

## RESEARCH ARTICLE

# Variable annuities valuation under a mixed fractional Brownian motion environment with jumps considering mortality risk

Nitu Sharma<sup>1</sup>  | Dharmaraja Selvamuthu<sup>1</sup>  | Selvaraju Natarajan<sup>2</sup> 

<sup>1</sup>Indian Institute of Technology Delhi, Delhi, India

<sup>2</sup>Indian Institute of Technology Guwahati, Guwahati, India

## Correspondence

Dharmaraja Selvamuthu, Indian Institute of Technology Delhi, Delhi, India.  
Email: [dharmar@maths.iitd.ac.in](mailto:dharmar@maths.iitd.ac.in)

## Funding information

Department of Science and Technology, Ministry of Science and Technology, India

## Abstract

Pricing a variable annuity (VA) is traditionally based on models with standard Brownian motion (BM). The assumption of independent increments in the BM is not always realistic as the price process of the risky assets exhibit a long-range dependency. To model such assets, a fractional Brownian motion (FBM) with Hurst parameter  $\left(\frac{1}{2}, 1\right)$  is an appropriate model. However, due to arbitrage in an FBM model and jumps in stock returns, we consider in this work a mixed FBM (MFBM) model with jumps and evaluate variable annuities with different riders. We analyze the proposed model numerically to see the impact of mortality risk. We perform a comparison between eight stochastic models to obtain the best-suited mortality model, for a dataset of the US male population. Finally, we obtain the price of VA guarantees using the forecasted values from the fitted mortality model.

## KEYWORDS

fractional Brownian motion, guarantees, mortality modeling, variable annuities

## 1 | INTRODUCTION

Variable annuities (VA) are insurance products whose benefit base rely on the market indices. They consist of features of both insurance and investment. In a VA, a policyholder usually pays a single premium at the beginning. Then, this premium is invested in one or several mutual funds chosen by the policyholder from a variety of different mutual funds. VAs are very popular insurance products in the USA and Japan.<sup>1,2</sup> They are now spreading across Europe. According to the Insured Retirement Institute (IRI) and the Life Insurance and Market Research Association (LIMRA) Secure Retirement Institute, VA sales for 2019 in the USA were more than \$101.9 billion marking a 2% increase compared to 2018.<sup>3</sup> VA products come with either a death benefit rider or a living benefit rider or both. A rider in a VA product provides insurance in terms of downside movement of underlying fund value with a survival or death benefit. VA products are usually embedded with two types of riders: a guaranteed minimum death benefit rider (GMDB) and a guaranteed minimum living benefit rider (GMLB). As the name suggests, a GMDB rider provides a guaranteed minimum amount irrespective of the underlying fund performance at the instant of death of the insured to the insured's nominee. The GMLB riders can be further classified into four riders: guaranteed minimum accumulation benefit (GMAB), guaranteed minimum income benefit (GMIB), guaranteed minimum withdrawal benefit (GMWB), and guaranteed lifelong withdrawal benefit (GLWB) riders. The GMIB and GMAB riders offer a guaranteed minimum amount, irrespective of the account value at the maturity of the contract. With GMIB, this guarantee is applicable only if the insured annuitize the account value, at the time of maturity. GMWB and GLWB riders provide a regular income to the insured irrespective of the fund value till a fixed

maturity time in case of GMWB rider and for a lifetime in case of GLWB rider. GMDB, GMAB, and GMIB riders are the oldest VA riders available in the market. In this work, we consider the pricing problem of VA with GMDB, GMAB, and GMIB riders only.

Valuation of a VA involves consideration of mainly three risks: investment risk, mortality risk, and interest rate risk. There is vast literature involving modeling investment risk in a VA contract using geometric Brownian motion (GBM) model and its variants. Some of the popular variations of the GBM model involve consideration of jumps (denoted by jump-diffusion Brownian motion (JDBM) models), stochastic volatility, stochastic interest rate, and correlated jump-diffusion processes. Some of the recent literature in VA pricing using variants of GBM model includes articles by Ballotta et al.,<sup>3</sup> Gerber et al.,<sup>4</sup> Ko and Bae,<sup>5</sup> Escobar et al.,<sup>6</sup> Yu et al.,<sup>7</sup> Sharma et al.,<sup>8</sup> and Sharma et al..<sup>9</sup> Gerber et al.<sup>4</sup> valued the GMDB rider using a JDBM model. Ko and Bae<sup>5</sup> priced a GMDB rider with a roll-up and fixed-guarantee features under Black-Scholes framework. They assumed a phase-type distribution for the time to death. Escobar et al.<sup>6</sup> priced GMDB and GMAB riders considering the lapse behavior of insured in pricing. They modeled investment risk in a Black-Scholes framework with stochastic interest rate risk. Yu et al.<sup>7</sup> valued GMDB products using the COS method under a general Lévy framework. Ballotta et al.<sup>3</sup> proposed pricing of VAs with GMDB and GMAB riders with modeling investment risk using Lévy processes. In their work, they modeled three risks: financial risk, mortality risk, and surrender risk. They used the Gompertz–Makeham model for mortality modeling by assuming independence between investment and demographic risk. Sharma et al.<sup>8</sup> priced a GLWB rider using a Generalized Autoregressive Conditional Heteroskedasticity (GARCH) model. The correlated jump-diffusion model is used for modeling investment risk in an equity-indexed annuities (EIAs) contract by Sharma et al..<sup>9</sup> EIAs are insurance contracts similar to VAs. Under EIA, the rate of return is linked to the returns of the underlying index. The earnings under EIA is lower than that in VA as EIAs have lesser risk compared to VAs.

The GBM model and its variants assume returns to be normally distributed with independent increments. Many empirical studies show that the stock returns are not symmetric and disobey the normal distribution assumption.<sup>10</sup> Additionally, returns are not independent of past returns and have a long-range dependency feature.<sup>11,12</sup> To incorporate these features of returns in risky asset modeling, one of the models in the literature is a fractional Brownian motion (FBM) (Refer to Biagini et al.<sup>13</sup> for the theory of FBM). Peter<sup>14</sup> pointed out that the long memory, thick tail, self-similarity and other characteristics of the FBM model can describe the changes in the price of the underlying asset in the financial market. Karatzas and Shreve<sup>15</sup> show that the financial asset returns have asymmetry, peak fat tails, unforeseen jumps, and many other characteristics which the traditional Brownian motion (BM) model cannot model. FBM models have a long-range dependency property, fat tail, and asymmetry, and their increments are non-independent. Therefore, FBM models can describe some distributional characteristics of asset returns. However, pricing with FBM results in arbitrage opportunities. Androschuk and Mishura<sup>16</sup> showed that using a linear combination of independent Brownian motion and fractional Brownian motion called mixed fractional Brownian motion (MFBM) results in an arbitrage-free model. Therefore, we also consider modeling with MFBM. The positive and negative dependency in a FBM is given by the Hurst index  $H$ . The increments of fractional process are positively correlated and long-range dependent for  $H > \frac{1}{2}$  and negatively correlated for  $H < \frac{1}{2}$ . For  $H = \frac{1}{2}$ , the FBM reduces to a Wiener process. The stock price returns are positively long-range dependent, therefore, we assume  $H > \frac{1}{2}$ .<sup>17</sup> Another feature present in stock returns is the presence of infrequent jumps. Therefore, to capture sudden and infrequent jumps, and long-range dependency in stock returns, we have used a combination of MFBM and Poisson jumps denoted by jump-diffusion mixed fractional Brownian motion (JDMFBM) model.

Due to various benefits of FBM over BM, this model has been applied in finance for pricing of currency options, lookback options, European call options and so forth.<sup>18–22</sup> Wang et al.<sup>19</sup> priced discrete-time options with transaction cost by the MFBM model. Sun<sup>18</sup> priced European currency options using the MFBM model. Ouyang et al.<sup>23</sup> studied the pricing problem of vulnerable European options using the MFBM model. Valuation of options using the MFBM model with jumps is proposed by Murwaningtyas et al.<sup>24</sup> Wang et al.<sup>22</sup> considered the pricing problem of one asset and three assets European call option and Greeks under the FBM model.

To the best of our knowledge, there is not much literature on VA valuation with the FBM model. Some of the recent literature in insurance pricing using FBM models include the works by Zimbidis,<sup>25</sup> Xu et al.,<sup>26</sup> and Wang et al..<sup>27</sup> Zimbidis<sup>25</sup> proposed pricing of defined benefit pension products modeling financial risk by a multidimensional FBM model, whereas Xu et al.<sup>26</sup> proposed pricing for EIA using the FBM framework. Wang et al.<sup>27</sup> priced equity-linked securities (ELS) option by FBM model. Concerning VA pricing, there exists no research work which involves financial risk modeling via a JDMFBM model.

Apart from investment risk, another risk affecting the variable annuity business is mortality risk. There are many works in the literature involving consideration of mortality risk. Milevsky and Posner<sup>28</sup> proposed a valuation framework for GMDB riders using the Gompertz–Makeham law of mortality. The Gompertz–Makeham model assumes that mortality increases at a constant rate. The benefit of the Gompertz–Makeham model is its simplicity in modeling mortality which further makes it a less reliable model in terms of fitting over the ages.<sup>29</sup> Mortality models play a crucial role in modeling longevity risk. Their performance depends on the different mortality data patterns in distinct countries. Therefore, parametric models could be considered better than generalized models for mortality. Most of the research in the literature uses historical mortality data, which provides an incorrect estimation of future mortality. Due to improvements in medicines and medical facilities and various other reasons, there has been a decrease in mortality and an increase in life expectancy over the past few decades. An increase in life expectancy implies that an individual will live longer on average. This increase in life expectancy creates difficulties in pricing insurance products, mainly pension and retirement plans where living benefits are provided. Without considering this trend, the life insurance companies will generally be underfunded, resulting in losses to them (refer to Mortality Risk Modeling Fall 2016 Actuarial Science Undergraduate Research).

This work considers the valuation of a variable annuity contract modeling two main risks: financial risk and mortality risk. To consider a simplified model and obtain an analytical solution, we have not considered interest rate risk modeling. Further, the maturity time of the GMDB, GMAB, and GMIB riders is considered to be less than ten years. Therefore, we assume that there will not be too much fluctuation in the risk-free interest rate. The financial risk is modeled using a JDMFBM model. Furthermore, the mortality risk is modeled using eight different parametric mortality models. These eight models are compared to obtain the best-suited mortality model for the considered mortality data. The aim of this article is two-fold. First, we study the impact of long-range dependency in investment modeling. Second, we demonstrate the impact of considering the stochastic mortality model compared to using the actual mortality probabilities. Therefore, we have considered a simplified contract with a constant interest rate and no additional benefits. Additionally, the VA contracts considered for pricing are short-duration contracts, usually of a duration less than ten years. Hence, interest rate risk may not impact the pricing remarkably. We performed the pricing of a VA contract with GMDB, GMAB, and GMIB riders. We obtained a closed-form expression for pricing as a function of survival and death probabilities for the three riders. Then, we performed a numerical analysis for pricing using the stochastic mortality model. Following Cairns et al.<sup>30</sup> and Carfora et al.,<sup>31</sup> we have compared eight different models for mortality. For analysis purposes, we considered the US male data of yearly death probabilities from the year 1933 to 2017 obtained from the Human Mortality Database. The mortality considered for pricing of VA riders is the forecasted probabilities obtained from the best-fitted model to the considered mortality datasets. Then, we used the forecasted probabilities from the best-fitted model to obtain the expected present value (E(PV)) of GMDB, GMAB, and GMIB riders.

The rest of the article is organized as follows. Section 2 considers the modeling of investment risk and mortality risk and pricing of the riders. In this section, the pricing equation for the three riders as a function of the fee charged by insurer and survival probabilities are derived. Section 3 consists of an empirical analysis of the proposed model considering US male mortality data. Section 4 gives concluding remarks with the possible future works.

