MTL 106 (Introduction to Probability Theory and Stochastic Processes)
Tutorial Sheet No. 1 (Basic Probability)

1. Items coming off a production line are marked defective (D) or non-defective (N). Items are observed and their condition noted. This is continued until two consecutive defectives are produced or four items have been checked, which ever occurs first. Describe the sample space for this experiment.

2. Let $\Omega = \{0, 1, 2, \ldots\}$. Let $\mathcal{F}$ be the collection of subsets of Ω that are either finite or whose complement is finite. Is $\mathcal{F}$ a σ-field? Justify your answer.

3. Consider $\Omega = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$. Let $\mathcal{F}$ be the largest σ-field over Ω. Define, for any event R, $P(R) =$ area of $R = (b-a)(d-c)$ where R is the rectangular region that is a subset of Ω of the form $R = \{(u, v) : a \leq u < b, c \leq v < d\}$. Let T be the triangular region $T = \{(x, y) : x \geq 0, y \geq 0, x + y < 1\}$. Show that T is an event, and find $P(T)$, using the axiomatic definition of probability.

4. Let $A_1, A_2, \ldots, A_N$ be a system of completely independent events (i.e., $P(\cap_{j=1}^{r} A_{i_j}) = \prod_{j=1}^{r} P(A_{i_j}), \quad r = 2, 3, \ldots, N$). Assume that $P(A_n) = \frac{1}{n+1}, \quad n = 1, 2, \ldots, N$.
   (a) Find the probability that exactly one of the A_i's occur?
   (b) Find the probability that atmost two A_i's occur?

5. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $A_1, A_2, \ldots, A_n \in \mathcal{F}$ with $P(\cap_{i=1}^{n-1} A_i > 0)$. Prove that
$$P\left(\cap_{i=1}^{n} A_i\right) = P(A_1)P(A_2 \mid A_1)P(A_3 \mid A_1 \cap A_2)\ldots P\left(A_n \mid \cap_{i=1}^{n-1} A_i\right)$$

6. If $A_1, A_2, \ldots, A_n$ are n events, then show that $\sum_{i=1}^{n} P(A_i) - \sum_{i \neq j, i=1}^{n} P(A_i \cap A_j) \leq P\left(\cup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i)$.

7. Let $\Omega = \{a, b, c, d\}$, $\mathcal{F} = \{\emptyset, \{a\}, \{b, c\}, \{d\}, \{a, b, c\}, \{b, c, d\}, \{a, d\}, \Omega\}$ and P a function from $\mathcal{F}$ to $[0, 1]$ with $P(\{a\}) = \frac{2}{7}$, $P(\{b, c\}) = \frac{3}{5}$ and $P(\{d\}) = \beta$. The value of β such that P to be a probability on $(\Omega, \mathcal{F})$.

8. Prove that, for any two events A and B, $P(A \cap B) \geq 1 - P(A) - P(B)$.

9. Consider a gambler who on each independent bet either wins 1 with probability $\frac{1}{3}$ or losses 1 with probability $\frac{2}{3}$. The gambler will quit either when he or she is winning a total of 10 or after 50 plays. The probability the gambler plays exactly 14 times?

10. Let $\Omega = \{4, 3, 2, 1\}$.
    (a) Find three different σ-algebras $\{\mathcal{F}_n\}$ for $n = 1, 2, 3$ such that $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}_3$.
    (b) Further, create a set function $P : \mathcal{F}_3 \to \mathcal{R}$ such that, $(\Omega, \mathcal{F}_3, P)$ is a probability space.

11. Suppose that the number of passengers for a limousine pickup is thought to be either 1, 2, 3, or 4, each with equal probability, and the number of pieces of luggage of each passenger is thought to be 1 or 2, with equal probability, independently for different passengers. What is the probability that there will be five or more pieces of luggage?

