MTL 106 (Introduction to Probability Theory and Stochastic Processes)
Tutorial Sheet No. 4 (Random Vector)

1. Let
$$F(x,y) = \begin{cases} 0, & x < 0, y < 0 \text{ or } x + y < 1 \\ 1, & \text{otherwise} \end{cases}$$
Show that F is not a distribution function in R^2.

2. Suppose two cards are drawn from a deck of 52 cards. Let X = number of aces obtained and Y = number of queens obtained. Discuss whether or not r.v.s X, Y are independent.

3. Let the joint CDF function of X and Y is
$$F(x,y) = \begin{cases} (1 - e^{-2x})(1 - e^{-2y}) & x > 0, y > 0 \\ 0 & \text{otherwise} \end{cases}.$$
What is the joint PDF of X and Y, and the $P(1 < X < 4, 1 < Y < 3)$?

4. Let the joint PMF function of X and Y is
$$p(x,y) = \begin{cases} \frac{1}{33}(2x + y) & x = 1,2,3, y = 1,2 \\ 0 & \text{otherwise} \end{cases}.$$
What are the marginals of X and Y ?

5. For what value of c is the function
$$p(x,y) = \begin{cases} c(x + 2y) & x = 1,2, y = 1,2 \\ 0 & \text{otherwise} \end{cases}.$$
a joint PMF?

6. Let the joint PDF function of X and Y is
$$f(x,y) = \begin{cases} 2e^{-(2x+y)} & 0 \leq x, y < \infty \\ 0 & \text{otherwise} \end{cases}.$$
what is $P(X \geq Y \geq 3)$?

7. Let the joint PDF function of X and Y is
$$f(x,y) = \begin{cases} \frac{2}{9}x & 0 < x < y < 3 \\ 0 & \text{otherwise} \end{cases}.$$
What is the marginal PDF of Y?

8. Let the joint PDF function of X and Y is
$$f(x,y) = \begin{cases} \frac{2}{9}(3 - x - y) & 0 < x; y < 3; 0 < x + y < 3 \\ 0 & \text{otherwise} \end{cases}.$$
What is the conditional probability $P(X < 1 | Y < 1)$?

9. A girl and boy decide to meet between 5:00 and 6:00 p.m. Assume they all arrive at separate times that are uniformly distributed at random within this time span. Each will wait no longer than 10 minutes for the other, after which they will both depart. What is the likelihood that they actually leave the house?

10. Let X and Y be two independent uniformly distributed r.v.s on $[0,1]$. Calculate, $P(Y \geq \frac{1}{2} | Y \geq 1 - 3X)$.

11. Verify that the normal distribution, geometric distribution, and Poisson distribution have reproductive property, but the uniform distribution and exponential distributions do not.

12. Suppose a two dimensional r.v. has a joint PDF:

$$f_{X,Y}(x,y) = \begin{cases} kx(x-y), & 0 < x < 2, -x < y < x \\ 0, & \text{otherwise} \end{cases}$$

(a) Evaluate the constant k

(b) Compute $P(X + Y < 1),\ P(XY < 1)$

(c) Compute $P(Y > -1/2 | X = 1)$

13. Let X_1, X_2 be i.i.d. r.v.s each having PMF $P(X = \pm 1) = \frac{1}{2}$. Define $X_3 = X_1 X_2$. Show that random variables X_1, X_2, X_3 are pairwise independent but not mutually independent.

14. Suppose X, Y are independent r.v.s each having binomial distribution with parameters n and p, $(0 < p < 1)$. Find the joint PMF of $(X + Y, X - Y)$.

15. Suppose X, Y are independent Poisson r.v.s, show that the conditional distribution of X given $Z = X + Y$, is binomial.

16. Let X, Y be i.i.d each having standard normal distribution. Show that $U = \sqrt{X^2 + Y^2}$ and $V = \frac{X}{Y}$ are independent.

17. Let X, Y be i.i.d r.v.s with common PDF

$$f(x) = \begin{cases} e^{-x}, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases}$$

Find the PDF of r.v.s $\min\{X, Y\}$, $\max\{X, Y\}$. Let $U = X + Y$ and $V = X - Y$. Find the conditional PDF of V, given $U = u$ for some fixed $u > 0$.

18. A point (u, v) is chosen as follows. First a u is chosen at random in the interval $(0, 1)$, then a point v is chosen at random on the interval $(0, u)$. Find the joint PDF of (X, Y), defined by $X(u, v) = u,\ Y(u, v) = v$. Also find the conditional PDF of X, given $Y = y$ and that of Y, given $X = x$.

19. Let (Ω, S, P) be the space formed by independent r.v.s X and Y. With X having a uniform distribution on $(-a, a), a > 0$, and Y being a continuous type r.v. with density f, where f is continuous and positive on $\mathbb{R}$, we can say that X is uniformly distributed. Let F be Y's CDF. If $u_0 \in (-a, a)$ is a fixed number, show that

$$f_{Y|X+Y}(y|u_0) = \begin{cases} \frac{f(y)}{F(u_0+a) - F(u_0-a)} & \text{if } u_0 - a < y < u_0 + a \\ 0 & \text{otherwise} \end{cases}$$

where $f_{Y|X+Y}(y|u_0)$ is the conditional density function of Y, given $X + Y = u_0$.