## 2 | MATHEMATICAL MODEL

In this section, we introduce the stochastic equity risk model and the mortality models. Later, we obtain the pricing equation for the three riders: GMDB, GMAB, and GMIB.

### 2.1 | Investment modeling

Consider a finite time horizon  $T > 0$ . Let  $(\Omega, \mathcal{F}, P)$  be the probability space and  $E$  denotes the expectation operator under the measure  $P$ . Consider that the market consists of only two traded assets, a risky asset with price process  $\{S_t, t \geq 0\}$  and a risk-free asset with price process  $\{R_t, t \geq 0\}$ . The dynamics of the index price  $S_t$  at time  $t$  under the real-world probability measure  $P$  is given by the stochastic differential equation (SDE)

$$dS_t = S_t ((\mu - m\lambda) dt + \sigma (dB_t^H + dB_t) + dJ_t), \quad (1)$$

with  $S_0 \geq 0$  denoting the price of risky asset at time point 0, and the dynamics of the risk-less security  $R_t$  is given by

$$dR_t = rR_t dt, \quad R_0 = 1, \quad (2)$$

where the parameters  $r, \sigma, \mu, m, T$ , and  $\lambda$  are constants. Here,  $r$  is the risk-free rate,  $\sigma$  is the volatility of the risky asset,  $\mu$  is the drift rate,  $\{B_t^H, t \geq 0\}$  is a fractional Brownian motion (FBM) process,  $\{B_t, t \geq 0\}$  is a Brownian motion (BM) process, and  $J_t$  is the jump process. Further, the jump process is a compound Poisson process given by  $J_t = \sum_{k=0}^{N_t} (Y_k - 1)$ , where  $\{N_t, t \geq 0\}$  is a Poisson process with parameter  $\lambda$  and  $Y_k$  for  $k = 1, 2, 3, \dots$  are independent and identically distributed random variables. Assume that  $Y_k$  follows log-normal distribution with parameters  $\alpha$  and  $\gamma^2$ , that is,  $\log(Y_k) \sim N(\alpha, \gamma^2)$ . And,  $m$  is expectation of  $Y_k - 1$  given as

$$m = E[Y_k - 1] = \exp\left(\alpha + \frac{\gamma^2}{2}\right) - 1.$$

Moreover, assume that the three processes  $\{B_t^H, t \geq 0\}$ ,  $\{B_t, t \geq 0\}$ , and  $\{N_t, t \geq 0\}$  are independent. Also, assume that the jump size distribution  $Y_k$  are independent of  $\{N_t, t \geq 0\}$  for all  $k$ .

Since no arbitrage opportunities exist under a JDMFBM model, therefore, there exists a unique risk-neutral measure  $Q$  under which the price dynamics follows

$$dS_t = S_t((r - m\lambda)dt + \sigma(d\hat{B}_t^H + d\hat{B}_t) + dJ_t), \quad (3)$$

where  $\hat{B}_t^H + \hat{B}_t = B_t^H + B_t - \frac{\mu-r}{\sigma}t$  which is also a MFBM under the measure  $Q$ . Using Itô's lemma,<sup>13</sup> the solution of Equation (3) is

$$S_t = S_0 \exp\left((r - m\lambda)t - \frac{1}{2}\sigma^2(t + t^{2H}) + \sigma(\hat{B}_t^H + \hat{B}_t) + \sum_{k=0}^{N_t} \log(Y_k)\right). \quad (4)$$

## 2.2 | Mortality models

Consider the distribution of lifetime of an individual aged  $x$  in the year  $t_1$ . The age  $x$  is insured's age last birthday and hence an integer value. Let  $\tau_x$  be the random variable denoting the residual or remaining lifetime of the person aged  $x$ . The cumulative distribution function of  $\tau_x$  is given by

$$F_{\tau_x}(t) = P(\tau_x \leq t) = {}_tq_x, \quad \text{for } 0 \leq t \leq \omega - x, \quad (5)$$

where  ${}_tq_x$  denotes the probability of the remaining lifetime being atmost  $t$  for an individual aged  $x$  and  $\omega$  is the maximum lifetime of the individual. Let  $K_x = \lfloor \tau_x \rfloor$ , the integer part of the random variable  $\tau_x$ . Then, the probability mass function of the random variable  $K_x$  is given by

$$\begin{aligned} P(K_x = k) &= P(k \leq \tau_x < k + 1) \\ &= F_{\tau_x}(k + 1) - F_{\tau_x}(k) \\ &= {}_{k+1}q_x - {}_kq_x \\ &= {}_kp_x - {}_{k+1}q_{x+k}, \quad \text{for } k = 0, 1, 2, \dots, \lfloor \omega \rfloor - x, \end{aligned} \quad (6)$$

where  ${}_kp_x$  denotes the probability of a life aged  $x$  surviving till the age  $x + k$ . Note that we will write the one year death and survival probabilities  ${}_1q_{x+k}$  and  ${}_1p_{x+k}$  simply as  $q_{x+k}$  and  $p_{x+k}$  respectively. The modeling of these probabilities is discussed as follows.

Traditionally, deterministic models have been used by actuaries for modeling the age dynamics of human mortality. The deterministic models include historical mortality tables or classical mortality models such as Gompertz and Makeham. Stochastic models have been adopted for the last two decades to describe the uncertainty linked to mortality. The majority of the work in mortality modeling has been concentrated on short-rate models for mortality. They model either

the spot mortality rates,  $q(t, x)$ , or the spot rate of mortality,  $\mu(t, x)$ . In this work, we will be modeling the spot mortality rates,  $q(t, x)$ . Here,  $q(t, x)$  denotes the probability of a person aged  $x$  in year  $t$  dying in next one year, that is, between year  $t$  to  $t + 1$ . For example, if a person is aged 25 years in the year 2020, then  $q(2020, 25)$  denotes the probability of dying in the next year. Similarly, this person will be aged 26 years in 2021; therefore,  $q(2021, 26)$  will also denote the probability of the person aged 25 in the year 2020 dying in the year 2021–2022, provided he/she survives the year 2020–2021. Hence, the probability of a person aged 25 in the year 2020, dying in year 2021–2022 is  $(1 - q(2020, 25))q(2021, 26)$ .

Consider  $x_1, x_2, \dots, x_k$  to be the age range and  $t_1, t_2, \dots, t_n$  to be the time range (in years). Let  $c$  be the cohort index. Then the cohort range, that is, range of  $c$  will be from  $t_1 - x_k$  to  $t_n - x_1$ . All the mortality models are functions of these three parameters, that is, age, time, and cohort. Assume that the force of mortality  $\mu(t, x)$  is constant from age  $x$  to  $x + 1$  and year  $t$  to  $t + 1$ . Assume that the number of deaths over a year follows a binomial distribution. Therefore, we consider a *logit* link function. Note that the data available is of death rate over a year. Also, the *logit* of  $q(t, x)$  will be replaced with *log* of  $\mu(t, x)$  if deaths are assumed to follow a Poisson distribution.

One of the most effective approaches to the stochastic modeling of mortality rates was proposed by Lee and Carter.<sup>32</sup> They gave a regression model for *logit* of  $q(t, x)$  as a function of parameters dependent on age and time. The model is given as follows:

$$\text{logit}(q(t, x)) = \log \left( \frac{q(t, x)}{1 - q(t, x)} \right) = a_x + b_x k_t + \epsilon_{t,x}, \quad (7)$$

where  $a_x$  and  $b_x$  are age-specific coefficients,  $k_t$  is a time-varying parameter and  $\epsilon_{t,x}$  is the error term. The product  $b_x k_t$  denotes the age-period effect on mortality. However, there is an issue with the parameterization in the above model, as it is not unique. For instance, if we re-parametrize it by dividing  $b_x$  by a non zero constant  $\alpha$  and multiply  $k_t$  with  $\alpha$ , then the new set of parameters will also give the same values of  $q(t, x)$  for any non-zero  $\alpha$ . Therefore, to guarantee the identifiability of the model, the following conditions are imposed:

$$\sum_x b_x = 1, \quad (8)$$

$$\sum_t k_t = 0. \quad (9)$$

The basic Lee–Carter model has been improved by considering different effects such as incorporating cohort effect, age-period effect and so forth. Renshaw and Haberman<sup>33</sup> extended the Lee–Carter model by including a cohort effect, represented by the product of a parameter dependent on the year of birth  $t - x$  ( $\gamma_{t-x}$ ) with an age-dependent parameter ( $b_x^0$ ). The model is given as:

$$\text{logit}(q(t, x)) = a_x + b_x k_t + b_x^0 \gamma_{t-x} + \epsilon_{t,x} \quad (10)$$

with constraints

$$\sum_x b_x = \sum_x b_x^0 = 1, \quad (11)$$

$$\sum_t k_t = \sum_c \gamma_c = 0. \quad (12)$$

In the same work,<sup>33</sup> they found issues in capturing the cohort effects with the model in Equation (10) and hence proposed a simpler substructure of the above model by substituting  $b_x^0 = 1$ . The new model is denoted as RH model. The RH model assumes independence between the period and cohort parameters. The equation for the RH model is given as

$$\text{logit}(q(t, x)) = a_x + b_x k_t + \gamma_{t-x} + \epsilon_{t,x}. \quad (13)$$

The constraints for the identifiability of the parameters are given by

$$\sum_x b_x = 1, \quad (14)$$

$$\sum_t k_t = \sum_c \gamma_c = 0. \quad (15)$$

Note that for consistency we have used a logit link function whereas Renshaw and Haberman<sup>33</sup> have used a log link function. Another popular substructure of RH model is the Age-period cohort (APC) model<sup>30</sup> obtained by substituting  $b_x = 1$  in the RH model given in Equation (13). The model equation is

$$\text{logit}(q(t, x)) = a_x + k_t + \gamma_{t-x} + \epsilon_{t,x}. \quad (16)$$

For the identifiability of the parameters, the following constraints were imposed

$$\sum_t k_t = \sum_c \gamma_c = \sum_c c\gamma_c = 0. \quad (17)$$

Another famous improvement of Lee–Carter model is given by Cairns et al.<sup>34</sup> They fitted the following model (CBD model in Table 1) to mortality rates

$$\text{logit}(q(t, x)) = k_t^{(1)} + k_t^{(2)}(x - \bar{x}) + \epsilon_{t,x}, \quad (18)$$

where  $\bar{x}$  is the mean age in the sample range. The term  $k_t^{(2)}(x - \bar{x})$  in the above model captures the age-period effect as in Lee–Carter model. The only difference is that, it is a pre-specified function of age here, whereas, in the Lee–Carter model, there are non-parametric terms  $b_x$  which does not have any prior structure. The above model does not have any identifiability issue. Hence, it does not require any constraints. Cairns et al.<sup>30</sup> considered three extensions to the original CBD model. They incorporated combinations of a cohort effect term and a quadratic age term. The so formed three models, model M6, M7, and M8 are given in Table 1. The model M6 (Table 1)

$$\text{logit}(q(t, x)) = k_t^{(1)} + k_t^{(2)}(x - \bar{x}) + \gamma_{t-x} + \epsilon_{t,x}, \quad (19)$$

is an extension of CBD (2006)<sup>34</sup> model by including a cohort parameter with following constraints

$$\sum_c \gamma_c = 0, \quad \sum_c c\gamma_c = 0. \quad (20)$$

Model M7 (Table 1) is an extension of model M6 by including a quadratic age parameter. The model equation is given by

$$\text{logit}(q(t, x)) = k_t^{(1)} + k_t^{(2)}(x - \bar{x}) + k_t^{(3)}((x - \bar{x})^2 - \hat{\sigma}^2) + \gamma_{t-x} + \epsilon_{t,x}, \quad (21)$$

with the following constraints

$$\sum_c \gamma_c = 0, \quad \sum_c c\gamma_c = 0, \quad \sum_c c^2\gamma_c = 0, \quad (22)$$

where  $\hat{\sigma}^2$  is the variance of the ages considered. The third extended model (Model M8 given in Table 1) is obtained by including an age-cohort parameter with a pre-specified function  $\beta - x$  of age parameter. The model equation is given as

$$\text{logit}(q(t, x)) = k_t^{(1)} + k_t^{(2)}(x - \bar{x}) + (\beta - x)\gamma_{t-x} + \epsilon_{t,x}, \quad (23)$$

where  $\beta$  is a constant to be estimated. There is only one constraint for the above model given as  $\sum_c \gamma_c = 0$ . Plat<sup>35</sup> proposed a combination of CBD model with some attributes of the Lee–Carter model. The model equation is given as

$$\text{logit}(q(t, x)) = a_x + k_t^{(1)} + k_t^{(2)}(x - \bar{x}) + \gamma_{t-x} + \epsilon_{t,x}. \quad (24)$$

The constraints for the model are given by

$$\sum_t k_t^{(1)} = 0, \quad \sum_t k_t^{(2)} = 0, \quad \sum_c \gamma_c = 0. \quad (25)$$

Table 1 has the models considered above along with the corresponding constraints.

