MTL 106 (Introduction to Probability Theory and Stochastic Processes)
Tutorial Sheet No. 2 (Random Variable)

1. For $f : \Omega \to \Omega'$, show that $A \subset f^{-1}(f(A))$, if $A \subset \Omega$ and $f(f^{-1}(B)) \subset B$, if $B \subset \Omega'$.

2. Let $f : R \to R$ where $f(x) = \cos(x)$. Describe the $\sigma-$field induced by f and verify that it is a sub $\sigma-$field on R.

3. If A_1, A_2 are measurable sets and $f : \Omega \to R$ is defined by

$$f(w) = \begin{cases} -1, & w \in A_1 \\ 1, & w \in A_1^c \cap A_2 \\ 0, & w \in A_1^c \cap A_2^c \end{cases}$$

   Examine whether f is a measurable function over (a) $\sigma(A_1)$ (b) $\sigma(A_2)$ (c) $\sigma(\{A_1, A_2\})$.

4. Consider measurable space (Ω, F), where $\Omega = \{a, b, c, d\}$ and $F = \{\phi, \Omega, \{a, b\}, \{c, d\}\}$. Examine if the function $X : \Omega \to R$ defined by $X(a) = X(b) = 1,\ X(c) = 1,\ X(d) = 2$ is $F-$measurable. Find the minimal $\sigma-$field w.r.t which X is measurable.

5. Let f, g be Borel functions on (R, β). Define for $A \in \beta$:

$$h(w) = \begin{cases} f(w), & w \in A \\ g(w), & w \in A^c \end{cases}$$

   Show that h is a Borel function.

6. Give an example to show that $|f|$ is measurable function does not imply that f is measurable function.

7. If X is a r.v. on a probability space (Ω, F, P), then show that $a + bX,\ |X|$,

$$X^+ = \begin{cases} X, & X \geq 0 \\ 0, & X < 0 \end{cases}$$

   are r.v.s on this probability space.

8. In a bombing attack, there is 50% chance that a bomb can strike the target. Two hits are required to destroy the target completely. How many bombs must be dropped to give a 99% chance or better of completely destroying the target?

9. Five-digit codes are chosen at random from the collection $\{0, 1, 2, \ldots, 9\}$ with replacement. Let X be the r.v. that indicates how many zeros there are in the selected codes. Find the PMF of r.v. X?

10. There are 10 coins in an urn, 4 of which are fake. One coin at a time is taken out of the urn until all counterfeit coins have been discovered. Let X be the r.v. that indicates how many coins must be eliminated in order to locate the first fake coin. Find the PMF of X?

11. For what values of α and p does the following function represent a PMF $p_X(x) = \alpha p^x,\quad x = 0, 1, 2, \ldots$.

12. A r.v. X has the following PMF

$X = x$	0	1	2	3	4	5	6	7	8
$P(X = x)$	k	$3k$	$5k$	$7k$	$9k$	$11k$	$13k$	$15k$	$17k$

   1. Determine the value of k.
   2. Find $P(X < 4),\ P(X \geq 5),\ P(0 < X < 4)$.
   3. Find the CDF of X.
   4. Find the smallest value of x for which $P(X \leq x) = 1/2$.

13. Let X be a r.v. with PMF

$$p_X(x) = \frac{4c}{5^x}, x = 1, 2, \ldots$$

for some constant c. What is the value of c? Compute the probability that X is even?

14. A random number is chosen from the interval $[0, 1]$ by a random mechanism. What is the probability that (i) its first decimal will be 3 (ii) its second decimal will be 3 (iii) its first two decimal will be 3's?

15. Suppose X, the random number of eggs laid by a bird is a r.v. having PMF:

$$p_X(x) = \frac{e^{-\lambda}\lambda^x}{x!},\ x = 0, 1, 2, \ldots$$

where $\lambda > 0$ is a known constant. The probability of an egg developing is p. Find the PMF of r.v. Y, the number of eggs developing. Also find the PMF of $|Y - 4|$.

