Department of Mathematics
MTL 106 (Introduction to Probability Theory and Stochastic Processes)
Tutorial Sheet No. 9

1. Consider a time-homogeneous CTMC $\{X(t), t \geq 0\}$ with the infinitesimal generator matrix $\mathbf{Q} = \begin{pmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 4 & -5 \end{pmatrix}$ and initial distribution $(0, 0, 1)$. Find $P(\tau > t)$ where τ denotes the first transition time of the Markov chain.

2. Let's say a business employs four operators to handle a single phone number. A busy signal will be heard if someone calls while all four operators are occupied. Let $X(t)$ be the number of active operators at time t. Assume that each individual service process and the arrival processes are Poisson processes with rates λ and μ, respectively. Determine the forward Kolmogorov equations for the Markov Process and the infinitesimal generator matrix $\mathbf{Q}$.

3. In orbit, there are two communication satellites. Each satellite has an exponential lifetime with a mean of $\frac{1}{\mu}$. A replacement is sent up if one fails. The preparation and sending up of a replacement takes an exponential amount of time, with mean $\frac{1}{\lambda}$. Let X_t represents the number of satellites in orbit at the given moment. Consider $\{X_t, t \geq 0\}$ as a CTMC with the state space $\{0, 1, 2\}$. Show that the infinitesimal generator matrix is given by
$$\mathbf{Q} = \begin{pmatrix} -\lambda & \lambda & 0 \\ \mu & -(\lambda+\mu) & \lambda \\ 0 & 2\mu & -2\mu \end{pmatrix}.$$
Write down the Kolmogorov forward and backward equations for the above process.

4. Consider a stochastic process $\{X(t), t \geq 0\}$ which take the value 0 and 1 with probabilities $p_0(t)$ and $p_1(t)$, respectively. Also
$$\mathrm{P}\{X(t + \triangle t) = 1 | X(t) = 0\} = \alpha \triangle t + o(\triangle t)$$
and
$$\mathrm{P}\{X(t + \triangle t) = 0 | X(t) = 1\} = \beta \triangle t + o(\triangle t).$$
As $\triangle t \to 0$, $o(\triangle t) \to 0$ and α and β are positive constants. Assume $p_0(0) = 1$.
   (a) Write the forward Kolmogorov equations for the Markov process $\{X(t), t \geq 0\}$.
   (b) Derive the transient or time-dependent probability distribution of the process.

5. Suppose that the probability of an accident happening in time t to $t + \triangle t$ is $(\nu + j\lambda)\triangle t + o(\triangle t)$ given that $j \geq 0$ accidents have occurred previous to time t, for ν, $\lambda > 0$. Let $X(t)$ denote number of accidents happen up to and till time t.
   (a) Write the forward Kolmogorov equations for the Markov process $\{X(t), t \geq 0\}$.
   (b) Prove that
$$P_{0n+1}(t) = \frac{\nu(\nu+\lambda)\ldots(\nu+n\lambda)}{(n+1)!\lambda^{n+1}} e^{-\nu t} \left(1 - e^{-\lambda t}\right)^{n+1}, \quad n = 0, 1, \ldots$$

6. Two categories of jobs, designated as 1 and 2, arrive at a processor according to independent Poisson processes with rates λ_1 and λ_2 respectively. Let $X(t)$ represent the kind of the latest job arrival prior to time t. Show that that $\{X(t), t \geq 0\}$ constitutes a CTMC and describe its transition rates. Show that $P_{ij}(t) = a + be^{-(\lambda_1+\lambda_2)t}$, and identify the coefficients a and b. Prove that $\{X(t), t \geq 0\}$ is ergodic and determine its stationary distribution.

7. Consider a CTMC $\{X(t), t \geq 0\}$ with state space $\{0, 1, 2\}$ and
$$\mathbf{Q} = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix}.$$

(a) Find its transition probability matrix $\mathbf{P}(t)$.