TABLE 1 Mortality models

| Model                   | Equation                                                                                                                                   | Constraint                                                          |
|-------------------------|--------------------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------|
| Lee–Carter (LC)         | $\text{logit}(q(t, x)) = a_x + b_x k_t + \epsilon_{t,x}$                                                                                   | $\sum_x b_x = 1, \sum_t k_t = 0$                                    |
| Renshaw–Haberman (RH)   | $\text{logit}(q(t, x)) = a_x + b_x k_t + \gamma_{t-x} + \epsilon_{t,x}$                                                                    | $\sum_x b_x = 1, \sum_t k_t = 0, \sum_c \gamma_c = 0$               |
| Age-period-cohort (APC) | $\text{logit}(q(t, x)) = a_x + k_t + \gamma_{t-x} + \epsilon_{t,x}$                                                                        | $\sum_t k_t = 0, \sum_c \gamma_c = 0, \sum_c c\gamma_c = 0$         |
| Cairns–Blake–Dowd (CBD) | $\text{logit}(q(t, x)) = k_t^{(1)} + k_t^{(2)}(x - \bar{x}) + \epsilon_{t,x}$                                                              | No constraint                                                       |
| M6                      | $\text{logit}(q(t, x)) = k_t^{(1)} + k_t^{(2)}(x - \bar{x}) + \gamma_{t-x} + \epsilon_{t,x}$                                               | $\sum_c \gamma_c = 0, \sum_c c\gamma_c = 0$                         |
| M7                      | $\text{logit}(q(t, x)) = k_t^{(1)} + k_t^{(2)}(x - \bar{x}) + k_t^{(3)}((x - \bar{x})^2 - \hat{\sigma}^2) + \gamma_{t-x} + \epsilon_{t,x}$ | $\sum_c \gamma_c = 0, \sum_c c\gamma_c = 0, \sum_c c^2\gamma_c = 0$ |
| M8                      | $\text{logit}(q(t, x)) = k_t^{(1)} + k_t^{(2)}(x - \bar{x}) + (\beta - x)\gamma_{t-x} + \epsilon_{t,x}$                                    | $\sum_c \gamma_c = 0$                                               |
| Plat                    | $\text{logit}(q(t, x)) = a_x + k_t^{(1)} + k_t^{(2)}(x - \bar{x}) + \gamma_{t-x} + \epsilon_{t,x}$                                         | $\sum_t k_t^{(1)} = 0, \sum_t k_t^{(2)} = 0, \sum_c \gamma_c = 0$   |

Assume that  $t_1$  is the year in which an insured aged  $x$  enters into the VA contract and that all probabilities considered henceforth are based on this assumption. Therefore, we can say that  $q_x = q(t_1, x)$  and  $p_x = 1 - q(t_1, x)$ . The death and survival probabilities at the age  $x + k$  using the above mentioned notations are calculated as

$$q_{x+k} = q(t_1 + k, x + k), \quad (26)$$

$$p_{x+k} = 1 - q_{x+k}, \quad (27)$$

$${}_k p_x = \prod_{j=0}^{k-1} p_{x+j}, \quad (28)$$

for  $k = 0, 1, 2, \dots, [\omega] - x$ .

## 2.3 | Pricing model

In this section, we will consider pricing a VA contract with the GMDB, GMAB and, GMIB riders. A GMDB rider provides a guaranteed benefit in event of death of the policyholder before maturity time ( $T$ ). There is no guaranteed benefit if the insured survives till  $T$ . Whereas, in the case of GMAB and GMIB rider, there is no guaranteed death benefit if the insured dies before time  $T$ . The guarantee is applicable only if the insured survives the maturity. The reader can refer to Bacinello et al.<sup>36</sup> and Bauer et al.<sup>37</sup> for the basic pricing equation of the three riders.

Assume that the fee for the riders is charged continuously as a  $\delta$  percentage of the fund value. Also, assume that the guaranteed benefits are paid at the end of the year of death. First we model the underlying fund of these riders. Let  $T$  be the maturity time of the contract, and  $W_t$  be the underlying fund value of the VA contract at time  $t$ . Then, the evolution of  $W_t$  in  $0 \leq t \leq T$  is given by the following SDE

$$dW_t = W_t((r - \delta - m\lambda)dt + \sigma(d\hat{B}_t^H + d\hat{B}_t) + dJ_t), \quad (29)$$

whose solution (with  $W_0 = w_0$ ) is

$$W_t = w_0 \exp \left( (r - \delta - m\lambda)t - \frac{1}{2}\sigma^2(t + t^{2H}) + \sigma(d\hat{B}_t^H + d\hat{B}_t) + \sum_{k=0}^{N_t} \log(Y_k) \right). \quad (30)$$

Conditioning on  $N_t = n$ ,  $\log \left( \frac{W_t}{w_0} \right)$  is a normally distributed random variable with conditional mean and variance given as

$$\mu(t, n) = E^Q \left( \log \frac{W_t}{w_0} \middle| N_t = n \right) = (r - \delta - m\lambda)t - \frac{1}{2}\sigma^2(t + t^{2H}) + n\alpha, \quad (31)$$

$$\sigma^2(t, n) = \text{Var}^Q \left( \log \frac{W_t}{w_0} \middle| N_t = n \right) = \sigma^2(t + t^{2H}) + n\gamma^2, \quad (32)$$

where  $E^Q$  and  $\text{Var}^Q$  denotes mean and variance under the risk-neutral measure  $Q$ . Then, for  $A \subset \mathcal{R}$  (set of real numbers)

$$P \left( \log \left( \frac{W_t}{w_0} \right) \in A \right) = \sum_{n=0}^{\infty} P \left( \log \left( \frac{W_t}{w_0} \right) \in A \middle| N_t = n \right) P(N_t = n). \quad (33)$$

Hence, the related probability density function is

$$f_{W_t}(y) = \exp(-\lambda t) \sum_{n=0}^{\infty} \frac{(\lambda t)^n \exp \left\{ -\frac{(y - \mu(t, n))^2}{2\sigma^2(t, n)} \right\}}{n! \sqrt{2\pi\sigma^2(t, n)}}, \quad y \geq 0. \quad (34)$$

The fee for a VA rider is obtained by equating the E(PV) of inflows to the E(PV) of outflows. The inflows to the insured are the living benefits (LB) and death benefits (DB). And outflows usually consists of a lump sum premium paid in the beginning ( $w_0$ ). The pricing of GMDB, GMAB, and GMIB riders is considered in details as follows.

### GMDB pricing

A guaranteed minimum death benefit rider provides insured a minimum guaranteed death benefit if he dies before the maturity. The insured is given the fund value if he survives till maturity. The insured is charged a fee ( $\delta$ ) as a percentage of fund value continuously. If  $\tau$  is the time point of death, then the value of death benefit for a GMDB rider is given by<sup>37</sup>

$$X_\tau = \max\{W_\tau, GDB(\tau)\}, \text{ for } \tau < T, \quad (35)$$

where  $GDB(\tau)$  is the guaranteed amount applicable at time point  $\tau$ , which can be a constant or a function of time.  $W_\tau$  is the fund value at time point  $\tau$ , which is the value of the investment with a continuous deduction of guarantee fee. Hence, the expected present value (E(PV)) of the death benefit is given by

$$\begin{aligned} D_0 &= E(E^Q(\exp(-r\tau) \max\{W_\tau, GDB(\tau)\} | \tau < T)) \\ &= \sum_{k=0}^{T-1} E^Q(\exp(-r(k+1)) \max\{W_{k+1}, GDB(k+1)\}) P(K_x = k). \end{aligned} \quad (36)$$

Using distribution of  $\log(W_t/w_0)$ , the value of death benefit is given as

$$D_0 = \sum_{k=0}^{T-1} \exp(-r(k+1)) E^Q(\max\{W_{k+1}, GDB(k+1)\})_k p_x q_{x+k}, \quad (37)$$

where

$$\begin{aligned} E^Q(\max\{W_{k+1}, GDB(k+1)\}) &= \sum_{n=0}^{\infty} \left[ GDB(k+1) \phi \left( \frac{\log(GDB(k+1)) - \log(w_0) - \mu(k+1, n)}{\sqrt{\sigma^2(t, n)}} \right) \right. \\ &\quad \left. + w_0 \left( 1 - \phi \left( \frac{\log(GDB(k+1)) - \log(w_0) - (\mu(k+1, n) + \sigma^2(k+1, n))}{\sqrt{\sigma^2(k+1, n)}} \right) \right) \right] \\ &\quad \times \exp \left( \mu(k+1, n) + \frac{1}{2} \sigma^2(k+1, n) \right) P(N_t = n). \end{aligned} \quad (38)$$

$$\times \exp \left( \mu(k+1, n) + \frac{1}{2} \sigma^2(k+1, n) \right) P(N_t = n). \quad (39)$$

The E(PV) of the living benefit is

$$\begin{aligned} L_0 &= E^Q(\exp(-rT) W_T)_T p_x \\ &= \sum_{n=0}^{\infty} w_0 \exp \left( -rT + \mu(T, n) + \frac{1}{2} \sigma^2(T, n) \right) p_x P(N_t = n). \end{aligned} \quad (40)$$

Therefore, the E(PV) of the GMDB guarantee ( $X_0$ ) will be

$$X_0 = D_0 + L_0. \quad (41)$$

### GMAB pricing

Under this rider, the insured's family gets the fund value if he dies before maturity and insured gets maximum of a guaranteed maturity amount and fund value if he survives till maturity. This guarantee is usually the premium amount accumulated with a predefined interest rate. Conditional on  $N_t = n$ , the E(PV) of the death benefit for the GMAB rider is given by<sup>37</sup>

$$\begin{aligned} D_0 &= E(E^Q(\exp(-r\tau)W_\tau | \tau < T)) \\ &= \sum_{k=0}^{T-1} \exp(-r(k+1))E^Q(W_{k+1})P(K_x = k), \end{aligned} \quad (42)$$

and living benefit is

$$\begin{aligned} L_0 &= E^Q(\exp(-rT) \max\{W_T, GAB\} | \tau > T) \\ &= E^Q(\exp(-rT) \max\{W_T, GAB\})_T p_x. \end{aligned} \quad (43)$$

