MTL 106 (Introduction to Probability Theory and Stochastic Processes)
Tutorial Sheet No. 7 (Introduction to Stochastic Processes)

1. Trace the path of the following stochastic processes:

   (a) $\{W_k, k \in T\}$ where W_k be the waiting time of the k^{th} customer in the system before receiving service and $T = \{1, 2, \ldots\}$.

   (b) $\{X_t, t \in T\}$ where X_t be the number of jobs in system at time t, and $T = [0, \infty)$.

   (c) $\{Y_t, t \in T\}$ where Y_t denotes the cumulative service requirements of all jobs in system at time t, and $T = [0, \infty)$. ANs (a) Discrete time discrete state space.
   (b) Continuous time discrete state space.
   (c) Continuous time continuous state space.

2. Let $\{X_t, 0 \le t \le T\}$ be a stochastic process such that $E[X_t] = 0$ and $E\left[X_t^2\right] = 1$ for all $t \in [0, T]$. Find the upper bound of $|E[X_t X_{t+h}]|$ for any $h > 0$ and $t \in [0, T-h]$.

3. Let $\{X_t, t \ge 0\}$ be a stochastic process with $X_t = A_0 + A_1 t + A_2 t^2$, where $A_i's$ are uncorrelated r.vs. with mean 0 and variance 1. Find the mean function and covariance function of X_t. Ans $E[X_t] = 0$ and $Cov[X_t, X_s] = 1 = st + s^2 t^2$.

4. Prove that $\{X_t, t \in \mathbb{R}\}$ is a second-order process if and only if $m(t)$ is independent of t and $C(s,t)$ depends only on $|t - s|$.

5. Consider the process $\{X_t, t \ge 0\}$ with $X_t = A\cos(wt) + B\sin(wt)$ where w is a positive constant and A and B are uncorrelated r.v. with mean 0 and variance 1. Check whether $\{X_t, t \ge 0\}$ wide-sense stationary or not? Ans Yes

6. In a communication system, the carrier signal at the receiver is modeled by the process $\{Y_t, t \ge 0\}$ which is defined as $Y_t = X_t \cos(2\pi wt + \Theta)$ where $\{X_t, t \ge 0\}$ is a zero-mean and wide-sense stationary process, and Θ is a uniform distributed random variable in the interval $(-\pi, \pi)$ and w is a positive constant. Assume that, Θ is independent of the process $\{X_t, t \ge 0\}$. Is $\{Y_t, t \ge 0\}$ wide-sense stationary? Ans Yes

7. Let Y_1, Y_2 be two i.i.d. r.v's where each follows $Y_i \sim N(\mu, \sigma^2)$. Define $X_t = Y_1\cos(\lambda t) + Y_2 \sin(\lambda t),\ -\infty < t < \infty$ where λ is a real constant. Show that X_t is a second-order process. Also, show that $\{X_t, -\infty < t < \infty\}$ is a Gaussian process.

8. Let Y_1, Y_2 be two i.i.d. r.v's where $P(Y_i = 1) = P(Y_i = -1) = \frac{1}{2}, i = 1, 2$. Define $X_t = Y_1\cos(\lambda t) + Y_2\sin(\lambda t),\ -\infty < t < \infty$ where λ is a real constant. Show that $\{X_t, -\infty < t < \infty\}$ is a second order stationary process which is not strictly stationary.

9. Prove that $\{X_t, -\infty < t < \infty\}$ is a second-order stationary process if for every number τ, the second-order process Y_t defined by $Y_t = X_{t+\tau}, -\infty < t < \infty$ has the same mean and covariance function as of X_t.

10. Consider the random telegraph signal, denoted by $X(t)$, jumps between two states, 0 and 1, according to the following rules. At time $t = 0$, the signal $X(t)$ start with equal probability for the two states, i.e., $P(X(0) = 0) = P(X(0) = 1) = 1/2$, and let the switching times be decided by a Poisson process $\{Y(t), t \ge 0\}$ with parameter λ independent of $X(0)$. At time t, the signal

$$X(t) = \frac{1}{2}\left(1 - (-1)^{X(0)+Y(t)}\right), t > 0.$$

    Is $\{X(t),\ t \ge 0\}$ covariance/wide sense stationary? Ans Yes

11. Prove that every real-valued stochastic process $\{X_t, t \ge 0\}$ with independent increments is a Markov process.

12. Let X and Y be two i.i.d. random variables each having uniform distribution on interval $[-\pi, \pi]$. Define, for $t \ge 0$, $Z_t = \sin(Xt + Y)$. Check whether $\{Z_t, t \ge 0\}$ is covariance stationary or not? Ans Yes

13. Let A be a positive random variable that is independent of a strictly stationary random process $\{X_t, t \geq 0\}$. Show that $\{Y_t, t \geq 0\}$ where $Y_t = AX_t$ is also strictly stationary random process.

14. Is the stochastic process $\{X_t, t \geq 0\}$ stationary, whose probability distribution under a certain condition, for $n = 0, 1, \ldots$, given by

$$P\{X_t = n\} = \begin{cases} \frac{at}{(1+at)} & n = 0 \\ \frac{(at)^{n-1}}{(1+at)^{n+1}} & n = 1, 2, \ldots. \end{cases} \quad AnsYes$$

15. Let $\{N(t), t \geq 0\}$ be a Poisson process with parameter λ. Show that

$$\lim_{t\to\infty} N(t) = \infty \text{ a.s. } .$$

Hint: Find the limit of $P\{N(k) \geq n\}$ as $k \to \infty$. Write $\{\lim_{t\to\infty} N(t) = \infty\}$ in terms of $\{N(k) \geq n\}$.

16. Let $\{N(t), t \geq 0\}$ be a Poisson process with parameter λ. Show that

$$P\{N(t) \text{ is odd}\} = e^{(-\lambda t)} \sinh(\lambda t),$$
$$P\{N(t) \text{ is even}\} = e^{(-\lambda t)} \cosh(\lambda t).$$

Hint: Find the probability that $N(t) = 2n + 1$ and compare this to the n-th term of the Taylor expansion of $\sinh(\lambda t)$.

17. Suppose that $X_1, X_2, \ldots$ are i.i.d. random variables each having $N\left(0, \sigma^2\right)$. Consider, $\{S_n, n = 1, 2, \ldots\}$ is a stochastic process where $S_n = \exp\left(\sum_{i=1}^{n} X_i - \frac{n}{2}\sigma^2\right)$. Find $E\left[S_n\right]$ for all n. ANs 1