16. Two balls are chosen at random from an urn of five balls with labels numbered 1 through 5 without replacement. The r.v. X should indicate the total of the numbers on the two balls. What is the the PMF of X?

17. The sum of the two six-sided dice's numbers is determined after rolling them. Let X, the r.v. that represents the total of the numbers rolled, be the variable. What is the the PMF of X?

18. A die is tossed two times. Define X =sum of scores and Y =absolute difference of scores on the two tosses. Identify the probability space and verify that X, Y are r.v.s on this probability space. Write down the events: $\{X = 3\},\ \{Y \leq 1\},\ \{X > 5\},\ -1 < X \leq 4.5$ and determine their probabilities. Also find the PMF of X and the distribution of Y. Using these determine $P(X \leq 7 | X > 4)$ and $P(Y > 2.5 | Y \leq 4)$.

19. Let the PDF of X be

$$f(x) = \begin{cases} c(x - 5x^2), & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}.$$

    1. What is the value of c?
    2. What is the distribution of X?
    3. Obtain $P\left(\frac{1}{4} < X < \frac{3}{4}\right)$.

20. The life length of a certain equipment in hours is a continuous r.v. with PDF

$$f_X(x) = \begin{cases} xe^{-kx}, & x > 0 \\ 0, & \text{elsewhere} \end{cases}$$

Determine the constant k.

(i) Given that the equipment has been operative for the last 10 hours, what is the probability that it will not fail during next 15 hours.

(ii) Ten pieces of equipment are tested independently for 50 hours and Y= the number of pieces that fail during the testing time. Find PMF of the r.v. Y.

(iii) If n pieces of the equipment are hooked up in series and operate independently, find the PDF of the life of series system. Find n so that the reliability of the series system for 20 hours is 80%.

21. A bombing plane flies directly above a railroad track. Assume that if a larger(small) bomb falls within 40(15) feet of the track, the track will be sufficiently damaged so that traffic will be disrupted. Let X denote the perpendicular distance from the track that a bomb falls. Assume that

$$f_X(x) = \begin{cases} \frac{100-x}{5000}, & if\ 0 < x < 100 \\ 0, & \text{otherwise} \end{cases}.$$

1. Find the probability that a larger bomb will disrupt traffic.
2. If the plane can carry three large (eight small) bombs and uses all three(eight), what is the probability that traffic will be disrupted?

22. Let

$$F_X(x) = \begin{cases} 0, & x < 0 \\ 1 - 2e^{-x} + e^{-2x}, & x \geq 0 \end{cases}.$$

Is F_X a distribution function? What type of r.v. is X? Find the PMF/PDF of X?

23. A continuous type r.v. X has the PDF

$$f(x) = \begin{cases} \frac{3x^2}{8} & 0 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}.$$

Compute the probability that X is greater than its 75^{th} percentile?

24. Let the CDF of X be

$$F(x) = \begin{cases} 0 & -\infty < x < 0 \\ 1 - \sum_{k=0}^{4} \frac{x^k e^{-x}}{k!} & 0 \leq x < \infty \end{cases}.$$

Find $P(X > 0)$?

25. Let X be a r.v. with CDF

$$F(x) = \begin{cases} 0 & -\infty < x < 0 \\ 1 - e^{-2x} & 0 \leq x < \infty \end{cases}.$$

Find $P(0 \leq e^X \leq 5)$?

26. The r.v. X has PDF

$$f(x) = \begin{cases} (k+1)x^3 & 0 < x < 2 \\ 0 & \textit{otherwise} \end{cases},$$

where k is a constant. Compute the probability of X between the first and third quartiles?

