Department of Mathematics
MTL 106 (Introduction to Probability Theory and Stochastic Processes)
Tutorial Sheet No. 7

1. Trace the path of the following stochastic processes,
   (a) $\{X_k, k \in T\}$ where X_k is the waiting time of the kth student arriving at a bus stop and $T = \{1, 2, \ldots\}$.
   (b) $\{X_t, t \in T\}$ where X_t be the number of cars parked in a parking garage at a shopping mall at time t, and $T = [0, \infty)$.
   (c) $\{Y_t, t \in T\}$ where Y_t denotes the exchange rates of the Indian rupee (INR) against the US dollar (USD) at time t, and $T = [0, \infty)$.
   (d) $\{X_t, t \in T\}$ where X_t denotes the number of passengers in a bus at tth trip, and $T = \{1, 2, \ldots\}$.
2. Let $\{X_t, t \geq 0\}$ be a stochastic process with $X_t = A_0 + A_1 t + A_2 t^2$, where $A_i's$ are uncorrelated r.vs. with mean 0 and variance 1. Find the mean function and covariance function of X_t.
3. Let $\{N(t), t \geq 0\}$ be a Poisson process with rate λ. Find its covariance function $C(t_1, t_2)$.
4. Let Y_1, Y_2 be two IID RVs where each follows $Y_i \sim N(\mu, \sigma^2), i = 1, 2$. Define $X_t = Y_1 \cos(\lambda t) + Y_2 \sin(\lambda t)$, $-\infty < t < \infty$ where λ is a real constant. Show that $\{X_t, -\infty < t < \infty\}$ is a second-order process.
5. Suppose that $X_1, X_2, \ldots$ are IID RVs each having normal distribution with mean 0 and variance σ^2. Consider the stochastic process $\{S_n\, ,\ n = 1, 2, \ldots\}$ where $S_n = \exp\left(\sum_{i=1}^{n} X_i - \frac{n}{2}\sigma^2\right)$. Find $E\left[S_n\right]$ for all n.
6. Let $\{X_t, 0 \leq t \leq T\}$ be a stochastic process such that $E[X_t] = 0$ and $E\left[X_t^2\right] = 1$ for all $t \in [0, T]$. Find the upper bound of $|E[X_t X_{t+h}]|$ for any $h > 0$ and $t \in [0, T - h]$.
7. Consider the process $\{X_t, t \geq 0\}$ with $X_t = A\cos(wt) + B\sin(wt)$ where w is a positive constant and A and B are uncorrelated RV with mean 0 and variance 1. Check whether $\{X_t, t \geq 0\}$ is wide-sense stationary or not.
8. In a communication system, the carrier signal at the receiver is modeled by the process $\{Y_t, t \geq 0\}$ which is defined as $Y_t = X_t \cos(2\pi wt + \Theta)$ where $\{X_t, t \geq 0\}$ is a zero-mean and wide-sense stationary process, and Θ is a uniform distributed RV in the interval $(-\pi, \pi)$ and w is a positive constant. Assume that, Θ is independent of the process $\{X_t, t \geq 0\}$. Is $\{Y_t, t \geq 0\}$ wide-sense stationary?
9. Consider Problem 4 with $P(Y_i = 1) = P(Y_i = -1) = \frac{1}{2}, i = 1, 2$ instead of normal distribution. Show that $\{X_t, -\infty < t < \infty\}$ is a second order stationary process which is not strict-sense stationary.
10. Prove that $\{X_t, -\infty < t < \infty\}$ is a second-order stationary process if for every number τ, the second-order process Y_t defined by $Y_t = X_{t+\tau}, -\infty < t < \infty$ has the same mean and covariance function as of X_t.
11. Consider the random telegraph signal, denoted by X_t, which jumps between two states, 0 and 1, according to the following rules. At time $t = 0$, the signal X_t start with equal probability for the two states, i.e., $P(X_0 = 0) = P(X_0 = 1) = 1/2$, and let the switching times be decided by a Poisson process $\{Y_t, t \geq 0\}$ with parameter λ independent of X_0. At time t, the signal
$$X_t = \frac{1}{2}\left(1 - (-1)^{X_0 + Y_t}\right), t > 0.$$
Is $\{X_t,\ t \geq 0\}$ wide-sense stationary.
12. Prove that every real-valued stochastic process $\{X_t, t \in T\}$ with independent increments is a Markov process.
13. Prove or disprove the following statement: If $\{X_t, t \in T\}$ is a Markov process, then $\{X_t, t \in T\}$ has independent increment property.
14. Let X and Y be two IID RVs each having uniform distribution on interval $[-\pi, \pi]$. Define, for $t \geq 0$, $Z_t = \sin(Xt + Y)$. Check whether $\{Z_t, t \geq 0\}$ is covariance stationary or not.

15. Let A be a positive RV that is independent of a strict-sense stationary random process $\{X_t, t \geq 0\}$. Show that $\{Y_t, t \geq 0\}$ where $Y_t = AX_t$ is also strict-sense stationary random process.

16. Let $\{X_t, t \geq 0\}$ be a stochastic process where the PMF of X_t is given by

$$P\{X_t = n\} = \begin{cases} \frac{at}{(1+at)} & n = 0 \\ \frac{(at)^{n-1}}{(1+at)^{n+1}} & n = 1, 2, \ldots. \end{cases}$$

Check whether $\{X_t, t \geq 0\}$ is covariance stationary or not.

17. Let $\{N(t), t \geq 0\}$ be a Poisson process with parameter λ. Show that

$$\lim_{t \to \infty} N(t) = \infty \text{ a.s. } .$$

Hint: Find the limit of $P\{N(k) \geq n\}$ as $k \to \infty$. Write $\{\lim_{t\to\infty} N(t) = \infty\}$ in terms of $\{N(k) \geq n\}$.

18. Let $\{N(t), t \geq 0\}$ be a Poisson process. Prove that the process $\{X(t), t \geq 0\}$, where

$$X(t) = N(t + L) - N(t), \ L > 0,$$

is covariance or wide-sense stationary.

19. Let $\{N(t), t \geq 0\}$ be a Poisson process with parameter λ. Show that

$$P\{N(t) \text{ is odd}\} = e^{(-\lambda t)} \sinh(\lambda t),$$
$$P\{N(t) \text{ is even}\} = e^{(-\lambda t)} \cosh(\lambda t).$$

Hint: Find the probability that $N(t) = 2n + 1$ and compare this to the n-th term of the Taylor expansion of $\sinh(\lambda t)$.