Using the distribution of  $\log\left(\frac{W_t}{w_0}\right)$ , the E(PV) of the death benefit is

$$D_0 = \sum_{k=0}^{T-1} \sum_{n=0}^{\infty} w_0 \exp\left(-r(k+1) + \mu(k+1, n) + \frac{1}{2}\sigma^2(k+1, n)\right) P(N_t = n) {}_k p_x q_{x+k}, \quad (44)$$

and the living benefit is

$$L_0 = \sum_{n=0}^{\infty} \exp(-rT) E^Q(\max\{W_T, GAB\})_T p_x P(N_T = n).$$

Hence, the E(PV) of the GMAB ( $X_0$ ) guarantee will be

$$X_0 = D_0 + L_0. \quad (45)$$

### GMIB pricing

This rider is similar to GMAB rider, but under this, the guarantee is applicable only if the insured annuitize the Guaranteed amount. Conditional on  $N_t = n$ , the E(PV) of the death benefit for the rider is given by<sup>37</sup>

$$\begin{aligned} D_0 &= E(E^Q(\exp(-r\tau)W_\tau | \tau < T)) \\ &= \sum_{k=0}^{T-1} \exp(-r(k+1))E^Q(W_{k+1}) {}_k p_x q_{x+k}, \end{aligned} \quad (46)$$

and the living benefit is

$$\begin{aligned} L_0 &= E^Q(\exp(-rT) \max\{W_T, GMB g a(T)\} | \tau > T) \\ &= E^Q(\exp(-rT) \max\{W_T, GMB g a(T)\})_T p_x. \end{aligned} \quad (47)$$

Using the distribution of  $\log\left(\frac{W_t}{w_0}\right)$ , the E(PV) of the death benefit is

$$D_0 = \sum_{k=0}^{T-1} \sum_{n=0}^{\infty} w_0 \exp\left(-r(k+1) + \mu(k+1, n) + \frac{1}{2}\sigma^2(k+1, n)\right) P(N_t = n) {}_k p_x q_{x+k}, \quad (48)$$

TABLE 2 Mortality model fitting

|                      | LC       | APC      | RH       | CBD      | M6       | M7       | IM8      | PLAT     |
|----------------------|----------|----------|----------|----------|----------|----------|----------|----------|
| Number of parameters | 197      | 276      | 332      | 174      | 308      | 394      | 309      | 361      |
| AIC                  | 168701.9 | 122747.2 | 85296.29 | 300155.7 | 104405.9 | 92994.97 | 104397.4 | 85400.82 |
| BIC                  | 169980.2 | 124538.1 | 87450.57 | 301284.8 | 106404.4 | 95551.56 | 106402.5 | 87743.27 |
| LL                   | −84154   | −61097.6 | −42316.1 | −149904  | −51894.9 | −46103.5 | −51889.7 | −42339.4 |

and the living benefit is

$$L_0 = \sum_{n=0}^{\infty} \exp(-rT) E(\max\{W_T, GMB g a(T)\})_T p_x P(N_T = n).$$

Therefore, the E(PV) of the GMIB guarantee ( $X_0$ ) will be

$$X_0 = D_0 + L_0. \quad (49)$$

### 3 | EMPIRICAL ANALYSIS

For analysis purposes, we consider a mortality data of the US male population from the year 1933 to 2017 (obtained from the Human Mortality Database). Let the policyholder's age at inception, that is, in year 2017, be 55 years. In this section, we first consider modeling the mortality dataset with the models mentioned in Section 3.1. After comparing the eight mortality models, we identify the best-fitted model and forecast future mortality probabilities for life aged 55 years with that model. Then, we value the GMDB, GMAB, and GMIB riders using the predicted probabilities from the best-fitted model. For the numerical experiment, we compare E(PV) of the guarantees for the GBM (see Appendix A), JDBM (see Appendix B) and JDMFBM model.

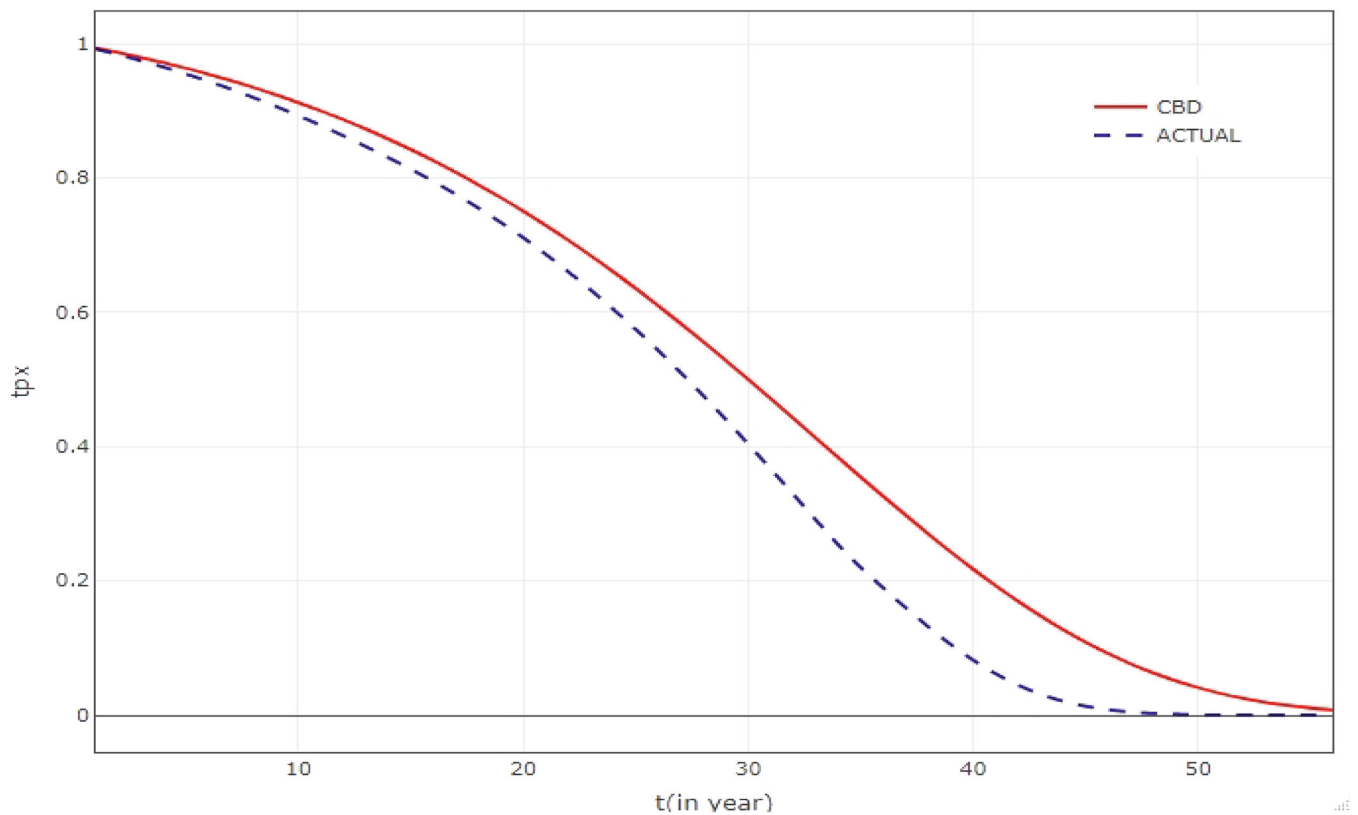
#### 3.1 | Mortality modeling

Mortality improvement patterns differ from country to country. Therefore, models that provide a good fit for a particular country need not be equally suitable for another country. VA products are considered mainly as pension products, therefore, we considered data for the higher age group, that is, for the people aged 55–110 of the US male population. The data consists of the yearly death probabilities,  $q(t, x)$ , for the years 1933–2017. In this section, we compare various mortality models and determine which are best-suited to forecast mortality at higher ages.

To find the best-fitted model to the dataset, we apply various standard criteria such as Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC) also known as Schwartz Information Criterion (SIC), and log-likelihood (LL). AIC and BIC are ways of penalizing high number of parameters. Their values are based on LL values and number of parameters. The difference between AIC and BIC is that, the later penalizes free parameters more strongly compared to the former. The best-fitted model is chosen by minimizing AIC, BIC, and by maximizing the log-likelihood function values. The AIC, BIC, and log-likelihood values obtained from fitting the mortality dataset by the different models mentioned in Table 1 are shown in Table 2. (We have used the R Software<sup>38</sup> for computation of these values.)

From Table 2, we observe that the AIC, BIC values are minimum corresponding to the CBD model and the LL value corresponding to the CBD model is maximum. Therefore, CBD is the best-fitted model to the considered dataset. Note that, the three criteria need not give the same result for all mortality data. Therefore, one can choose between LL, AIC, and BIC depending upon whether the model requires to minimize the number of parameters or not and to what extent the parameters are required to be penalized.

Figure 1 displays the graph of actual and forecasted survival probabilities for a male aged 55 in the year 2017. The actual probabilities are the historical table values for the year 2017. And the forecasted probabilities are the probabilities forecasted by the CBD model for the next 56 years of the individual aged 55 in the year 2017. Figure 1 shows that the predicted probabilities are comparatively higher than the actual probabilities. The corresponding life expectancy using



**FIGURE 1** Survival probabilities after age 55 generated by the CBD stochastic mortality model and by the US mortality tables.

**TABLE 3** Parameters used in analysis

| Model  | $r$  | $\sigma$ | $H$             | $\lambda$ | $\alpha$ | $\gamma$ |
|--------|------|----------|-----------------|-----------|----------|----------|
| GBM    | 0.03 | 0.1      |                 |           |          |          |
| JDBM   | 0.03 | 0.1      |                 | 1.25      | 0.005    | 0.01     |
| JDMFBM | 0.03 | 0.1      | {0.6, 0.7, 0.8} | 1.25      | 0.005    | 0.01     |

actual and forecasted probabilities is 25.33 and 28.68, respectively. This evidence supports the claim of a gradual increase in life expectancy over the years.

### 3.2 | Guarantee valuation

Consider the parameter values for the GBM, JDBM, and JDMFBM model as given in Table 3. The guarantees are 100%, 80%, and 80% respectively for GMDB, GMAB, GMIB contract. Further, in the case of GMIB, a guaranteed income of 6% of the guarantee base is provided after maturity. Let the value of  $\omega$  be 110. The fee charged by the insurance company is the break-even point at which  $E(PV)$  equals the initial premium or lump sum premium paid by the insured. Corresponding to different fee ( $\delta$ ) values and taking parameters given in Table 3, we compute  $E(PV)$ s for GMDB, GMAB, and GMIB contracts under different models using Monte Carlo simulation method. The  $E(PV)$  for GMDB, GMAB, and GMIB contracts, given in Table 4 are obtained using forecasted probabilities from the CBD model. From Table 4, it is observed that an increase in the value of fee ( $\delta$ ) decreases  $E(PV)$  values. Similar observations are made by Bacinello et al.<sup>36</sup> and Bauer et al.,<sup>37</sup> showing that  $E(PV)$ s are a decreasing function of the fee. Table 5 shows the range for break-even fee for the three contracts GMDB, GMAB, and GMIB, under the models GBM, JDBM, and JDMFBM. The fee for a GMDB contract lies between 10 and 15 b.p. when a JDMFBM model is applied. Whereas, using GBM and JDBM models, it becomes less than 10 b.p. Similar results are obtained in case of GMAB and GMIB contracts. From Table 5, it is observed that the BM and JDBM models under