27. Let X be a continuous type r.v. having CDF $F(x)$. What is the CDF of $Y = max(0, X)$?

28. What is the PMF of the r.v. X if its CDF is given by

$$F(x) = \begin{cases} 0 & x < 3 \\ 0.4 & 3 \leq x < 4 \\ 0.6 & 4 \leq x < 5 \\ 1 & x \geq 5 \end{cases}.$$

29. Let $\Omega = [0, 1]$. Define $X :\to R$ by

$$X(\omega) = \begin{cases} \omega, & 0 \leq \omega \leq \frac{1}{2} \\ \omega - \frac{1}{2}, & \frac{1}{2} < \omega \leq 1 \end{cases}$$

For any interval $I \subseteq [0, 1]$, define $P(I) = \int_I 2x dx$. Identify the probability space and verify that X is a random variable on this space. Determine the distribution function of X and use this to find $P(X > \frac{1}{2}),\ P(\frac{1}{4} < X < \frac{1}{2}),\ P(X < \frac{1}{2} | X > \frac{1}{4})$.

30. Let $f(x)$ be a PDF of the continuous type r.v. X. Show that for every $-\infty < \mu < \infty$ and $\sigma > 0$, the function $\frac{1}{\sigma} f(\frac{x-\mu}{\sigma})$ is also a PDF of some continuous type r.v., say Z.

31. A point is chosen at random inside a square D with vertices at (0,0),(1,0),(0,1) and (1,1) in R^2. Define $X(u,v) = u+v,\ (u,v) \in D$. For any quadrilateral or triangle E in D, define $P(E)$ =area of E. Identify the probability space and verify that X is a r.v. on this space. Find the distribution function of X.

32. An urn contains n cards numbered $1, 2, \ldots, n$. Let X be the least number on the m cards drawn randomly without replacement from the urn. Find probability distribution of r.v. X. Compute $P(X \geq \frac{3}{2})$.

33. Let $F(x),\ x \geq 0$ be a distribution for a non-negative random variable and

$$G(x) = \begin{cases} 0, & x \leq 0 \\ 1 - exp[\alpha(1 - F(x))], & x > 0 \end{cases}$$

where $\alpha > 0$ is a constant. Show that $G(x)$ is a distribution function and compute $P(X = 0 | X \leq 1)$ assuming $G(x)$ is the distribution function of the r.v. X.

34. Let X be a continuous r.v. taking values in the interval $[0,1]$. If $P(x < X \leq y)$, for all $x, y,\ 0 \leq x < y \leq 1$ depends only on $(y - x)$, then show that X has uniform probability distribution on the interval $[0,1]$.

35. Let X be a r.v. such that $P(X > \frac{1}{2}) = \frac{7}{8}$ and its PDF is:

$$f_X(x) = \begin{cases} ax, & 0 \leq x < 1 \\ b - x, & 1 \leq x < 2 \\ 0, & \text{otherwise} \end{cases}$$

Determine a, b and find the distribution function of X.

36. Let X be a continuous type r.v. having PDF

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \ -\infty < x < \infty.$$

Find PDF of:

(i) $Y = a + bX$ (ii) $Y = \ln X$ (iii) $Z = X^2$ (iv) $Y = \begin{cases} X^{1/2}, & X > 0 \\ -|X|^{1/2}, & X \leq 0 \end{cases}$.

37. Let X be a continuous type r.v. with PDF

$$f_X(x) = \begin{cases} 6x(1-x), & 0 < x < 1 \\ 0, & \text{elsewhere} \end{cases}.$$

Find the PDF of

(i) $Y = \frac{X}{1+X}$ (ii) $Z = \begin{cases} 2X, & X < \frac{1}{4} \\ \frac{1}{2}, & \frac{1}{4} \leq X < \frac{3}{4} \\ \frac{2}{3}X, & X \geq \frac{3}{4} \end{cases}$.

38. Let X and Y be two r.v.s such that their MGFs exist. Then, prove the following:

   1. If $M_X(t) = M_Y(t),\ \forall\ t$, then X and Y have same distribution.
   2. If $\Psi_X(t) = \Psi_Y(t),\ \forall\ t$, then X and Y have same distribution.

