MTL 106 (Introduction to Probability Theory and Stochastic Processes)
Tutorial Sheet No. 8 (DTMC)

1. Suppose that a machine can be in two states $\{0, 1\}$ where 0 implies working and 1 implies out of order on a day. The probability that a machine is working on a particular day depends on the state of the machine during two previous days. Specifically assume that $P(X(n+1) = 0|X(n-1) = j, X(n) = k) = q_{jk} \quad j, k = 0, 1$ where $X(n)$ denotes the state of the machine on nth day.

   (a) Show that $\{X(n), n = 1, 2, \ldots\}$ is not a discrete-time Markov chain.

   (b) Define a new state space for the problem by taking the pairs (j, k) where j and k are 0 or 1. It is said that machine is in state (j, k) on day n if the machine is in state j on day $(n - 1)$ and in state k on nth day. Show that in this case the state space of the system is a discrete-time Markov chain.

   (c) It is given that the machine was working on Monday and Tuesday. Find the probability that it will be working on Thursday?

2. The one step transition probability matrix of DTMC $\{X_n, n = 0, 1, \ldots\}$ with state space $\{1, 2, 3\}$ is

$$\mathbf{P} = \begin{pmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{pmatrix}$$

   and the initial distribution is $\pi_0 = (0.7, 0.2, 0.1)$. Find

   (a) $P(X_2 = 3)$

   (b) $P(X_3 = 2, X_2 = 3, X_1 = 3, X_0 = 2)$ Ans a) 0.279 b) 0.0168

3. Consider the DTMC $\{X_n, n = 0, 1, \ldots\}$ with three states $S = \{1, 2, 3\}$ that has the following transition matrix:

$$\mathbf{P} = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}.$$

   Assume that $P(X_0 = 1) = P(X_o = 2) = \frac{1}{4}$ and $P(X_o = 3) = \frac{1}{2}$. Find $P(X_0 = 3, X_1 = 2, X_2 = 1)$ ANs 1/2.

4. For a Markov chain $\{X_n, n = 0, 1, \ldots\}$ with state space $E = \{0, 1, 2, 3, 4\}$ and transition probability matrix P given below, classify the states of the chain. Also determine the closed communicating classes.

   (a) $\mathbf{P} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & 0 & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}.$ (b) $\mathbf{P} = \begin{pmatrix} 0 & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{2}{3} & 0 & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & \frac{1}{3} & 0 & \frac{2}{3} \end{pmatrix}$ a) 0, 1, 2, 4-transient; 3-recurrent

   b) $\{0\}, \{1\}, \{3\}, \{2, 4\}$-communicating class; $\{3\}$-closed communicating class.

5. Consider the chain with transition matrix

$$\mathbf{P} = \begin{pmatrix} 1/6 & 1/3 & 1/2 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 1/6 & 1/3 & 1/2 & 0 \\ 0 & 1/6 & 1/3 & 1/2 \end{pmatrix}.$$

   Identify the closed class and compute transition probability matrix.

6. Assume $\{Y_n, n = 0, 1, \ldots\}$ is a sequence of i.i.d. random variables taking values on Z_+, with probability distribution $P[Y_n = i] = a_i$ $(i = 0, 1, \ldots)$. Let $X_{n+1} = (X_n - 1)^+ + Y_n$. Prove that $\{X_n, n = 0, 1, \ldots\}$ is a Markov chain and write down its transition probability matrix.

7. Let X_0 be an integer-valued random variable, $P(X_0 = 0) = 1$, that is independent of the i.i.d. sequence $Z_1, Z_2, \ldots$, where $P(Z_n = 1) = p$, $P(Z_n = -1) = q$ and $P(Z_n = 0) = 1 - (p + q)$. Let $X_n = max\{0, X_{n-1} + Z_n\}$, $n = 1, 2, \ldots$. Prove that $\{X_n, n = 0, 1, \ldots\}$ is a DTMC. Write the one-step TPM or draw the state transition diagram for this Markov chain.

8. The owner of the local one-chair barbershop is thinking about expanding because there appear to be too many people waiting there. Observations show that in the time required to cut one person's hair, there may be 0, 1, and 2 arrivals, with probabilities of 0.3, 0.4, and 0.3 correspondingly. There are constantly six clients in the salon getting their hair cut. Let $X(t)$ be the total number of customers at any given time t and $X_n = X(t_n^+)$ be the total number of customers at the moment the nth person's hair cut is finished. Using the assumption of i.i.d. arrivals, demonstrate that $\{X_n, n = 1, 2, \ldots\}$ is a Markov chain. Find the probability matrix for its single transition.