TABLE 4 E(PV) using CBD forecasted probabilities

| T=5   | $\delta$ (in basis points) |          |          |          |          |          |             |          |          |          |          |          |          |          |          |          |
|-------|----------------------------|----------|----------|----------|----------|----------|-------------|----------|----------|----------|----------|----------|----------|----------|----------|----------|
|       | 1                          | 3        | 5        | 10       | 15       | 20       | 25          | 35       | 45       | 55       | 65       | 75       | 85       | 95       | 100      |          |
| H=0.6 | GMDB                       | 100.1671 | 100.0697 | 99.97235 | 99.72944 | 99.48715 | 99.24548    | 99.00442 | 98.52414 | 98.04631 | 97.5709  | 97.09792 | 96.62734 | 96.15916 | 95.69336 | 95.46135 |
|       | GMAB                       | 101.7701 | 101.6819 | 101.5938 | 101.3742 | 101.1555 | 100.9376    | 100.7206 | 100.2892 | 99.86129 | 99.43681 | 99.01579 | 98.59822 | 98.18411 | 97.77345 | 97.56941 |
|       | GMIB                       | 101.6408 | 101.552  | 101.4634 | 101.2425 | 101.0223 | 100.8031    | 100.5847 | 100.1505 | 99.71964 | 99.29226 | 98.86829 | 98.44775 | 98.03063 | 97.61692 | 97.41135 |
| H=0.7 | GMDB                       | 100.1935 | 100.0961 | 99.99876 | 99.75584 | 99.51353 | 99.27185    | 99.03077 | 98.55046 | 98.07259 | 97.59715 | 97.12412 | 96.65349 | 96.18525 | 95.7194  | 95.48736 |
|       | GMAB                       | 102.4366 | 102.35   | 102.2634 | 102.0477 | 101.8328 | 101.6187    | 101.4056 | 100.9818 | 100.5614 | 100.1444 | 99.73079 | 99.3206  | 98.91381 | 98.51041 | 98.30998 |
|       | GMIB                       | 102.2832 | 102.196  | 102.1089 | 101.8918 | 101.6755 | 101.4601    | 101.2455 | 100.8189 | 100.3956 | 99.97577 | 99.55927 | 99.14615 | 98.7364  | 98.33001 | 98.12807 |
| H=0.8 | GMDB                       | 100.2254 | 100.1279 | 100.0306 | 99.78764 | 99.54532 | 99.30361    | 99.06251 | 98.58216 | 98.10424 | 97.62874 | 97.15565 | 96.68497 | 96.21666 | 95.75074 | 95.51866 |
|       | GMAB                       | 103.3473 | 103.2623 | 103.1774 | 102.9656 | 102.7548 | 102.5447    | 102.3355 | 101.9196 | 101.5071 | 101.0978 | 100.6919 | 100.2893 | 99.89003 | 99.49403 | 99.29726 |
|       | GMIB                       | 103.1662 | 103.0807 | 102.9952 | 102.7821 | 102.5699 | 102.3585    | 102.1479 | 101.7293 | 101.3139 | 100.9019 | 100.4932 | 100.0877 | 99.68558 | 99.2867  | 99.08849 |
| JDBM  | GMDB                       | 100.0455 | 99.94807 | 99.85075 | 99.60787 | 99.36563 | 99.1240061  | 98.88301 | 98.40287 | 97.9252  | 97.44999 | 96.97723 | 96.5069  | 96.03899 | 95.5735  | 95.34165 |
|       | GMAB                       | 100.172  | 100.0773 | 99.98278 | 99.74701 | 99.51206 | 99.27791893 | 99.0446  | 98.58043 | 98.11956 | 97.66201 | 97.20779 | 96.75692 | 96.30942 | 95.8653  | 95.64452 |
|       | GMIB                       | 100.1295 | 100.0345 | 99.93958 | 99.70285 | 99.46692 | 99.23178568 | 98.99746 | 98.53121 | 98.0682  | 97.60843 | 97.15192 | 96.69869 | 96.24876 | 95.80213 | 95.58007 |
| BM    | GMDB                       | 100.2109 | 100.1134 | 100.0159 | 99.77272 | 99.53016 | 99.28821962 | 99.04691 | 98.56614 | 98.08786 | 97.61203 | 97.13866 | 96.66772 | 96.19921 | 95.73311 | 95.50097 |
|       | GMAB                       | 100.3036 | 100.2087 | 100.1139 | 99.87751 | 99.64193 | 99.40715532 | 99.1732  | 98.70776 | 98.24562 | 97.78679 | 97.33129 | 96.87914 | 96.43036 | 95.98496 | 95.76354 |
|       | GMIB                       | 100.2622 | 100.1669 | 100.0718 | 99.83442 | 99.59786 | 99.36210982 | 99.12716 | 98.65967 | 98.1954  | 97.73438 | 97.27661 | 96.82212 | 96.37092 | 95.92303 | 95.70033 |

TABLE 4 (Continued)

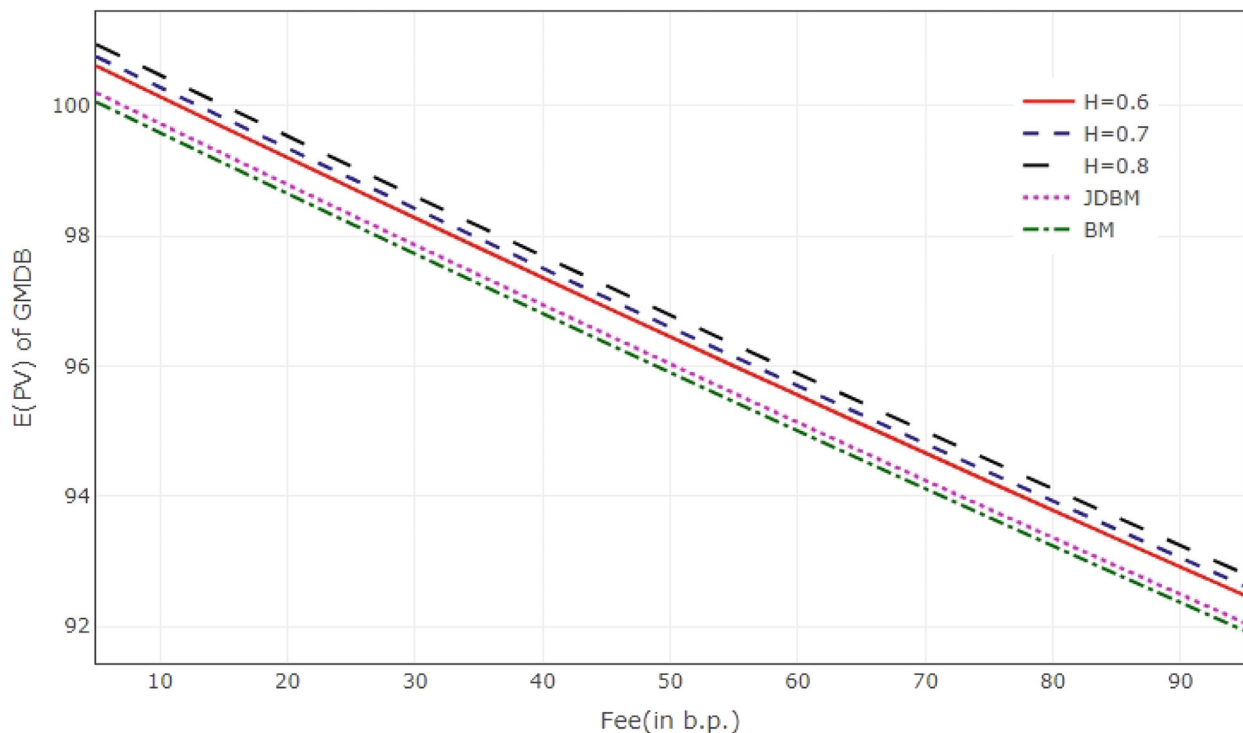
| T=10  |      | $\delta$ (in basis points) |          |          |          |          |             |          |          |          |          |          |          |          |          |          |
|-------|------|----------------------------|----------|----------|----------|----------|-------------|----------|----------|----------|----------|----------|----------|----------|----------|----------|
|       |      | 1                          | 3        | 5        | 10       | 15       | 20          | 25       | 35       | 45       | 55       | 65       | 75       | 85       | 95       | 100      |
| H=0.6 | GMDB | 100.9829                   | 100.7933 | 100.6041 | 100.1329 | 99.66398 | 99.19749    | 98.73339 | 97.81229 | 96.90057 | 95.99816 | 95.10495 | 94.22086 | 93.3458  | 92.47969 | 92.04995 |
|       | GMAB | 102.9132                   | 102.7387 | 102.5646 | 102.1315 | 101.7012 | 101.2739    | 100.8495 | 100.0093 | 99.18069 | 98.36347 | 97.55761 | 96.76305 | 95.97973 | 95.20756 | 94.82564 |
|       | GMIB | 101.6852                   | 101.5037 | 101.3227 | 100.872  | 100.4242 | 99.9791     | 99.53676 | 98.66031 | 97.79477 | 96.94009 | 96.09619 | 95.26301 | 94.44049 | 93.62857 | 93.22657 |
| H=0.7 | GMDB | 101.1274                   | 100.9379 | 100.7487 | 100.2775 | 99.80868 | 99.34225    | 98.8782  | 97.95717 | 97.04549 | 96.14307 | 95.24982 | 94.36565 | 93.49048 | 92.6242  | 92.19437 |
|       | GMAB | 104.3558                   | 104.1847 | 104.0142 | 103.5897 | 103.1681 | 102.7494    | 102.3334 | 101.5101 | 100.6979 | 99.89678 | 99.10672 | 98.32762 | 97.55938 | 96.80192 | 96.42721 |
|       | GMIB | 102.7365                   | 102.5585 | 102.381  | 101.9389 | 101.4997 | 101.0632    | 100.6294 | 99.77    | 98.92141 | 98.08355 | 97.25636 | 96.43975 | 95.63365 | 94.83798 | 94.44403 |
| H=0.8 | GMDB | 101.3153                   | 101.1258 | 100.9366 | 100.4654 | 99.99657 | 99.53011    | 99.06602 | 98.14487 | 97.23305 | 96.33045 | 95.43698 | 94.55255 | 93.67708 | 92.81048 | 92.38047 |
|       | GMAB | 106.522                    | 106.3542 | 106.1868 | 105.7702 | 105.3564 | 104.9454    | 104.537  | 103.7285 | 102.9307 | 102.1436 | 101.3671 | 100.6011 | 99.84546 | 99.10015 | 98.73133 |
|       | GMIB | 104.4297                   | 104.2552 | 104.0812 | 103.648  | 103.2175 | 102.7896    | 102.3645 | 101.5221 | 100.6903 | 99.86903 | 99.05812 | 98.25751 | 97.46713 | 96.68687 | 96.30052 |
| JDBM  | GMDB | 100.5734                   | 100.3835 | 100.1939 | 99.72173 | 99.25201 | 98.78471712 | 98.31985 | 97.39735 | 96.48441 | 95.58094 | 94.68685 | 93.80206 | 92.92647 | 92.06    | 91.63015 |
|       | GMAB | 100.5954                   | 100.4092 | 100.2233 | 99.7608  | 99.30113 | 98.84431495 | 98.39036 | 97.49102 | 96.60311 | 95.72664 | 94.8616  | 94.008   | 93.16585 | 92.33516 | 91.92411 |
|       | GMIB | 100.281                    | 100.0907 | 99.90071 | 99.42762 | 98.95712 | 98.48920642 | 98.02387 | 97.10089 | 96.18818 | 95.28568 | 94.39337 | 93.51123 | 92.63923 | 91.77735 | 91.3502  |
| BM    | GMDB | 100.4306                   | 100.2411 | 100.0519 | 99.58079 | 99.1121  | 98.64584094 | 98.18201 | 97.26157 | 96.35069 | 95.44927 | 94.55724 | 93.6745  | 92.80096 | 91.93653 | 91.50771 |
|       | GMAB | 100.3556                   | 100.1693 | 99.98351 | 99.521   | 99.06135 | 98.60455763 | 98.15061 | 97.25128 | 96.36336 | 95.48684 | 94.62174 | 93.76806 | 92.92582 | 92.095   | 91.68389 |
|       | GMIB | 100.0509                   | 99.86064 | 99.67077 | 99.1979  | 98.7276  | 98.25988509 | 97.79474 | 96.87214 | 95.95976 | 95.05758 | 94.16557 | 93.2837  | 92.41194 | 91.55028 | 91.12323 |

