MTL 106 (Introduction to Probability Theory and Stochastic Processes)
Tutorial Sheet No. 3 (Function of a Random Variable)

1. A discrete r.v. X has a uniform probability distribution on the set $\{-k, -(k-1), \ldots, -1, 0, 1, 2, \ldots, r\}$. Find the probability distribution of (a) $|X|$ and (b) $(X+1)^2$.

2. Let X be a r.v. with Poisson distribution with parameter λ. Show that the characteristic function of X is $\Psi_X(t) = exp\left[\lambda(e^{it}-1)\right]$. Hence, compute $E(X^2)$, $Var(X)$ and $E(X^3)$.

3. Show that the PGFs of the geometric, negative binomial, and Poisson distribution exist and hence calculate them.

4. Prove that, the r.v. X has exponential distribution and satisfies a memoryless property or Markov property which is given as
$$P(X > x + s / X > s) = P(X > x) \quad x, s \in \mathbf{R}^+. \tag{1}$$

5. Suppose that diameters of a shaft s manufactured by a certain machine are normal r.v. with mean 10 and s.d. 0.1. If for a given application the shaft must meet the requirement that its diameter falls between 9.9 and 10.2 cm. What proportion of shafts made by this machine will meet the requirement?

6. A machine automatically packs a chemical fertilizer in polythene packets. It is observed that 10% of the packets weigh less than 2.42 kg while 15% of the packets weigh more than 2.50 kg. Assuming that the weight of the packet is normal distributed, find the mean and variance of the packet.

7. Demonstrate that any continuous nonnegative exponential r.v. is an exponential r.v.

8. Give α and β positive real numbers with the relationship $\alpha < \beta$. What is the probability that the distance between two randomly chosen points along a straight line segment of length β is at least α?

9. Let X have a beta distribution i.e. its PDF is
$$f_X(x) = \frac{1}{\beta(a,b)} x^{a-1}(1-x)^{b-1} \ , \ 0 < x < 1$$
and Y given $X = x$ has binomial distribution with parameters (n, x). Find the PDF of Y.

10. One of the inputs to a certain program is a r.v. whose value is a nonnegative real number, call it Y. The PDF of Y is given by $f_Y(y) = ye^{-y}$, $y > 0$. Conditioned on $Y = y$, the execution time X of the program is an exponentially distributed r.v. with parameter y.
    (a) Compute the PDF of the program execution time X.
    (b) Find the probability that the input value is at least 200 given that the program execution time is 99 hours.

11. An airline knows that 5 percent of the people making reservations on a certain flight will not show up. Consequently, their policy is to sell 52 tickets for a flight that can hold only 50 passengers. Assume that passengers who come to the airport are independent of each other. What is the probability that there will be a seat available for every passenger who shows up?

12. The number of weekly breakdowns of a computer is a r.v. having a Poisson distribution with $\alpha = .03$. (a) What is the probability that the computer will have an even number of breakdowns during the given week? (b) If the computer is run for 10 consecutive weeks what is the probability that (i) at least two weeks have no breakdown (ii) 10th is the first week to have a breakdown?.

13. The number of times that an individual contracts viral infection in a given year is a Poisson r.v. with parameter $\lambda = 5$. Using a new health scheme, 75% of the population reduces the Poisson parameter λ to 1. If an individual does not get a viral infection for a year, what is the probability that he/she followed the new health scheme?

14. Knowing that the bus will come evenly dispersed at any moment between 6:00 AM and 6:20 AM, a student arrives at the bus stop on time at 6:00 AM. How likely is it that the student will have to wait more than five minutes? What is the likelihood that the kid will have to wait for at least another five minutes if the bus isn't there by 6:10 AM?

15. If $X \sim U(0,1)$. What is the PDF of $Y = -lnX$?

16. Let $X \sim U(-7,7)$. What is the probability that the quadratic equation $114t^2 + 25tX + 3X = 0$ has complex solutions?

17. Take into account two electronic devices, D_1 and D_2, whose expected lifespans are $N(40,36)$ and $N(45,9)$, respectively. Which gadget should be selected if it needs to be utilized for 45 hours? What if it must be utilized for 42 hours?

18. Let $Y \sim N(\mu, \sigma^2)$ where $\mu \in \mathbf{R}$ and $\sigma^2 < \infty$. Let X be another r.v. such that $X = e^Y$. Find the distribution function of X. Also, verify that $E(\log(X)) = \mu$ and $Var(\log(X)) = \sigma^2$.

19. Let X be a r.v. with $N(0,\sigma^2)$. Find the moment-generating function for the r.v. X. Deduce the moments of order n about zero for the r.v. X from the above result.

20. The MGF of a r.v. X is given by $M_X(t) = exp(\mu(e^t - 1))$. (a) What is the distribution of X? (b) Find $P(\mu - 2\sigma < X < \mu + 2\sigma)$, given $\mu = 4$.

21. What is the chance that the r.v. will be larger than its square, or $P(X > X^2)$, if the r.v. X has a uniform distribution over the range $[0,a]$?

22. Let $X \sim Exp(\lambda)$. What is the PDF of the r.v. $Y = X\sqrt{X}$?

23. Let a, b, c, d be any four real values, all of which should be positive. What is the PDF of the r.v. $Y = (b-a)X + a$ if $X \sim Beta(c,d)$?

24. Suppose that the life length of an item is exponentially distributed with parameter 0.5. Assume that ten such items are installed successively so that the ith item is installed immediately after the $(i-1)$th item has failed. Let T_i be the time to failure of the ith item $i = 1, 2, \ldots, 10$ and is always measured from the time of installation. Let S denote the total time of functioning of the 10 items. Assuming that $T_i's$ are independent, evaluate $P(S \geq 15.5)$.

25. A certain industrial process yields a large number of steel cylinders whose lengths are distributed normal with a mean 3.25 inches and a standard deviation 0.05 inches. If two such cylinders are chosen at random and placed end to end what is the probability that their combined length is less than 6.60 inches?

26. A small industrial unit has 30 machines whose lifetimes are independent exponential distributed with mean 100 months. If all the machines are under use at a time, find the probability that even after 200 months there are at least five machines working.

27. Suppose that 30 electronic devices say $D_1, D_2, \ldots, D_{30}$ are used in the following manner. As soon as D_1 fails, D_2 becomes operative. When D_2 fails, D_3 becomes operative, etc. Assume that the time to failure of D_i is an exponentially distributed r.v. with parameter $= 0.1\ (\text{h})^{-1}$. Let T be the total time of operation of the 30 devices. What is the probability that T exceeds 350 h?