9. A particle starts at the origin and moves to and fro on a straight line. At any move it jumps either 1 unit to the right or 1 unit to the left each with probability 1/3 or 1/3 respectively. With probability 1/3, the particle may stay at the same position in any move. Model this process as a stochastic process. Write the stochastic process with state space and parameter space. Verify whether this process satisfies Markov property.

10. Show that if a Markov Chain is irreducible and $P_{ii} > 0$ for some state i then the chain is aperiodic.

11. Consider the Markov chain $\{X_n, n = 0, 1, 2\}$with $S = \{0, 1, 2\}$ and transition probability matrix given as:

$$\mathbf{P} = \begin{pmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}.$$

   (a) Classify the chain as irreducible, aperiodic and find the stationary distribution.

   (b) Consider a DTMC with transition probability matrix $\begin{pmatrix} 0.4 & 0.6 & 0 \\ 0.4 & 0 & 0.6 \\ 0 & 0.4 & 0.6 \end{pmatrix}$. Find the stationary distribution for this Markov chain.

   (c) Assume $X_0 = 1$ and let R be the first time that the chain returns to state 1, i.e., $R = min\{m \geq 1 : X_n = 1\}$. Find $E[R|X_0 = 1]$. Ans a) $\pi = (6/16 \quad 3/16 \quad 7/16)$
   b) $\pi = (4/19 \quad 6/19 \quad 9/19)$

12. Consider the chain with transition matrix

$$\mathbf{P} = \begin{pmatrix} 1/6 & 1/3 & 1/2 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 1/6 & 1/3 & 1/2 & 0 \\ 0 & 1/6 & 1/3 & 1/2 \end{pmatrix}.$$

   Identify the closed class and compute transition probability matrix.

13. Consider a Markov Chain with state space $S = \{0, 1, 2, 3, 4\}$ and transition probability matrix

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0.25 & 0.75 & 0 & 0 \\ 0 & 0.5 & 0.5 & 0 & 0 \\ 0.25 & 0.25 & 0 & 0.25 & 0.25 \\ 0 & 0 & 0 & 0.5 & 0.5 \end{pmatrix}.$$

   (a) Classify the states of the chain.

   (b) Determine the stationary distribution for states 1 and 2.

(c) For the transient states, calculate μ_{ij}, the expected number of visits to transient state j, given that the process started in a transient state i.

(d) Find $P(X_5 = 2 \mid X_3 = 1)$.

14. A mathematics professor has 2 umbrellas. He keeps one of them at home and the other in the office. Every morning, when he leaves home, he checks the weather and takes an umbrella with him if it rains. In case both the umbrellas are in the office, he gets wet. The same procedure is repeated in the afternoon when he leaves the office to go home. The professor lives in a tropical region, therefore, the chance of rain in the afternoon is higher than in the morning; it is 1/5 in the afternoon and 1/20 in the morning. Whether it rains or not is independent of whether it rained the last time he checked. On day 0, there is an umbrella at home, and 1 in the office. Note that, there are two trips each day. What is the expected number of days that will pass before the professor gets wet? What is the probability that the first time he gets wet it is on his way home from the office?

15. Consider a DTMC on non-negative integers where the chain, moves from i, to $i+1$ with probability p, to state 0 with probability $1-p$, and then return to i with probability $0 < p < 1$. Show that this DTMC is irreducible and recurring and that it has a unique stationary distribution, also, find the stationary distribution.

16. A random walker walks on the integers. At each step, he moves one step right with probability $\frac{1}{3}$, two steps left with probability $\frac{1}{3}$, and stays on his place with probability $\frac{1}{3}$. What is the probability that the walker will return to his original location after three steps?

17. Show that the DTMC with countable state space $S = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ and transition probabilities $p_{i,i+1} = p_{i,i-1} = p, i \in S$ is recurrent when $p = \frac{1}{2}$ and transient otherwise.

18. Assume $\{Y_n, n = 0, 1, \ldots\}$ is a sequence of i.i.d. random variables taking values on Z_+, with probability distribution $P[Y_n = i] = a_i$ $(i = 0, 1, \ldots)$. Let $X_{n+1} = (X_n - 1)^+ + Y_n$. Prove that $\{X_n, n = 0, 1, \ldots\}$ is a Markov chain and write down its transition probability matrix.