**TABLE 5** Break-even fee range (in basis points) for contracts with maturity 10 years,  $H = 0.7$ ,  $r = 3\%$ ,  $\sigma = 10\%$ ,  $\lambda = 1.25$ ,  $\mu_f = 0.005$ ,  $\sigma_f = 0.01$  and using CBD forecasted probabilities.

|      | JDMFBM | JDBM | BM   |
|------|--------|------|------|
| GMDB | 10–15  | 5–10 | 5–10 |
| GMAB | 45–55  | 5–10 | 3–5  |
| GMIB | 25–35  | 3–5  | 1–3  |

**TABLE 6** Difference between E(PV)s obtained by subtracting E(PV)s obtained using actual probabilities from E(PV)s obtained using CBD forecasted probabilities.

| Fee | GMDB     |          |          | GMAB     |          |          | GMIB     |          |          |
|-----|----------|----------|----------|----------|----------|----------|----------|----------|----------|
|     | H = 0.6  | H = 0.7  | H = 0.8  | H = 0.6  | H = 0.7  | H = 0.8  | H = 0.6  | H = 0.7  | H = 0.8  |
| 5   | −0.23413 | −0.26159 | −0.29706 | 0.000926 | 0.030471 | 0.074755 | 0.348334 | 0.52203  | 0.748933 |
| 15  | −0.24527 | −0.27275 | −0.30822 | −0.00502 | 0.02488  | 0.069483 | 0.353362 | 0.528181 | 0.755389 |
| 25  | −0.25634 | −0.28384 | −0.31929 | −0.01081 | 0.019439 | 0.064353 | 0.358777 | 0.534649 | 0.762086 |
| 35  | −0.26734 | −0.29485 | −0.33029 | −0.01644 | 0.014146 | 0.059363 | 0.364579 | 0.541431 | 0.769021 |
| 45  | −0.27827 | −0.30579 | −0.34119 | −0.02192 | 0.009002 | 0.054513 | 0.370768 | 0.548525 | 0.776192 |
| 55  | −0.28913 | −0.31665 | −0.35202 | −0.02725 | 0.004004 | 0.049801 | 0.377342 | 0.555929 | 0.783596 |
| 65  | −0.29993 | −0.32743 | −0.36277 | −0.03242 | −0.00085 | 0.045225 | 0.384301 | 0.56364  | 0.791229 |
| 75  | −0.31065 | −0.33814 | −0.37343 | −0.03744 | −0.00555 | 0.040785 | 0.391645 | 0.571655 | 0.79909  |
| 85  | −0.32131 | −0.34878 | −0.38402 | −0.04232 | −0.01012 | 0.036478 | 0.39937  | 0.579973 | 0.807174 |
| 95  | −0.33191 | −0.35935 | −0.39452 | −0.04704 | −0.01454 | 0.032305 | 0.407478 | 0.588589 | 0.815479 |



**FIGURE 2** Comparison of E(PV) of GMDB for different Hurst parameters and under JDBM and BM models.

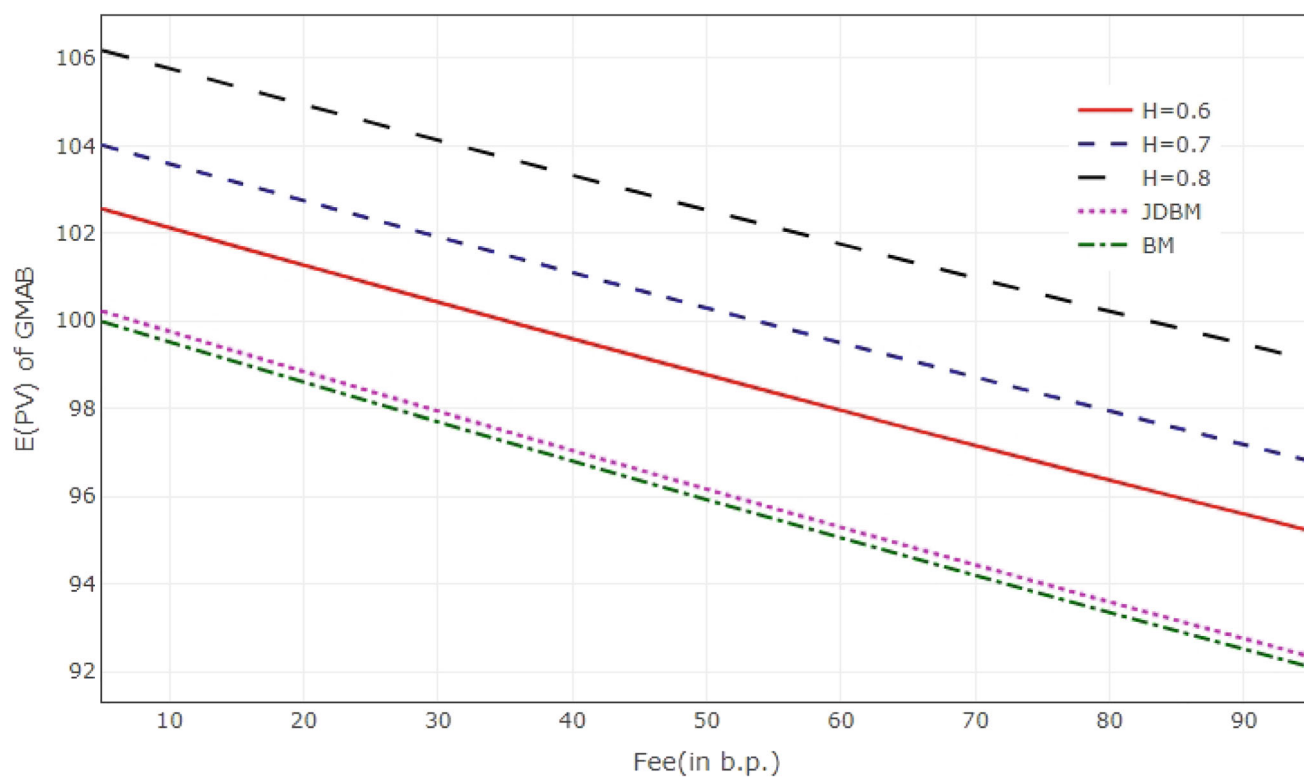


FIGURE 3 Comparison of E(PV) of GMAB for different Hurst parameters and under JDBM and BM models.

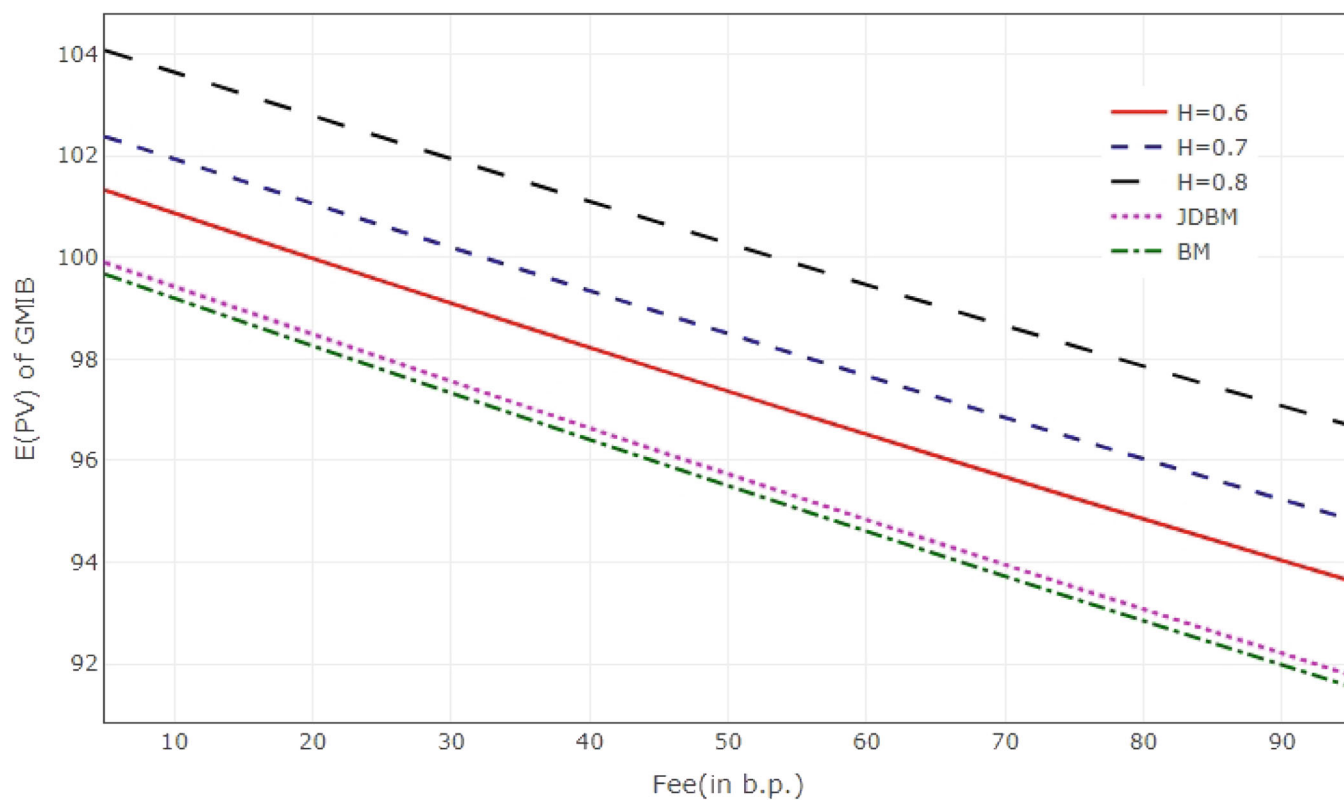


FIGURE 4 Comparison of E(PV) of GMIB for different Hurst parameters and under JDBM and BM models.

