MTL 106 (Introduction to Probability Theory and Stochastic Processes)
Tutorial Sheet No. 5 (Moments)

1. Let (X, Y) be uniformly distributed on the region $\{(x, y) : 0 < x < y < 1\}$.

   (a) Find $V(2X + 3Y - 4),\ Cov(X + Y, X - Y),\ E(X^3 + XY^2 - X^2Y)$.

   (b) Find regression of Y on X and of X on Y.

   (c) Find MGF of (X, Y) and use this to compute correlation coefficient between X, Y.

2. Let $X_1, X_2 \ldots, X_{m+n}$ be i.i.d. r.v.s each having a finite second order moment. Find the correlation coefficient between S_n and $S_{m+n} - S_n,\ n > m$, where $S_k = \sum_{i=1}^{k} X_i,\ k = 1, 2, \ldots, m + n$.

3. Let X, Y be discrete r.v.s with respective PMFs given by

$$p_X(x_1) = p_1,\ p_X(x_2) = 1 - p_1,\ p_Y(y_1) = p_2,\ p_Y(y_2) = 1 - p_2$$

   Show that X, Y are independent iff X, Y are uncorrelated.

4. Using the concept of conditional expectation, prove that:

$$Var(X) = Var(E(X|Y)) + E(Var(X|Y))$$

5. Let $X_1, X_2, \ldots, X_n$ be the sequence of i.i.d. r.v.s and N be discrete r.v. taking positive integer values. Suppose $X's$ and N are independent. Define random sum: $S_N = X_1 + X_2 + \ldots + X_N$.

   (a) Show that $E(S_N) = E(N)E(X_1)$ and $Var(S_N) = (E(X_1))^2 Var(N) + E(N)Var(X_1)$.

   (b) Find MGF of S_N in terms of MGF's of X_i and N.

6. Let the joint PMF function of X and Y is

$$p(x, y) = \begin{cases} \binom{7}{y} x^y (1 - x)^{7-y} & y = 0, 1, 2, \ldots, 7 \\ 0 & \text{otherwise} \end{cases}.$$

   Find the conditional expectation of Y given $X = x$?

7. Let the joint PDF function of X and Y is

$$f(x, y) = \begin{cases} \frac{3}{4}x & 0 < y < x < 1 \\ 0 & \text{otherwise} \end{cases}.$$

   Find the conditional expectation of Y given $X = 0.5$?

8. Let the joint PDF function of X and Y is

$$f(x, y) = \begin{cases} \frac{6}{7}x & 1 \le x + y \le 2, x \ge 0, y \ge 0 \\ 0 & \text{otherwise} \end{cases}.$$

   Find the marginal PDF of Y and the conditional variance of X given $Y = \frac{1}{2}$?

9. Let the joint PDF of X and Y is

$$f(x, y) = \begin{cases} 4x & 0 < x < \sqrt{y} < 1 \\ 0 & \text{otherwise} \end{cases}.$$

   Find the conditional variance of Y for $X = x$?

10. The joint PDF of (X, Y) is given by:

$$f_{X,Y}(x, y) = \begin{cases} \frac{1}{4}(1 + xy(x^2 - y^2)), & |x| \leq 1,\ |y| \leq 1 \\ 0, & \text{otherwise} \end{cases} .$$

Compute

(a) $E(Y|X),\ E(XY^2 + Y|X),\ Cov(X^2, Y^2)$

(b) $MGF(X, Y),\ MGF(X + Y)$.