19. Assume that $P = (p_{ij})$ is a finite irreducible double stochastic matrix, i.e., $\sum_{k=1}^{m} p_{kj} = \sum_{k=1}^{m} p_{ik} = 1$ for each $i = 1, 2, \ldots, m$. Prove that the stationary distribution for this Markov chain is the uniform distribution on $\{1, 2, \ldots, m\}$.

20. Let X_0 be an integer-valued random variable, $P(X_0 = 0) = 1$, that is independent of the i.i.d. sequence $Z_1, Z_2, \ldots$, where $P(Z_n = 1) = p$, $P(Z_n = -1) = q$ and $P(Z_n = 0) = 1 - (p + q)$. Let $X_n = max\{0, X_{n-1} + Z_n\}$, $n = 1, 2, \ldots$. Prove that $\{X_n, n = 0, 1, \ldots\}$ is a DTMC. Write the one-step TPM or draw the state transition diagram for this Markov chain.

21. Show that the birth and death chain is recurrent if and only if

$$\sum_{k=0}^{\infty} \gamma_k = \infty$$

where, $\gamma_0 = 1$ and

$$\gamma_k = \frac{q_1 \cdots q_k}{p_1 \cdots p_k}, \qquad k = 1, 2, \ldots$$
$$q_i = p_{i,i-1}, \qquad i = 1, 2, \ldots$$
$$p_i = p_{i,i+1}, \qquad i = 1, 2, \ldots.$$

22. Show that an irreducible birth and death chain on $S = \{0, 1, \ldots\}$ is recurrent if and only if

$$\sum_{k=0}^{\infty} \frac{q_1 \cdots q_k}{p_1 \cdots p_k} = \infty$$

where

$$q_i = p_{i,i-1} \text{ and } p_i = p_{i,i+1}, \quad i = 1, 2, \ldots$$

23. Consider the birth and death chain on $\{0, 1, \ldots\}$ with

$$p_k = \frac{k+2}{2(k+1)} \text{ and } q_k = \frac{k}{2(k+1)}, k \geq 0$$

determine whether the chain is recurrent or transient.

24. Prove that $\mu_{ij} = \frac{F_{ij}}{1-F_{ij}} \quad , i, j \in S$.

25. Prove that if i is a recurrent state and suppose $p_{ij}^{(n)} > 0$ for some n, then state j is recurrent and $F_{ij} = F_{ji} = 1$.

26. Let

$$f_{jk}^{(n)} = P[X_n = k, X_m \neq j, 0 < m < n | X_0 = j]$$

be the probability that the chain starting from state j, the chain leads to state k at the nth step without hitting j in the middle. Prove that, if $p_{jk}^{(n)} > 0$, then there exists $m \leq n$ such that $f_{jk}^{(n)} > 0$.

27. Let $\{X_n, n = 0, 1, \ldots\}$ be a DTMC. Prove that the distribution of X_n is independent of n if and only if the initial distribution π is a stationary distribution.

28. The owner of the local one-chair barbershop is thinking about expanding because there appear to be too many people waiting there. Observations show that in the time required to cut one person's hair, there may be 0, 1, and 2 arrivals, with probabilities of 0.3, 0.4, and 0.3 correspondingly. There are constantly six clients in the salon getting their hair cut. Let $X(t)$ be the total number of customers at any given time t and $X_n = X(t_n^+)$ be the total number of customers at the moment the nth person's hair cut is finished. Using the assumption of i.i.d. arrivals, demonstrate that $\{X_n, n = 1, 2, \ldots\}$ is a Markov chain. Find the probability matrix for its single transition.

29. Consider a DTMC model arising in an insurance problem. To compute insurance or pension premiums for professional diseases such as silicosis, it is needed to compute the average degree of disability at pre-assigned time periods. Suppose that, m degrees of disability $S_1, S_2, \ldots, S_m$ are retained. Assume that an insurance policy holder can go from degree S_i to degree S_j with a probability p_{ij}. This strong assumption leads to the construction of the DTMC model in which $\mathbf{P} = [p_{ij}]$ is the one step transition probability matrix related to the degree of disability. Using real observations recorded in India, the following transition matrix $\mathbf{P}$ is considered :

$$\mathbf{P} = \begin{pmatrix} 0.90 & 0.10 & 0 & 0 & 0 \\ 0 & 0.95 & 0.05 & 0 & 0 \\ 0 & 0 & 0.90 & 0.05 & 0.05 \\ 0 & 0 & 0 & 0.90 & 0.10 \\ 0 & 0 & 0.05 & 0.05 & 0.90 \end{pmatrix}$$

(a) Classify the states of the chain as transient, positive recurrent or null recurrent along with period.