(b) Show that the limiting distribution of $P(t)$ as $t \to \infty$ is $\pi = (1/3, 1/3, 1/3)$ and

$$\frac{2}{3}e^{-3t/2} \leq \sup_i \sum_j \mid P_{ij}(t) - \pi_j \mid \leq \frac{4}{3}e^{-3t/2}.$$

8. Consider a CTMC $\{X(t), t \geq 0\}$ with

$$\mathbf{Q} = \begin{pmatrix} -3 & 1 & 2 \\ 2 & -4 & 2 \\ 2 & 1 & -3 \end{pmatrix}.$$

Prove that its transition probability matrix is

$$\mathbf{P}(t) = \frac{1}{5}\begin{pmatrix} 2+3e^{-5t} & 1-e^{-5t} & 2-2e^{-5t} \\ 2-2e^{-5t} & 1+4e^{-5t} & 2-2e^{-5t} \\ 2-2e^{-5t} & 1-e^{-5t} & 2+3e^{-5t} \end{pmatrix},$$

hence, the limiting distribution of $\mathbf{P}(t)$ is $\pi = (2/5, 1/5, 2/5)$ and

$$\sup_i \sum_j \mid P_{ij}(t) - \pi_j \mid = \frac{8}{5}e^{-5t}.$$

9. Consider a reversible CTMC $\{X(t), t \geq 0\}$ with state space S. Suppose $\mathbf{P}(t) = [P_{ij}(t)]$ has stationary distribution π. Then, prove that $\lim_{t\to\infty} P_{ij}(t) = \pi_j$.

10. Let $\{X(t), t \geq 0\}$ be a CTMC. Fix $s > 0$, define $Y(t) = X(s-t)$. Then, prove that,

    (a) $\{Y(t), 0 \leq t \leq s\}$ is a CTMC.

    (b) Show that $\{Y(t), 0 \leq t \leq s\}$ has homogeneous transition probability matrix if and only if $\{X(t), t \geq 0\}$ is stationary, i.e., the distribution of $X(t)$, $m_i = P(X(t) = i)$ is independent of t.
    Moreover, show that the transition probability matrix of $\{Y(t), 0 \leq t \leq s\}$ is given by $P^*_{ij}(t) = m_j P_{ij}(t)/m_i$ where $[P_{ij}(t)]$ is the transition probability matrix of $\{X(t), t \geq 0\}$.

11. The birth-death process is called a pure death process if $\lambda_i = 0$ for all i. Suppose $\mu_i = i\mu$, $i = 1, 2, \ldots$ and initially $X_0 = n$. Show that X_t has $B(n, p)$ distribution with $p = e^{-\mu t}$.

12. The birth-death process $\{X_t, t \geq 0\}$ is called a pure birth process if $\mu_i = 0$ for every i and suppose that $\lambda_i = i\lambda$ for $i = 1, 2, \ldots$. For $X_0 = n$, show that

$$P\{X_t = i | X_0 = n\} = P_{ni}(t) = \binom{i-1}{i-n} e^{-n\lambda t}\left(1 - e^{-\lambda t}\right)^{i-n}, \quad i = n, n+1, \ldots.$$

13. Consider a birth-death process with birth rate $\lambda_i = (i+1)\lambda$, $i \geq 0$ and $\mu_i = i\mu$, $i \geq 0$. Find the expected time to go from state 0 to 1.

14. Consider, the irreducible birth-death process, then the

    (a) state is transient if and only if

$$\sum_{n=1}^{\infty} \frac{\mu_1 \ldots \mu_n}{\lambda_1 \ldots \lambda_n} < \infty, \tag{1}$$

    (b) state is positive recurrent if and only if

$$\sum_{n=1}^{\infty} \frac{\lambda_1 \ldots \lambda_n}{\mu_1 \ldots \mu_n} < \infty, \tag{2}$$

(c) state is null recurrent if and only if neither (1) nor (2) hold.