39. Show that $E(|X|) = \displaystyle\int_{-\infty}^{0} F_X(x)dx + \int_{0}^{\infty} (1 - F_X(x))dx.$

40. Let X be a r.v. with mean μ and variance σ^2. Show that $E[(X-b)^2]$, as a function of b, is minimized when $b = a\mu$.

41. Let X be a continuous type r.v. with PDF

$$f_X(x) = \begin{cases} a + bx^2, & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}.$$

If $E(X) = \frac{3}{5}$, find the value of a and b.

42. Let A, B, C be events in a sample space with $P(A) = 0.1,\ P(B) = 0.4,\ P(C) = 0.3,\ P(A \cap B) = 0.1, A \cap C = B \cap C = \emptyset$. Find mean and variance of a r.v. $X = I_A + I_B - I_C$, where $I_{[.]}$ is the indicator function of the set [.].

43. Show that the matrix $\sum = (\mu_{i+j})_{k \times k}$ is non negative definite where μ_p is the central moment of order p of the r.v. X.

44. From a point on the circumference of the circle of radius r, a chord is drawn in a random direction. Show that the expected value of length of the chord is $\frac{4r}{\pi}$ and its variance is $2r^2\left(1 - \frac{8}{\pi^2}\right)$.

45. For the r.v. X with PMF $p_X(x) = \frac{e^{-\lambda}\lambda^x}{x!},\ x = 0, 1, 2, \ldots$. Show that $\mu_2 = \mu_3 = \lambda,\ \mu_4 = \lambda + 3\lambda^2$.

46. Show that for a r.v. with PDF $f_X(x) = \frac{k\alpha^k}{(x+\alpha)^{k+1}},\ x > 0$, the α^{th} absolute moment exists for X, for $\alpha < k$.

47. For the r.v. with PMF $p_X(x) = \binom{r+x-1}{x} p^r q^x,\ x = 0, 1, 2, \ldots$, where $q = 1 - p$, find the MGF and hence the mean and variance of X.

48. Find the mean, variance of the r.v. X having the distribution function

$$F_X(x) = \begin{cases} 0, & x < 0 \\ p + (1-p)(1 - e^{-\lambda x}), & 0 \le x < T \\ 1, & x \ge T \end{cases}.$$

49. For the Laplace distribution, i.e. the r.v. having PDF given by:

$$f_X(x) = \frac{1}{2\lambda} exp\left(\frac{-|X - \mu|}{\lambda}\right),\ x \in R,\ \lambda > 0,\ 0 < \mu < \infty.$$

Find MGF and its mean and variance.

50. Consider X with MGF $M(t),\ t > 0$. Show that

$$P(tX > s^2 + \log(M(t)) < e^{-s^2}, \quad t > 0.$$

51. Show that the sequence of moments determine the probability distribution of the r.v. uniquely if it is a bounded r.v..

52. Consider a r. v. X with $E(X) = 1$ and $E(X^2) = 1$. Find $E\left[(X - E(X))^4\right]$ if it exists.

53. Let X be a discrete r.v. with MGF

$$M_X(t) = \alpha + \beta e^{2t} + \gamma e^{4t}, \quad E(X) = 3, \quad Var(X) = 2.$$

(i) Find α, β and γ? (ii) Find the PMF of X? (iii) Find $E(2^X)$?

54. The MGF of a r.v. X is given by $M_X(t) = exp(\mu(e^t - 1))$.
(i) What is the distribution of X?
(ii) Find $P(\mu - 2\sigma < X < \mu + 2\sigma)$, given $\mu = 4$.

55. Let X be a r.v. with characteristic function given by $E[e^{iXt}] = \frac{1}{7}\left(2 + e^{-it} + e^{it} + 3e^{2it}\right)$. Determine (i) $P\left(-1 \le X \le \frac{1}{2}\right)$ (ii) $E(X)$.