(b) Find the limiting distribution for the degree of disability.

30. A particle starts at the origin and moves to and fro on a straight line. At any move it jumps either 1 unit to the right or 1 unit to the left each with probability 1/3 or 1/3 respectively. With probability 1/3, the particle may stay at the same position in any move. Model this process as a stochastic process. Write the stochastic process with state space and parameter space. Verify whether this process satisfies Markov property.

31. Consider the simple random walk on a circle. Assume that K odd number of points labeled $0, 1, \ldots, K-1$ are arranged on a circle clockwise. From i, the walker moves to $i+1$ (with K identified with 0) with probability p $(0 < p < 1)$ and to $i-1$ (with -1 identified with $K-1$) with probability $1-p$. Find the steady state distribution for this random walk, if it exist.

32. For $j = 0, 1, \ldots$, let $p_{jj+2} = v_j$ and $p_{j0} = 1 - v_j$, define the transition probability matrix of Markov chain. Discuss the character of the states of this chain.

33. Let, $\{X_n, n = 0, 1, \ldots\}$ be a time-homogeneous DTMC with state space $S = \{0, 1, 2, 3, 4\}$ and one-step transition probability matrix

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

   (a) Classify the states of the chain as transient, +ve recurrent or null recurrent.

   (b) When $P(X_0 = 2) = 1$, find the expected number of times the Markov chain visits state 1 before being absorbed.

   (c) When $P(X_0 = 1) = 1$, find the probability that the Markov chain gets absorbed in state 0 .

34. A rumor-spreading paradigm is one method of information dissemination over a network. Assume that there are 5 hosts connected to the network. One host starts off by sending a message. Every round, the message is sent by one host to another host who is selected independently and uniformly at random from the other 4 hosts. When all hosts have received the message, the process ends. Create a discrete-time model of this process with Markov chains

   (a) X_n be state of host $(i = 1, 2, \ldots, 5)$ who received the message at the end of nth round.

   (b) Y_n be number of hosts having the message at the end of nth round.

   Find one step transition probability matrix for the above DTMC. Classify the states of the chains as transient, positive recurrent or null recurrent.

35. Assume that $P = (p_{ij})$ is a finite irreducible double stochastic matrix, i.e., $\sum_{k=1}^{m} p_{kj} = \sum_{k=1}^{m} p_{ik} = 1$ for each $i = 1, 2, \ldots, m$. Prove that the stationary distribution for this Markov chain is the uniform distribution on $\{1, 2, \ldots, m\}$.

36. Let $\{X_n, n = 0, 1, \ldots\}$ be a DTMC. Prove that the distribution of X_n is independent of n if and only if the initial distribution π is a stationary distribution.

37. Consider the simple random walk on a circle. Assume that K odd number of points labeled $0, 1, \ldots, K-1$ are arranged on a circle clockwise. From i, the walker moves to $i + 1$ (with K identified with 0) with probability p $(0 < p < 1)$ and to $i - 1$ (with -1 identified with $K - 1$) with probability $1 - p$. Find the steady state distribution for this random walk, if it exist.

38. Let, $\{X_n, n = 0, 1, \ldots\}$ be a time-homogeneous DTMC with state space $S = \{0, 1, 2, 3, 4\}$ and one-step transition probability matrix

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

   (a) Classify the states of the chain as transient, +ve recurrent or null recurrent.

   (b) When $P(X_0 = 2) = 1$, find the expected number of times the Markov chain visits state 1 before being absorbed.

   (c) When $P(X_0 = 1) = 1$, find the probability that the Markov chain gets absorbed in state 0 .

39. A rumor-spreading paradigm is one method of information dissemination over a network. Assume that there are 5 hosts connected to the network. One host starts off by sending a message. Every round, the message is sent by one host to another host who is selected independently and uniformly at random from the other 4 hosts. When all hosts have received the message, the process ends. Create a discrete-time model of this process with Markov chains

   (a) X_n be state of host $(i = 1, 2, \ldots, 5)$ who received the message at the end of nth round.

   (b) Y_n be number of hosts having the message at the end of nth round.