15. Consider a birth-death process with state space $S = \{0, 1, 2\}$ and rates $\lambda_0 = \mu_2 = \lambda$ and $\lambda_1 = \mu_1 = 0$. Find $P\{X(t) = n | X(0) = 0\}, \quad n = 0, 1, 2.$

16. Consider a birth-death process $\{X(t), t \geq 0\}$ with state space $S = \{0, 1, \ldots\}$ and birth rates are of the form $\lambda_n = n\lambda, \quad n = 0, 1, \ldots$ and death rates are of the form $\mu_n = n\mu, \quad n = 0, 1, \ldots$ where λ and μ are non-negative constants. Assume $P_{ii}(j) = 1$ for fixed i. Take $E(X(t)) = \sum_{k=0}^{\infty} kP_{ik}(t)$. Prove that $E(X(t)) = ie^{(\lambda-\mu)t}$.

17. For a non–homogeneous Poisson process, the intensity function is given by

$$\lambda(t) = \begin{cases} t, & 0 < t \leq 5, \\ 4, & 5 < t < \infty \end{cases}.$$

If $S_4 = 3$, calculate the probability that $S_6 > 5$.

18. Let $\{\tilde{N}(t), t \geq 0\}$ be a non-homogeneous Poisson process with intensity function $\lambda(\cdot)$. Suppose $\lambda(t) > 0$ for all $t > 0$ and $\int_0^{\infty} \lambda(t)dt = \infty$, such that, the function $\Lambda(t) = \int_0^t \lambda(s)ds$ is a strictly increasing, continuous function on $(0, \infty)$ onto $(0, \infty)$ and therefore has an inverse function Λ^{-1}.

    Prove that, $\{N(t) = \tilde{N}(\Lambda^{-1}), t \geq 0\}$ can be seen to be a homogeneous Poisson process with rate 1 and $\tilde{N}(t) = N(\Lambda(t))$.

19. Patients see doctors according to a Poisson process at a rate of 8 per hour, and the average time it takes a doctor to examine a patient is 6 minutes. The doctor sees each new patient as they arrive. Calculate the likelihood that patients will have to wait when they arrive.

20. Suppose there are n identical machines operating independently and serviced by a single repair crew. If a machine breaks down while another is being repaired it must wait its turn before repairs can start. Assume each machine has an operating time exponential with mean $\frac{1}{\mu}$ and a repair time exponential with mean $\frac{1}{\lambda}$. Let $X(t) =$ no. of machines in operating condition at time t. Model $X(t)$ as a Markov chain with state space $E = \{0, 1, \ldots, N\}$. Determine rate matrix Λ and the forward Kolmogorov equations. Determine the limiting equilibrium probability distribution of the process.

21. Assume that the passengers arrive at a bus station is a Poisson process with rate λ. The bus only departs after a deterministic time T. Let W be the combined waiting time for all the passengers. Compute the expected value of W.

22. Let $\{N(t), t \geq 0\}$ be a Poisson process with rate λ. Let X_1 be the epoch of the first arrival and X_n be the inter-arrival time between the $n-1$th and nth arrival. Let S_{n-1} be the epoch of the $n-1$th arrival.

    (a) Show that $P(X_1 > x) = e^{-\lambda x}$.

    (b) Show that $P\{X_n > x | S_{n-1} = \tau\} = e^{-\lambda x}$.

23. Suppose the arrival at a counter form a time-homogeneous Poisson process with parameter λ and suppose each arrival is of type A or of type B with respective probabilities p and $1-p$. Let $X(t)$ be the type of the last arrival before time t. Write down the forward Kolmogorov equations for the stochastic process $\{X(t), t \geq 0\}$. Find the time-dependent system state probabilities.

24. Let $\{N(t), t \geq 0\}$ be a Poisson process with parameter λ. Let T_0 denote the time of the first event and T_n denote the inter-arrival time between nth and $(n+1)$th arrivals.

