MTL 106 (Introduction to Probability Theory and Stochastic Processes)
Tutorial Sheet No. 9 (CTMC)

1. Consider a time-homogeneous CTMC $\{X(t), t \geq 0\}$ with the infinitesimal generator matrix $\mathbf{Q} = \begin{pmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 4 & -5 \end{pmatrix}$ and initial distribution $(0, 0, 1)$. Find $P(\tau > t)$ where τ denotes the first transition time of the Markov chain.

2. In orbit, there are two communication satellites. Each satellite has an exponential lifetime with a mean of $\frac{1}{\mu}$. A replacement is sent up if one fails. The preparation and sending up of a replacement takes an exponential amount of time, with mean $\frac{1}{\lambda}$. Let X_t represent the number of satellites in orbit at the given moment. Consider $\{X_t, t \geq 0\}$ as a CTMC with a state space $\{0, 1, 2\}$. Show that the infinitesimal generator matrix is given by
$$\mathbf{Q} = \begin{pmatrix} -\lambda & \lambda & 0 \\ \mu & -(\lambda+\mu) & \lambda \\ 0 & 2\mu & -2\mu \end{pmatrix}.$$
Write down the Kolmogorov forward and backward equations for the above process.

3. Consider a CTMC $\{X(t), t \geq 0\}$ with state space $\{0, 1, 2\}$ and
$$\mathbf{Q} = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix}.$$
   (a) Prove that its transition probability matrix is
$$\mathbf{P(t)} = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + \frac{2}{3}R(t)$$
   where
$$\mathbf{R(t)} = \begin{pmatrix} \cos(\frac{\sqrt{3}}{2}) & \cos(\frac{\sqrt{3}}{2} - \frac{2\pi}{3}) & \cos(\frac{\sqrt{3}}{2} + \frac{2\pi}{3}) \\ \cos(\frac{\sqrt{3}}{2} + \frac{2\pi}{3}) & \cos(\frac{\sqrt{3}}{2}) & \cos(\frac{\sqrt{3}}{2} - \frac{2\pi}{3}) \\ \cos(\frac{\sqrt{3}}{2} - \frac{2\pi}{3}) & \cos(\frac{\sqrt{3}}{2} + \frac{2\pi}{3}) & \cos(\frac{\sqrt{3}}{2}) \end{pmatrix}.$$
   (b) Show that the limiting distribution of $P(t)$ as $t \to \infty$ is $\pi = (1/3, 1/3, 1/3)$ and
$$\frac{2}{3}e^{-3t/2} \leq \sup_i \sum_j \mid p_{ij}(t) - \pi_j \mid \leq \frac{4}{3}e^{-3t/2}.$$

4. Let
$$\mathbf{Q} = \begin{pmatrix} -3 & 1 & 2 \\ 2 & -4 & 2 \\ 2 & 1 & -3 \end{pmatrix}.$$
Prove that its transition probability matrix is
$$\mathbf{P(t)} = \frac{1}{5}\begin{pmatrix} 2+3e^{-5t} & 1-e^{-5t} & 2-2e^{-5t} \\ 2-2e^{-5t} & 1+4e^{-5t} & 2-2e^{-5t} \\ 2-2e^{-5t} & 1-e^{-5t} & 2+3e^{-5t} \end{pmatrix}.$$
hence, the limiting distribution of $P(t)$ is $\pi = (2/5, 1/5, 2/5)$ and
$$\sup_i \sum_j \mid p_{ij}(t) - \pi_j \mid = \frac{8}{5}e^{-5t}.$$

5. The birth-death process is called a pure death process if λ_i=0 for all i. Suppose $\mu_i = i\mu$, $i = 1, 2, 3, \ldots$ and initially $X_0 = n$. Show that X_t has $B(n, p)$ distribution with $p = e^{-\mu t}$.

6. Consider birth-death process with λ_i=0 for all i. Suppose $\mu_i = \mu$, $i = 1, 2, 3, \ldots$. Find the value of $P_{ij}(t)$ for this process.

7. The birth-death process is called a birth process if μ_n=0 for every n. Suppose $\lambda_n = n\lambda$, $n = 1, 2, 3, \ldots$. If $X_0 = i$ show that

$$P(X_t = n|X_0 = i) = P_{in}(t) = \begin{pmatrix} i-1 \\ i-n \end{pmatrix} e^{-n\lambda t}(1 - e^{-\lambda t})^{i-n}, \quad i \geq n.$$

8. Consider a birth-death process with birth rate $\lambda_i = (i + 1)\lambda$, $i \geq 0$ and $\mu_i = i\mu$, $i \geq 0$. Find the expected time to go from state 0 to 1.

9. Prove that the irreducible birth-death process is transient iff

$$\sum_{n=1}^{\infty} \frac{\mu_1 \ldots \mu_n}{\lambda_1 \ldots \lambda_n} < \infty, \tag{1}$$

positive recurrent iff

$$\sum_{n=1}^{\infty} \frac{\lambda_1 \ldots \lambda_n}{\mu_1 \ldots \mu_n} < \infty, \tag{2}$$

and null recurrent iff neither (1) nor (2) holds.

10. Consider a birth-death process with state space $S = \{0, 1, 2\}$ and birth rates $\lambda_0 = \mu_2 = \lambda$. Find $P(X(t) = n|X(0) = 0), \quad n = 0, 1, 2$.

11. Consider a birth-death process $\{X(t), t \geq 0\}$ with state space $S = \{0, 1, \ldots\}$ and birth and death rates of the form $\lambda_n = n\lambda$ and $\mu_n = n\mu, \quad n = 0, 1, \ldots$ where λ and μ are non- negative constants. Assume $P_{ii}(j) = 1$ for fixed i. Take $m(t) = \mathbb{E}(X(t)) = \sum_{k=0}^{\infty} kP_{ik}(t)$. Prove that $m(t) = ie^{(\lambda-\mu)t}$.

12. Suppose there are n identical machines operating independently and serviced by a single repair crew. If a machine breaks down while another is being repaired it must wait its turn before repairs can start. Assume each machine has an operating time exponential with mean $\frac{1}{\mu}$ and a repair time exponential with mean $\frac{1}{\lambda}$. Let $X(t)$ = no. of machines in operating condition at time t. Model $X(t)$ as a Markov chain with state space $E = \{0, 1, 2, \ldots, N\}$. Determine rate matrix Λ and the forward Kolmogorov equations. Determine the limiting equilibrium probability distribution of the process.

13. Every Wednesday night, free vision screenings are provided by the eye clinic at City Hospital. Three ophthalmologists are on call. The real-time is found to be spread with potential distribution around the average test time of 20 minutes. Patients are accepted on a first-come, first-served basis and clients arrive according to a Poisson process with a mean of 6/hr. The hospital planners are interested in knowing:

    (a) What is the mean number of people waiting?

    (b) What is the total amount of time a patient spends at the clinic?

    (c) What is the average percentage idle time of each of the doctors?