12. Let $\Omega = \mathcal{R}$ and $\mathcal{F}$ be the Borel σ-field on $\mathcal{R}$. For each interval $I \subseteq \mathcal{R}$ with end points a and b ($a \leq b$), let $P(I) = \int_a^b \frac{1}{\pi} \frac{1}{1+x^2} dx$. Does P define a probability on the measurable space $(\Omega, \mathcal{F})$? Justify your answer.

13. Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $\{A_n\}$ be a nondecreasing sequence of elements in $\mathcal{F}$. Prove that
$$P\left(\lim_{n \to \infty} A_n\right) = \lim_{n \to \infty} P(A_n).$$

$P\{s_4\} = \frac{3}{10}$. Define, $A_n = \begin{cases} \{s_1, s_3\} & \text{if n is odd} \\ \{s_2, s_4\} & \text{if n is even} \end{cases}$. Find $P(\liminf A_n)$, $P(\limsup A_n)$.

14. Let w be a complex cube root of unity with $w \neq 1$. A fair die is thrown three times. If x, y and z are the numbers obtained on the die. Find the probability that $w^x + w^y + w^z = 0$.

15. An urn contains balls numbered from 1 to N. A ball is randomly drawn.

16. What is the probability that the number on the ball is divisible by 3 or 4?

17. Pick a number x at random out of the integers 1 through 30. Let A be the event that x is even, B that x is divisible by 3 and C that x is divisible by 5. Are the events A, B and C pairwise independent? Further, are the events A, B and C mutually independent?

18. The first generation of particles is the collection of off-springs of a given particle. The next generation is formed by the off-springs of these members. If the probability that a particle has k off springs (splits into k parts) is p_k, where $p_0 = 0.4$, $p_1 = 0.3$, $p_2 = 0.3$, find the probability that there is no particle in second generation. Assume particles act independently and identically irrespective of the generation.

19. A and B throw a pair of unbiased dice alternatively with A starting the game. The game ends when either A or B wins. A wins if he throws 6 before B throws 7. B wins if he throws 7 before A throws 6. What is the probability that A wins the game? Note that "A throws 6" means the sum of values of the two dice is 6. Similarly "B throws 7".

20. In a meeting at the UNO 40 members from under-developed countries and 4 from developed ones sit in a row. What is the probability no two adjacent members are representatives of developed countries.

21. A random walker starts at 0 on the x-axis and at each time unit moves 1 step to the right or 1 step to the left with probability 0.5. Find the probability that, after 4 moves, the walker is more than 2 steps from the starting position.

22. The coefficients a, b and c of the quadratic equation $ax^2 + bx + c = 0$ are determined by rolling a fair die three times in a row. What is the probability that both the roots of the equation are real? What is the probability that both roots of the equation are complex?

23. An electronic assembly consists of two subsystems, say A and B. From previous testing procedures, the following probabilities assumed to be known: $P(A \text{ fails}) = 0.20$, $P(A \text{ and } B \text{ both fail}) = 0.15$, $P(B \text{ fails alone}) = 0.15$. Evaluate the following probabilities (a) $P(A \text{ fails}/B \text{ has failed})$ (b) $P(A \text{ fails alone } /A \text{ or } B \text{ fail})$.

24. An aircraft has four engines in which two engines in each wing. The aircraft can land using atleast two engines. Assume that the reliability of each engine is $R = 0.93$ to complete a mission, and that engine failures are independent.

    (a) Obtain the mission reliability of the aircraft.

    (b) If at least one functioning engine must be on each wing, what is the mission reliability?

25. Four lamps are located in circular. Each lamp can fail with probability q, independently of all the others. The system is operational if no two adjacent lamps fail. Obtain an expression for system reliability?.

26. A batch of N transistors is dispatched from a factory. To control the quality of the batch the following checking procedure is used; a transistor is chosen at random from the batch, tested and placed on one side. This procedure is repeated until either a pre-set number $n(n < N)$ of transistors have passed the test (in which case the batch is accepted) or one transistor fails (in this case the batch is rejected). Suppose that the batch actually contains exactly D faulty transistors. Find the probability that the batch will be accepted.