MTL 106 (Introduction to Probability Theory and Stochastic Processes)
Tutorial Sheet No. 6 (Limiting Probabilities)

1. (Jensens's Inequality) If g is a convex function and $E(X)$ exists, then show that $g(E(X)) \leq E(g(X))$. Hence show that $E(X) \leq (E(|X|^r))^{1/r}$ for $r \geq 1$.

2. Let $g(X) \geq 0, \forall \ x \in [0, \infty)$ be a non-decreasing even function. Show that for any r.v. X such that $E(g(X))$ exists and $P(|X| \geq \epsilon) \leq \frac{E(g(X))}{g(\epsilon)}$.

3. Consider a sequence of r.v.s $\{X_n\}$ with $E(X_n) = m$ and

$$Cov(X_i, X_j) = \begin{cases} \sigma^2, & i = j \\ a\sigma^2, & i = j \pm 1, \text{ where } |a| < 1, \ \sigma^2 > 0 \text{ are given contants} \\ 0, & \text{otherwise} \end{cases}$$

   Show that WLLN holds for $\{X_n\}$.

4. Consider a sequence of independent r.v.s $\{X_n\}$ such that

$$P(X_n = 0) = 1 - \frac{2}{n^3}, \ P(X_n = \pm n) = \frac{1}{n^3}, \ n > 1$$

   Does the sequence $\{X_n\}$ obey CLT ?

5. Consider a sequence $\{X_n\}$ of identically distributed r.v.s with the property that $nP(|X_i| > n) \to 0$ as $n \to \infty$. Show that $\frac{1}{n} \max_{1 \leq i \leq n} X_i \xrightarrow{p} 0$

6. Suppose $|X_n - X| \leq Y_n$, almost surely for some random variable X, then show that if $E(Y_n) \to 0$, then $E(X_n) \to E(X)$ and $X_n \xrightarrow{p} X$.

7. Show that $X_n \xrightarrow{2} X \ \Rightarrow \ E(X_n) \to E(X), \ E(X_n^2) \to E(X^2)$ as $n \to \infty$.

8. Let $\{X_i\}$ be a sequence of independent r.v.s, such that each X_i has mean 0 and variance 1. Show that

$$\sqrt{n}\frac{X_1 + X_2 + \cdots + X_n}{X_1^2 + X_2^2 + \cdots + X_n^2} \xrightarrow{l} Z \sim N(0,1).$$

9. Does WLLN hold for the following sequences
   (a) $P(X_k = \pm 2^k) = \frac{1}{2}$
   (b) $P(X_k = \pm \frac{1}{k}) = \frac{1}{2}$.

10. For what values of α, does the strong law of large numbers hold for the sequence $\{X_n\}$, where $P(X_k = \pm k^\alpha) = \frac{1}{2}, \ k = 1, 2, \ldots$.

11. Let $\{X_n\}$ be a sequence of independent r.v.s with the following probability distribution. In each case, does the Lindeberg Condition for CLT hold ?
    (a) $P(X_k = \pm \frac{1}{2^n}) = \frac{1}{2}$
    (b) $P(X_k = \pm \frac{1}{2^{n+1}}) = \frac{1}{2^{n+3}}, \ P(X_n = 0) = 1 - \frac{1}{2^{n+2}}$.

12. Suppose that the number of customers who visit SBI, IIT Delhi on a Saturday is a r.v. with $\mu = 75$ and $\sigma = 5$. Find the lower bound for the probability that there will be more than 50 but fewer than 100 customers in the bank?

13. A replacement ball is selected from an urn holding ten identical balls numbered 0 through 9.
    (a) What can be inferred about how zeros appear in n drawings from the law of large numbers?

(b) How many drawings are needed to ensure that the relative frequency of occurrence of 0's will be between 0.09 and 0.11 with a probability of at least 0.95?

14. Does the r.v. X exist for which
$$P\left[\mu - 2\sigma \leq X \leq \mu + 2\sigma\right] = 0.6.$$

15. Use CLT to show that
$$\lim_{n\to\infty} e^{-nt}\sum_{k=0}^{n}\frac{(nt)^k}{k!} = 1 = \begin{cases} 1, & 0 < t < 1 \\ 0.5, & t = 1 \\ 0, & t > 1 \end{cases}.$$

16. Let $X_1, X_2, \ldots, X_n$ be a sequence of r.v. from a uniform distribution on the interval from 0 to 5. What is the limiting moment generating function of $\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}$ as $n \to \infty$?

17. Let $X \sim B(n,p)$. Use the CLT to find n such that
$$P[X > n/2] \geq 1 - \alpha.$$
Calculate the value of n when $\alpha = 0.90$ and $p = 0.45$.

18. Construct a sequence of independent r.v.s $\{X_n\}$ with
$P(X_n = 0) = 1 - \frac{1}{n^\alpha},\ P(X_n = \pm n) = \frac{1}{2n^\alpha}$. Determine the values of α for which the sequence $\{X_n\}$ obeys WLLN.

19. The bulbs manufactured by a plant have life lengths which follow exponential distribution with mean 100 hours. A series of 50 independent life tests is performed. Each test consist of
1. choosing 4 bulbs at random from the plant
2. putting them into operation simultaneously and independently of each other
3. the life test ends as soon as any one of the four bulbs fails and time taken to complete the test is noted

Let $Y_1, Y_2, \ldots, Y_{50}$ respectively be the times of 50 life tests.

(a) What is the PDF of any Y_i

(b) What is the PDF of T, the total time of 50 tests

(c) Use central limit theorem to compute approximately $P(T > 1000)$

20. Suppose that X_i , $i = 1, 2, \ldots, 30$ are independent r.v. each having a Poisson distribution with parameter 0.01. Let $S = X_1 + X_2 + \cdots + X_{30}$.
(a) Using central limit theorem evaluate $P(S \geq 3)$.
(b) Compare the answer in (a) with exact value of this probability.

21. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. r.v. with mean 1 and variance 1600, and assume that these variables are non-negative. Let $Y = \sum_{k=1}^{100} X_k$. Use the central limit theorem to approximate the probability $P(Y \geq 900)$.

22. A machine requires 4 of the 6 independent parts to function. Let each component's lifetime be represented by $X_1, X_2, \ldots, X_6$. Assume they are all distributed exponentially with theta as the parameter. What is the machine lifetime PDF?

23. Let X be a r.v. having a Gamma distribution with parameters n and 1. How large must n be in order to guarantee that
$$P\left(\left|\frac{X}{n} - 1\right| > 0.01\right) < 0.01?$$

24. In order for the probability that the average number of heads obtained deviates from 0.5 by no more than 0.01 is at least 0.90, how many fair coin flips are required?

25. Let X be a nonnegative r.v. Prove:

$$E(X) \leq [E(X^2)]^{\frac{1}{2}} \leq [E(X^3)]^{\frac{1}{3}} \leq \ldots.$$

Verify the above result for $X \sim \ Geo(p)$.

26. Find out the nature of convergence of $\{X_n,\ n = 1, 2, 3, \ldots\}$ for different values of k for $n = 1, 2, \ldots$:

$$P(X_n = n^k) \ = \ \frac{1}{n};\ \ P(X_n = 0) = 1 - \frac{2}{n};\ \ P(X_n = -n^k) = \frac{1}{n}.$$

27. Show that the convergence in the rth mean does not imply almost sure convergence for the sequence $\{X_n,\ n = 1, 2, 3, \ldots\}$ defined below:

$$P(X_n = 0) \ = \ 1 - \frac{1}{n};\ \ P(X_n = -n^{1/2r}) = \frac{1}{n}.$$

28. For each $n \geq 1$, let X_n be an uniformly distributed r.v. over set $\{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}$, i.e.,

$$P\left(X_n = \frac{k}{n}\right) = \frac{1}{n+1}, k = 0, 1, \ldots, n.$$

Let U be a r.v. with uniform distribution in the interval $[0, 1]$. Show that $X_n \xrightarrow{d} U$.

29. Let $\{X_n,\ n = 1, 2, 3, \ldots\}$ be a sequence of i.i.d. r.v. with

$$E(X_1) = Var(X_1) = \lambda \in (0, \infty)$$

and $P(X_1 > 0) = 1$. Show that for $n \to \infty$,

$$\sqrt{n}\frac{Y_n - \lambda}{\sqrt{\lambda}} \xrightarrow{d} N(0, 1)$$

where

$$Y_n = \frac{1}{n}\sum_{k-1}^{n} X_k.$$

30. Let $\{X_n,\ n = 1, 2, 3, \ldots\}$ be a sequence of i.i.d. r.v. with

$$P(X_n = 1) = P(X_n = -1) = \frac{1}{2}.$$

Show that

$$\frac{1}{n}\sum_{j=1}^{n} X_j$$

converges in probability to 0.

31. Let $X, X_1, X_2, \ldots$ be a sequence of r.v. defined on a same probability space (Ω, S, P). Prove that

(a) If $X_n \xrightarrow{a.s.} X$ and $Y_n \xrightarrow{a.s.} Y$, then $X_n + Y_n \xrightarrow{a.s.} X + Y$.

(b) If $X_n \xrightarrow{a.s.} X$ and $Y_n \xrightarrow{a.s.} Y$, then $X_n Y_n \xrightarrow{a.s.} XY$.

(c) If $X_n \xrightarrow{P} X$ iff $X_n - X \xrightarrow{P} 0$.

(d) If $X_n \xrightarrow{P} X$ and $Y_n \xrightarrow{P} Y$, then $P(X = Y) = 1$.

(e) If $X_n \xrightarrow{P} X$ then $X_n - X_m \xrightarrow{P} 0$.

(f) If $X_n \xrightarrow{P} X$ and $Y_n \xrightarrow{P} Y$, then $X_n + Y_n \xrightarrow{P} X + Y$.

(g) If $X_n \xrightarrow{P} X$ and $Y_n \xrightarrow{P} Y$, then $X_n Y_n \xrightarrow{P} XY$.

32. A fair coin is tossed 200 consecutive times. Let X be the number of tails. Use Chebyschev's inequality to find a lower bound for the probability that $\frac{X}{200}$ differs from $\frac{1}{2}$ by less than 0.1.

33. Let $X_1, X_2, \ldots, X_n$ be n independent Poisson distributed r.v. with means $1, 2, \ldots, n$, respectively. Find an x in terms of t such that

$$P\left(\frac{S_n - \frac{n^2}{2}}{n} \leq t\right) \approx \Phi(x)$$

for sufficiently large n, where Φ is the CDF of $N(0,1)$.

34. Let Y be a gamma distributed r.v. with PDF

$$f(y) = \begin{cases} \frac{1}{\Gamma(r)} e^{-y} y^{r-1}, & y > 0 \\ 0, & \text{otherwise} \end{cases}.$$

Find the limiting distribution of $\frac{Y - E(Y)}{\sqrt{Var(Y)}}$ as $r \to \infty$.

35. Suppose that the MGF ($M_x(t)$) of a r.v. X exists. Use Markov's inequality to show that for any $a \in R$ the following holds:

- $P(X \geq a) \leq exp(-ta) M_x(t), \quad \forall \ \ t > 0$.
- $P(X \leq a) \leq exp(-ta) M_x(t), \quad \forall \ \ t < 0$.