac{1}{n};\quad P(X_n = 0) = 1 - \frac{2}{n};\quad P(X_n = -n^k) = \frac{1}{n}.$$

22. Show that the convergence in the rth mean does not imply almost sure convergence for the sequence $\{X_n,\ n = 1, 2, 3, \ldots\}$ defined below:
$$P(X_n = 0) = 1 - \frac{1}{n};\quad P(X_n = -n^{1/2r}) = \frac{1}{n}.$$

23. For each $n \ge 1$, let X_n be an uniformly distributed r.v. over set $\{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}$, i.e.,
$$P\left(X_n = \frac{k}{n}\right) = \frac{1}{n+1}, k = 0, 1, \ldots, n.$$
Let U be a r.v. with uniform distribution in the interval $[0,1]$. Show that $X_n \xrightarrow{d} U$.

24. Let $\{X_n,\ n = 1, 2, 3, \ldots\}$ be a sequence of iid r.v. with $E(X_1) = Var(X_1) = \lambda \in (0, \infty)$ and $P(X_1 > 0) = 1$. Show that for $n \to \infty$,

$$\sqrt{n}\frac{Y_n - \lambda}{\sqrt{\lambda}} \xrightarrow{d} N(0,1)$$

where

$$Y_n = \frac{1}{n}\sum_{k-1}^{n} X_k.$$

25. Let $\{X_n,\ n = 1, 2, 3, \ldots\}$ be a sequence of iid r.v. with $P(X_n = 1) = P(X_n = -1) = \frac{1}{2}$. Show that

$$\frac{1}{n}\sum_{j=1}^{n} X_j$$

converges in probability to 0.

26. Let $X, X_1, X_2, \ldots$ be a sequence of r.v. defined on a same probability space (Ω, S, P). Prove that

(a) If $X_n \xrightarrow{a.s.} X$ and $Y_n \xrightarrow{a.s.} Y$, then $X_n + Y_n \xrightarrow{a.s.} X + Y$.

(b) If $X_n \xrightarrow{a.s.} X$ and $Y_n \xrightarrow{a.s.} Y$, then $X_n Y_n \xrightarrow{a.s.} XY$.

(c) If $X_n \xrightarrow{P} X$ iff $X_n - X \xrightarrow{P} 0$.

(d) If $X_n \xrightarrow{P} X$ then $X_n - X_m \xrightarrow{P} 0$.

(e) If $X_n \xrightarrow{P} X$ and $Y_n \xrightarrow{P} Y$, then $X_n + Y_n \xrightarrow{P} X + Y$.

(f) If $X_n \xrightarrow{P} X$ and $Y_n \xrightarrow{P} Y$, then $X_n Y_n \xrightarrow{P} XY$.

27. A fair coin is flipped 100 consecutive times. Let X be the number of heads obtained. Use Chebyschev's inequality to find a lower bound for the probability that $\frac{X}{100}$ differs from $\frac{1}{2}$ by less than 0.1.

28. Let $X_1, X_2, \ldots, X_n$ be n independent Poisson distributed r.v. with means $1, 2, \ldots, n$, respectively. Find an x in terms of t such that

$$P\left(\frac{S_n - \frac{n^2}{2}}{n} \leq t\right) \approx \Phi(x)$$

for sufficiently large n, where Φ is the CDF of $N(0,1)$.

29. Let Y be a Gamma distributed r.v. with PDF

$$f(y) = \begin{cases} \frac{1}{\Gamma(p)} e^{-y} y^{p-1} & y > 0 \\ 0 & \text{otherwise} \end{cases}.$$

Find the limiting distribution of $\frac{Y - E(Y)}{\sqrt{Var(Y)}}$ as $p \to \infty$.