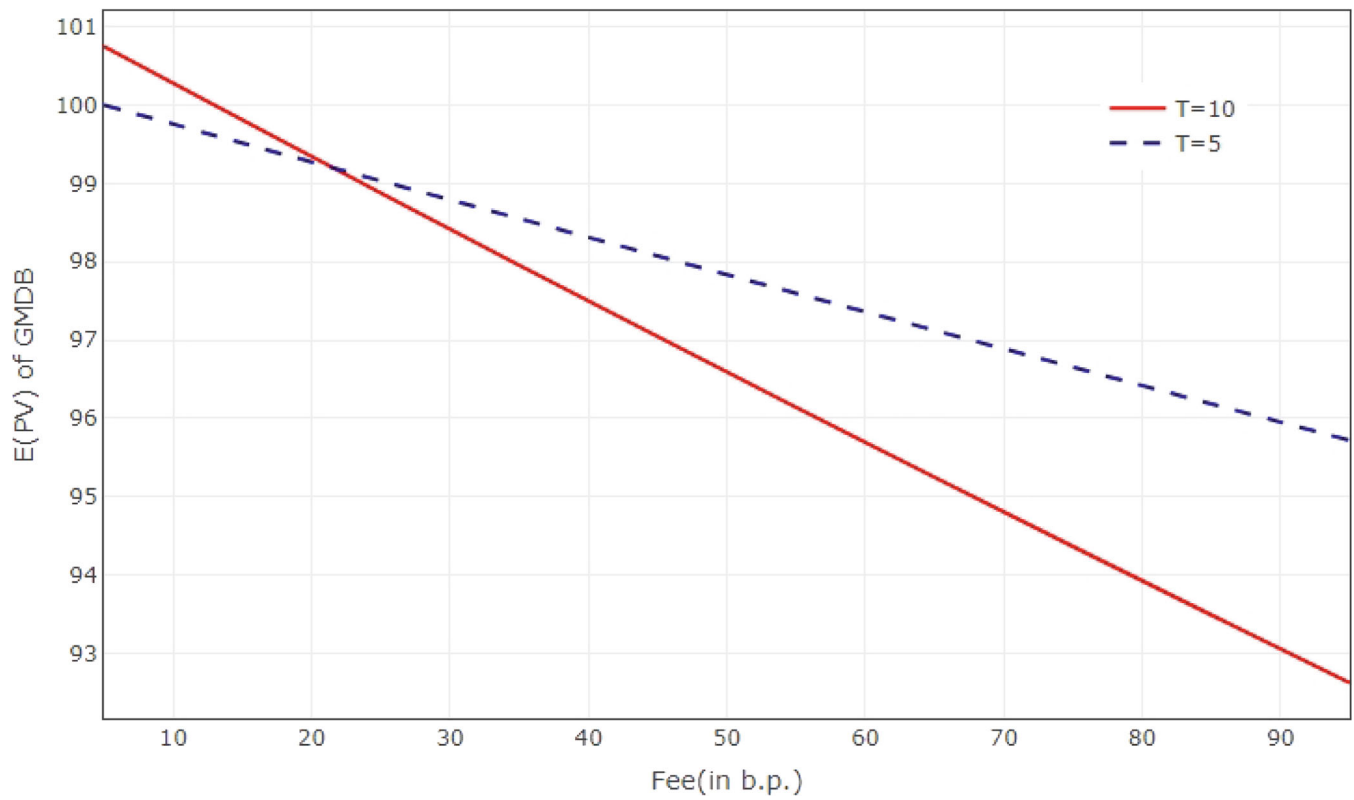


FIGURE 5 Comparison of  $E(PV)$  of GMDB for different time to maturity ( $T$ ), where  $H=0.7$ .

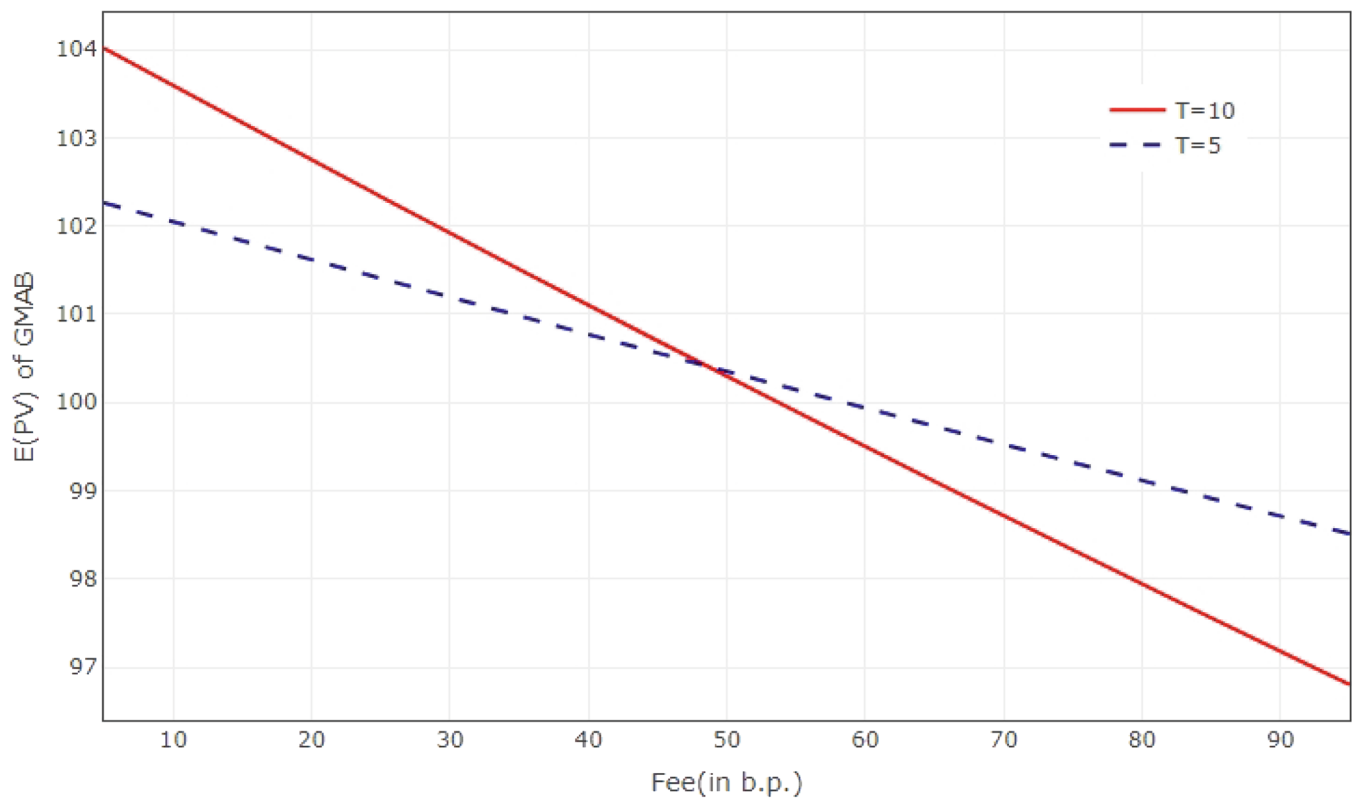


FIGURE 6 Comparison of  $E(PV)$  of GMAB for different time to maturity ( $T$ ), where  $H=0.7$ .

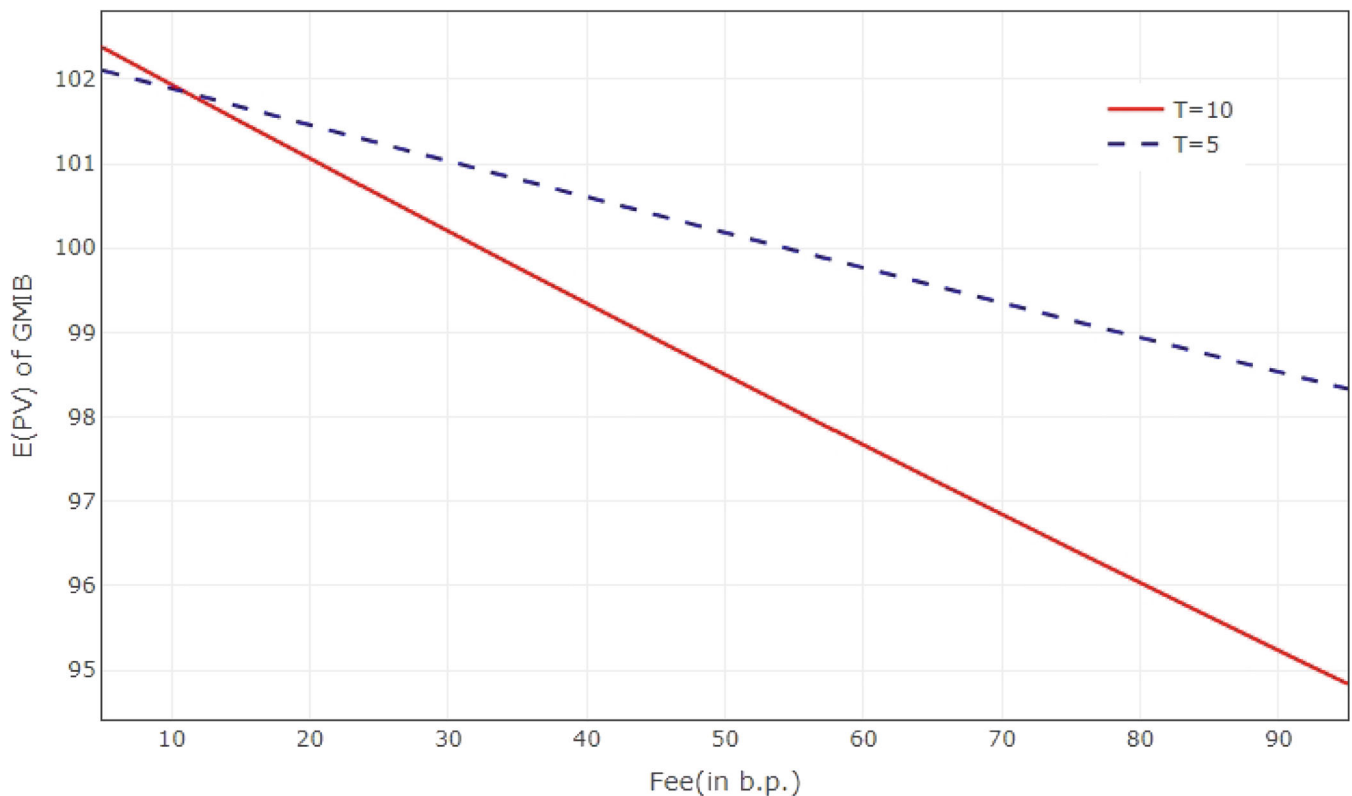


FIGURE 7 Comparison of E(PV) of GMIB for different time to maturity (T), where  $H=0.7$ .

value the E(PV)s of the three guarantees compared to JDMFBM model. Hence, a lower fee will be charged if we do not take long memory property of stock returns in consideration. The difference is smaller for a death benefit guarantee as premiums for death benefit guarantees are very low compared to living benefit guarantee premium.<sup>39</sup>

We have already shown in Section 3.1, the benefit of CBD forecasted probabilities over actual probabilities. The impact of these probabilities on GMDB, GMAB, and GMIB contracts is given in Table 6. Table 6 displays the difference in values of E(PV) of the three contracts obtained using CBD forecasted probabilities with that of E(PV)s obtained using actual probabilities. Compared to results from predicted probabilities, the use of real probabilities leads to over-valuation of GMDB E(PV). In the case of GMIB, usage of probability tables leads to under-valuation of E(PV)s. The values in Table 6 are for a small initial premium amount of 100; the difference will vary with change in the initial premium. For GMAB guarantees, the E(PV)s are sometimes underestimated and sometimes overestimated by using actual probabilities. In any case, the E(PV)s obtained by the CBD stochastic mortality model and by the US mortality tables are not the same.

Figures 2–4 show comparisons between the proposed model under different  $H$  values with GBM and JDBM models respectively for the three contracts GMDB, GMAB, and GMIB. From the figures, it is observed that an increase in the  $H$  values leads to a rise in the E(PV) for all the three contracts. The impact of changing  $H$  values is small in case of GMDB rider, but the same in case of GMAB and GMIB riders is not small. Therefore, a stock with higher dependence on the past values will have higher E(PV) and hence, a higher fee. Similar to the observation from Table 5, it is observed from Figures 2–4 that compared to the proposed model the BM and JDBM models underestimate the E(PV) of the three guarantees. Similar results are obtained by Ng et al.,<sup>39</sup> who shows that the fee obtained with GBM model is lesser compared to the fee obtained with the GARCH model.

The difference between the E(PV) of the three guarantees: GMDB, GMAB, and GMIB, for time to maturity 5 and 10 years is shown in Figures 5–7 respectively. After a certain fee value, E(PV) of guarantee for 10 years maturity becomes less than that for 5 years. As observed from Figures 5–7, this fee is around 10, 20, and 50 b.p. for GMDB, GMIB, and GMAB contracts respectively. Figure 5 shows that the break-even fee for a contract of 10 years is greater than that for a 5 years contract. Theoretically, the longer the contract duration, the higher the risk, so insurers should charge more premiums. In contrast, for living guarantees: GMAB and GMIB, longer duration reduces the probability of surviving the contract term, resulting in a reduction in fee as term increases. Figures 6 and 7 also conclude similar results for the break-even fee.