    (a) Prove that $T_n, n = 0, 1, ...$ are mutually independent RVs.

    (b) Find $P\{T_1 > t | T_0 = s\}$.

25. Let $\{X(t), t \geq 0\}$ be a time-homogeneous Poisson process with $X(0) = j$. Consider the RV
$$T_j = \inf\{t : X(t) = j + 1\}.$$
   (a) Find the distribution of T_j.
   (b) Find the joint distribution of $(T_{2023}, T_{2024}, T_{2025})$.

26. Assume that a line of taxis is waiting for passengers to arrive. According to a Poisson process, these taxis pick up an average of 60 people every hour. Once two passengers have been gathered or three minutes have passed since the first passenger boarded the cab, the taxi will depart. Let's say you are the first person to enter the taxi. How long does it often take you to wait for the departure?

27. Let $N(t)$ be the RV denoting the number of events that occur upto and including time t. Let $\{N(t), t \geq 0\}$ be a Poisson process with parameter λ. Let T_i be the inter-arrival times of the ith event $i = 1, 2, \ldots$. Which of the following statements are TRUE.
   (a) $\sum_{i=1}^{n} T_i$ follows $B(n, \lambda)$.
   (b) $P(T_2 \leq t) = 1 - e^{-\lambda t}$, $0 \leq t < \infty$.
   (c) T_i's are independent.
   (d) $Cov(T_i, T_j) > 0,$ for $i \neq j$.

28. Let us assume that cars arrive according to a Poisson process at rate 5 per hour. Assume each car will pick up a hitchhiker (usually stranger, for a ride in their car) with the probability $\frac{1}{5}$. You are third in line.
   (a) What is the probability that no car arrives in first 15 minutes?
   (b) What is the probability that you will have to wait for more than 2 hours?

29. Let $\{N(t), t \geq 0\}$ be a Poisson process with rate λ. Consider, the compound Poisson process $Z(t) = Y_1 + Y_2 + \cdots + Y_{N(t)}$ where $\{Y_n, n = 1, 2, \ldots\}$ is a sequence of independent and identically distributed RVs. Prove that, the increment $Z(t) - Z(s)$ has characteristic function $e^{(\varphi_{Y_1}(x) - 1)\lambda(t-s)}$, where $\varphi_{Y_1}(x)$ denotes the characteristic function of Y_1.

30. Let $\{N(t), t \geq 0\}$ be a Poisson process with rate 4. Find
   (a) Find $P(N(2) = 3, N(6) = 3)$.
   (b) Find the expected number of arrivals between time 2 and time 3 given that there was already 1 arrival by time 2.

31. Let $\{N(t), t \geq 0\}$ be a Poisson process.
   (a) Find the distribution of current life, δ_t and total life β_t.
   (b) Prove that the joint distribution of ν_t and δ_t is given by
   $$P(\nu_t > x, \delta_t > y) = \begin{cases} e^{-\lambda(x+y)}, & \text{if } x > 0, 0 < y < t \\ 0, & \text{if } y \geq t \end{cases}.$$

32. Consider a Poisson process with parameter λ. Let T_1 be the time of occurrence of the first event and let $N(T_1)$ denote the number of events occurred in the next T_1 units of time. Find the mean and variance of $N(T_1)T_1$.

33. Suppose $\{N(t), t \geq 0\}$ is a Poisson process and write $X(t) = N(t-)$. Is $N(1) - X(1-t)$ a Poisson process on $[0, 1]$? Why or why not?

34. Assume that passengers arrive at a bus station as a Poisson process with mean 6 minutes. A particular bus departs after a deterministic time $T = 5$ minutes. Let W be the combined waiting time for all passengers. Prove that, $E(W) = 75$ minutes. Suppose, assume T, the bus arrival time follows exponential distribution with mean 5 minutes, independent of the passengers' arrivals. Prove that, $E(W) = 150$ minutes.
Hint: Find, conditional expectation of W given N, then find the unconditional expectation of W.