MTL 106 (Introduction to Probability Theory and Stochastic Processes)
Tutorial Sheet No. 10 (Simple Markov Queueing Models)

1. Consider the Queueing model where customers arrives as a Poisson process with rate of 5 per minute and service time follows exponential distribution with rate μ service is provided to one customer at a time. Given that 90% of time the queue contains less than or equal to 4 customers, then find the value of μ.

2. In an airport, there is a separate runway for arrivals only. An arriving aircraft join a single queue for the runway. Here Poisson arrivals occur with the rate 20 arrivals per hour and service time is exponential distribution with rate 35 arrivals per hour. Find the mean waiting time in the queue.

3. According to a mobile repairman, the time duration on service of a job for one mobile unit has an exponential distribution, with a mean of 30 minutes. If he serve the mobile sets in the order they arrive and if the arrivals follow Poisson process, with an average rate of 10 every eight-hour day. Find the expected number of mobile units in the queue?

4. Consider an $M/M/1$ queue with mean service time as $\frac{1}{\mu}$ and customers arrival following Poisson process with rate λ, where $\lambda < \mu$. An arrived customer may leave the system directly with a probability of $\frac{i}{i+1}, i = 0, 1, \ldots,$ when they find there are already i customers in the system.
   (a) Find the steady-state probability distribution of the queueing system in terms of λ and μ.
   (b) Assuming that the queueing system is in steady-state, find the loss probability (probability that an arrived customer chooses not to join the queueing system).
   (c) If the mean service time μ^{-1} can be assigned as fast as possible (i.e., $\lambda << \mu$), recommend the service rate μ if the loss probability has to be less than 1%.

5. Consider a person running a hair salon alone, where the customers arrive as a Poisson process with rate 4 per hour and service rate is exponential distributed with parameter 6. Customers are ready to wait for the service but the waiting area can hold upto 5 customers only. What is the average time spent by customer in the system?

6. Consider a modified $M/M/1/N$ queueing system where the arrival and service rates are given by $\lambda_0 = \lambda + \sqrt{\lambda\mu}$, $\lambda_i = \lambda$ for $i = 1, 2, \ldots, N$, $\mu_i = \mu$ for $i = 1, 2, \ldots, N-1$, $\mu_N = \mu + \sqrt{\lambda\mu}$. Calculate the transient probabilities for this system.

7. Suppose M machines are subject to failures and repairs. For each machine, the failure times are independent exponential distributed with parameter λ, and repair times are exponential distributed with parameter μ. Let $X(t)$ denote the number of machines that are in satisfactory order at time t. If there is only one repairman, then under appropriate reasonable assumptions, let $\{X(t), t \geq 0\}$ be a birth and death process on $S = \{0, 1, \ldots, M\}$ with birth rates $\lambda_n = (M-n)\lambda, \quad n = 0, 1, \ldots, M$ and death rate $\mu_n = \mu, \quad n = 1, 2, \ldots, M$. Find the stationary distribution for this process.

8. A Poisson process with a rate of two per hour is formed when jobs arrive at a server. Such occupations' service times have an exponential distribution with a mean of 20 minutes. In the system, maximum four jobs can be processed due to storage space. Find the likelihood that a work will be rejected due to a lack of storage space, assuming that the percentage of processing power used by smaller jobs is insignificant. Find out how many jobs the system has on average at steady state as well.

9. Consider a message switching centre in Vodafone. Assume that traffic arrives as a Poisson process with average rate of 180 messages per minute. The line has a transmission rate of 600 characters per second. Suppose that the message length discipline follows an exponential distribution with an average length of 176 characters. The arriving messages are buffered for transmission where buffer capacity is, say, N. What is the minimum N to guarantee that $\pi_N < 0.005$.

10. In an $M/M/1$ queueing system with service rate μ and customers arrival follows Poisson process with rate λ, where $\lambda < \mu$. Suppose one extra identical server can be added, then which of the following scenarios is better with respect to mean waiting time in each system.

(a) Separate the two servers. Therefore, there will be two $M/M/1/\infty$ queues. In this case, it is assumed that an arriving customer can wither join the first queue or the second with equal probability.

(b) Join the two servers together. Therefore, it will be considered as $M/M/2/\infty$ queue.

11. Consider a system with two servers and having infinite capacity queue. The customers arrive with arrival rate λ and served with rate μ. Write down the Kolmogorov backward equation for the above process. Find the time-dependent and steady-state probability for the system.

12. Let $Y(t)$ denote the number of customers who arrive in the time period $(0, t]$. If a customer arrive a time $s \in (0, t]$, the probability that he is still in the process of being served at time t is $e^{-\mu(t-s)}$. Thus if a customer arrives at time chosen uniformly $(0, t]$, the probability that he is still in the process of being served at time t is denoted by q_t is:

$$
\begin{aligned}
q_t &= \frac{1}{t}\int_0^t e^{-\mu(t-s)} ds \\
&= \frac{1 - e^{-\mu t}}{\mu t}.
\end{aligned}
$$

Let $X(t)$ denote the number of customers arriving in $(0, t]$, that are still in the process of being served at time t. Prove that the conditional distribution of $X(t)$ given $Y(t) = k$ is $B\left(k, \frac{1-e^{-\mu t}}{t}\right)$.

13. Consider a parking lot with finite capacity c. The vehicles arrive with the arrival rate $\lambda = 5$ and the service time is exponential distributed with rate $\mu = 7$. Formulate the problem as a $M/M/c/c$ queueing system and find the minimum value of c such that $\pi_c < 0.05$.

14. Prove that, the PDF of the length of busy period, denoted by B, for $c = 2$ in $M/M/c$ queue is given by

$$
f_B(x) = \frac{e^{-(\lambda+2\mu)x}}{x} \sum_{r=0}^{\infty} (r+1) \left(\frac{\mu}{2\lambda}\right)^{\frac{r+1}{2}} I_{r+1}(2\sqrt{2\lambda\mu}), \quad x > 0.
$$

Further, show that

$$
E(B) = \frac{1}{\mu(1-\rho)}; \quad E(B^2) = \frac{2-\rho}{\mu^2(1-\rho)^3}.
$$

15. Users arrive at a nature park in cars according to a Poisson process at a rate of 20 cars per hour. They stay in the park for a random amount of time that is exponentially distributed with mean 3 hours and leave. Assuming the parking lot is sufficiently big so that nobody is turned away, then what is the expected number of cars in the lot in the long run equals.