## 4 | CONCLUSION AND FUTURE WORK

In this work, we modeled investment risk using the MFBM model and Poisson jump process. A GBM model with jumps, stochastic volatility or stochastic interest rates has independent increments. In comparison, stock returns have non-independent increments in general. JDMFBM models are an advancement over the GBM model and its variants, as they capture the long-range dependency and can have non-independent increments. We have also modeled the mortality risk to consider the mortality improvements over time, followed by a numerical analysis for pricing a VA contract.

For analysis purposes, we considered US male mortality data. Since mortality pattern varies from country to country, we modeled this data with eight demographic mortality models to obtain the best fitted one. We concluded that the CBD model is the most suited one for the considered dataset based on specific selection criteria. We forecasted mortality probabilities with the fitted CBD model and compared it with the mortality rates obtained from data for the year 2017. The life expectancy obtained with the CBD model is higher than that obtained with the actual probabilities. Therefore, the CBD model does consider the advancement in survival probabilities over time. We empirically analyzed our VA pricing model using these forecasted probabilities and compared the results with results obtained from GBM and JDBM models. From the analysis, we concluded that using GBM or JDBM models results in the under-valuation of the guarantees for stock returns having long-range dependence.

We also compared the guarantees  $E(PV)$  using the CBD model with those obtained using actual probabilities. The analysis for GMDb guarantees exhibits that the  $E(PV)$  is overestimated by using actual probabilities, resulting in charging a higher fee for the guarantee. In contrast, in a GMIB guarantee,  $E(PV)$  is underestimated by using actual probabilities, resulting in a lower price. In the case of GMAB guarantees, the  $E(PV)$  is sometimes underestimated and sometimes overestimated. Therefore, no generalized results are observed in the case of GMAB guarantees.

In this work, we have considered the pricing problem of GMDb, GMAB, and GMIB riders. The main contribution of this work is the development of the JDMFBM model in the valuation of a VA contract incorporating mortality risk for GMDb, GMAB, and GMIB riders. Our model can be generalized in many different ways. One of the generalizations will be to consider stochastic mortality. The volatility clustering phenomenon is repeatedly observed in markets, therefore, appending a stochastic volatility model as proposed by Sharma et al.<sup>8</sup> will capture such phenomenon. Another future research problem could be the calibration of parameters for JDMFBM models.

## ACKNOWLEDGMENTS

We thank the editor and the anonymous reviewers for their insightful comments, which helped in improving the article. This research work is supported by the Department of Science and Technology, India. One of the authors (Nitu Sharma) would like to thank UGC, India for providing her financial support through Senior Research Fellowship.

## DATA AVAILABILITY STATEMENT

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

## ORCID

Nitu Sharma  <https://orcid.org/0000-0002-3293-7207>

Dharmaraja Selvamuthu  <https://orcid.org/0000-0003-2892-0864>

Selvaraju Natarajan  <https://orcid.org/0000-0002-7184-4193>

## REFERENCES

1. Zhang L. Japanese variable annuity market: background, update and outlook. *Acta Mag.* 2006;2:45.
2. Ledlie MC, Corry DP, Finkelstein GS, Ritchie AJ, Su K, Wilson D. Variable annuities. *Br Actuar J.* 2008;14(2):327-389.
3. Ballotta L, Eberlein E, Schmidt T, Zeineddine R. Variable annuities in a Lévy-based hybrid model with surrender risk. *Quant Finance.* 2020;20(5):867-886.
4. Gerber HU, Shiu ES, Yang H. Valuing equity-linked death benefits in jump diffusion models. *Insur Math Econ.* 2013;53(3):615-623.
5. Ko B, Bae T. Pricing guaranteed minimum death benefit contracts under the phase-type law of mortality. *Lobachevskii J Math.* 2015;36(2):198-207.
6. Escobar M, Krazzler M, Ramsauer F, Saunders D, Zagst R. Incorporation of stochastic policyholder behavior in analytical pricing of GMABs and GMDbs. *Risks.* 2016;4(4):41.
7. Yu W, Yong Y, Guan G, Huang Y, Su W, Cui C. Valuing guaranteed minimum death benefits by cosine series expansion. *Mathematics.* 2019;7(9):835.

8. Sharma N, Dharmaraja S, Arunachalam V. A time series framework for pricing guaranteed lifelong withdrawal benefit. *Comput Econ*. 2021;58(4):1225-1261.
9. Sharma N, Pasricha P, Dharmaraja S. Valuation of equity-indexed annuities under correlated jump–diffusion processes. *J Comput Appl Math*. 2021;395:113575.
10. Mandelbrot BB, Van Ness JW. Fractional Brownian motions, fractional noises and applications. *SIAM Rev*. 1968;10(4):422-437.
11. Willinger W, Taqqu MS, Teverovsky V. Stock market prices and long-range dependence. *Finance Stochast*. 1999;3(1):1-13.
12. Plerou V, Gopikrishnan P, Rosenow B, Amaral LA, Stanley HE. Econophysics: financial time series from a statistical physics point of view. *Phys A Stat Mech Appl*. 2000;279(1-4):443-456.
13. Biagini F, Hu Y, Øksendal B, Zhang T. *Stochastic Calculus for Fractional Brownian Motion and Applications*. Springer Science & Business Media; 2008.
14. Peters EE. Fractal structure in the capital markets. *Financ Anal J*. 1989;45(4):32-37.
15. Karatzas I, Shreve S. *Brownian Motion and Stochastic Calculus*. Springer Science & Business Media; 2012.
16. Androshchuk T, Mishura Y. Mixed Brownian–Fractional Brownian model: absence of arbitrage and related topics. *Stoch Int J Probab Stoch Process*. 2006;78(5):281-300.
17. Zili M. On the mixed fractional Brownian motion. *J Appl Math Stoch Anal*. 2006;2006:1-9.
18. Sun L. Pricing currency options in the mixed fractional Brownian motion. *Phys A Stat Mech Appl*. 2013;392(16):3441-3458.
19. Wang XT, Zhu EH, Tang MM, Yan HG. Scaling and long-range dependence in option pricing II: pricing European option with transaction costs under the mixed Brownian–Fractional Brownian model. *Phys A Stat Mech Appl*. 2010;389(3):445-451.
20. Shokrollahi F, Kılıçman A. Actuarial approach in a mixed fractional Brownian motion with jumps environment for pricing currency option. *Adv Differ Equ*. 2015;2015(1):257.
21. Chen Q, Zhang Q, Liu C. The pricing and numerical analysis of lookback options for mixed fractional Brownian motion. *Fractals*. 2019;128:123-128.
22. Wang J, Wen S, Yang M, Shao W. Practical finite difference method for solving multi-dimensional black-Scholes model in fractal market. *Chaos Soliton Fract*. 2022;157:111895.
23. Ouyang Y, Yang J, Zhou S. Valuation of the vulnerable option price based on mixed fractional Brownian motion. *Discret Dyn Nat Soc*. 2018;2018:4047350 16 pages.
24. Murwaningtyas CE, Kartiko SH, Suryawan HP. Option pricing by using a mixed fractional Brownian motion with jumps. *J Phys Conf Ser*. 2019;1180(1):012011.
25. Zimbidis AA. Optimal management of a variable annuity invested in a Black–Scholes market driven by a multidimensional fractional Brownian motion. *Stoch Anal Appl*. 2010;29(1):61-77.
26. Xu L, Shen G, Yao D. Pricing of equity indexed annuity under fractional Brownian motion model. *Abstr Appl Anal*. 2014;2014:380718 9 pages.
27. Wang J, Yan Y, Chen W, Shao W, Tang W. Equity-linked securities option pricing by fractional Brownian motion. *Chaos Soliton Fract*. 2021;144:110716.
28. Milevsky MA, Posner SE. The titanic option: valuation of the guaranteed minimum death benefit in variable annuities and mutual funds. *J Risk Insur*. 2001;68(1):93-128.
29. Lindbergson M. Mortality among the elderly in Sweden 1988–1997. *Scand Actuar J*. 2001;2001(1):79-94.
30. Cairns AJ, Blake D, Dowd K, et al. A quantitative comparison of stochastic mortality models using data from England and Wales and the United States. *North Am Act J*. 2009;13(1):1-35.
31. Carfora MF, Cutillo L, Orlando A. A quantitative comparison of stochastic mortality models on Italian population data. *Comput Stat Data Anal*. 2017;112:198-214.
32. Lee RD, Carter LR. Modeling and forecasting US mortality. *J Am Stat Assoc*. 1992;87(419):659-671.
33. Renshaw AE, Haberman S. A cohort-based extension to the Lee–Carter model for mortality reduction factors. *Insur Math Econ*. 2006;38(3):556-570.
34. Cairns AJ, Blake D, Dowd K. A two-factor model for stochastic mortality with parameter uncertainty: theory and calibration. *J Risk Insur*. 2006;73(4):687-718.
35. Plat R. On stochastic mortality modeling. *Insur Math Econ*. 2009;45(3):393-404.
36. Bacinello AR, Millosovich P, Olivieri A, Pitacco E. Variable annuities: a unifying valuation approach. *Insur Math Econ*. 2011;49(3):285-297.
37. Bauer D, Kling A, Russ J. A universal pricing framework for guaranteed minimum benefits in variable annuities. *ASTIN Bull J IAA*. 2008;38(2):621-651.
38. R Core Team. *R: A Language and Environment for Statistical Computing*. R Foundation for Statistical Computing; 2017.
39. Ng ACY, Li JSH, Chan WS. Modeling investment guarantees in Japan: a risk-neutral GARCH approach. *Int Rev Financ Anal*. 2011;20(1):20-26.

**How to cite this article:** Sharma N, Selvamuthu D, Natarajan S. Variable annuities valuation under a mixed fractional Brownian motion environment with jumps considering mortality risk. *Appl Stochastic Models Bus Ind*. 2022;38(6):1019-1038. doi: 10.1002/asmb.2699

## APPENDIX A. GEOMETRIC BROWNIAN MOTION (GBM) MODEL

The stock price process in a GBM model under risk-neutral measure is given by

$$S_t = S_0 e^{\left(r - \frac{1}{2}\sigma_1^2\right)t + \sigma_1 B_t}, \quad (\text{A1})$$

where  $r$  is the continuously compounded risk-free rate,  $\sigma_1^2$  is the variance of the log price in unit time and  $B_t$  is a Wiener process. Note that the GBM model is a special case of the JDMFBM model with  $\lambda = 0$ ,  $H = \frac{1}{2}$  and,  $\sigma_1 = \frac{\sigma}{\sqrt{2}}$  in Equation (1) of Section 2.

## APPENDIX B. JUMP-DIFFUSION BROWNIAN MOTION (JDBM) MODEL

The stock price process for a JDBM model with Poisson jumps under risk-neutral measure is given by

$$S_t = S_0 e^{\left(r - \frac{1}{2}\sigma_1^2 - m\lambda\right)t + \sigma_1 B_t + \sum_{n=0}^{N_t} \log(Y_n)}, \quad (\text{B1})$$

where  $r$  is the continuously compounded risk-free rate,  $\sigma_1^2$  is the variance of the log price in unit time and  $B_t$  is a Wiener process. The Poisson process  $\{N_t, t \geq 0\}$ , jump size distribution  $Y_n$ , parameters  $\lambda$  and  $m$  are same as that defined in Equation (1) of Section 2. Note that the JDBM model is a special case of the JDMFBM model with  $H = \frac{1}{2}$  and  $\sigma_1 = \frac{\sigma}{\sqrt{2}}$  in Equation (1) of Section 2.