

RESEARCH ARTICLE

Time series model for GLWB with surrender benefit and stochastic interest rate: Dynamic withdrawal approach

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Abstract

This study investigates the pricing problem of a variable annuity (VA) contract embedded with a guaranteed lifetime withdrawal benefit (GLWB) rider. VAs are annuities in which the value is linked to a bond and equity sub-account fund. The guaranteed lifetime withdrawal benefit rider regularly provides a series of payments to the policyholder for the term of the policy while he/she is alive, regardless of portfolio performance. At the time of the policyholder's death, the remaining fund value is given to his nominee. Therefore, proper fund modeling is critical in the pricing of VA products. Several writers in the literature used a GBM model in which variance is considered to be constant to represent the fund value in a variable annuity contract. However, on the other hand, the returns on financial assets are non-normally distributed in real life. A bit much Kurtosis, leverage effect, and Non-zero Skewness characterize the returns. The generalized autoregressive conditional heteroscedastic (GARCH) models are also used for presenting a discrete framework for the pricing of GLWB. Still, the interest rate was kept constant without including the surrender benefit and the static withdrawal approach, which keeps the model far from the real scenario. Thus, in this research, the generalized GARCH models are used with surrender benefit and dynamic withdrawal strategy to develop a time series model for the pricing of annuity that overcomes the constraints of previous models. A numerical illustration and sensitivity analysis are used to examine the suggested model.

KEYWORDS

dynamic withdrawal, GARCH modeling, GLWB pricing, GLWB rider, surrender option, variable annuity

1 | INTRODUCTION

A contract between an individual and an insurance company, under which the insurer agrees to make regular payments to the insured, which can begin immediately or later, is known as a variable annuity. The policyholder can avail variable annuity contract by making either a single payment or a series of purchase payments. This policyholder's premium is then invested in one or more mutual funds depending on the choice made by the policyholder from the choices of mutual funds. The value of the policyholder's investment as a variable annuity owner can fluctuate according to the performance of the venture options they select. These variable annuity investment options are often mutual funds that invest in stocks,

securities, currency market instruments, or a combination of the three. Although variable annuities are contracted that put resources into mutual funds, they differ significantly.

Variable annuity consists of various embedded options/guarantees provided by the insurer for which the policyholder has to pay extra fees to avail them. These additional guarantees provide stability and safety as the policyholder approaches retirement. There are various categories into which these embedded options can be divided: the guaranteed minimum death benefit (GMDB) and guaranteed minimum living benefit (GMLB). Guaranteed minimum living benefits can be avail to the policyholder through various options. Some of the main options are guaranteed minimum withdrawal benefit (GMWB), guaranteed minimum income benefit (GMIB), guaranteed lifelong withdrawal benefit (GLWB), and guaranteed minimum accumulation benefit (GMAB) (Piscopo and Haberman¹). Annuitants with guaranteed minimum income benefit (GMIB) receive a minimum monthly payment regardless of market volatility, guaranteeing income in retirement. Conversely, a guaranteed minimum accumulation benefit (GMAB) rider ensures that a proprietor's agreement worth will be equivalent to a specific least level of the sum invested following a predetermined number of years, regardless of the actual performance. On account of the guaranteed minimum withdrawal benefit (GMWB), the rider ensures that a specific level of the sum contributed can be withdrawn yearly until the whole sum is recovered, regardless of the market performance.

One more GMWB rider that ensures withdrawals for life is the guaranteed lifetime withdrawal benefit (GLWB). GLWB is the lifelong version of GMWB and is very balanced for the low-risk takers with known returns. A specific percentage of the investment can be removed every year. However long the agreement holder lives, even assuming the fund value becomes zero. This percentage may differ depending on the withdrawal methodology selected by the policyholder. It may be static or dynamic. As a result, among the baby boomers approaching retirement, the popularity and the demand for these annuities will keep rising (Condron²). Hence, valuing a fair price for VA contracts with embedded Guarantees is critical.

GLWB was first introduced in 2004 and became the most popular guarantee in the VA market [Drinkwater et al.³]. However, the research on the valuation of GLWB is minimal, and the literature is very little. In this direction, a theoretical model was proposed by Piscopo and Haberman.¹ Their model assumed the sub-account fund to be GBM with drift and volatility as constant values. A similar assumption of the fund value process to be GBM was made by Dai et al.⁴ and Peng et al.⁵ in the valuation of variable annuity contracts embedded with GLWB or GMWB guarantees. Tian-Shyr Dai⁶ proposes a 3D tree that can examine the connection effects of the investment risk, the mortality risk, and Interest rate risk on the value and the fair charge of GMWB/GLWB with the assumption of constant volatility. However, the real-life scenarios are completely different. Modeling the fund value using GBM, in which volatility is considered constant, is less realistic. The GBM model failed to capture this leverage effect and volatility clustering with or without a stochastic interest rate. Hence, there is a need for a better model to capture all the features.

Recently Sharma et al.⁷ developed a time-series framework to obtain a fair price for the variable annuity embedded with GLWB. To overcome the limitations of the GBM model, they proposed a time series model for pricing an annuity using GARCH models. Their work assumed the constant interest rate with a static withdrawal strategy and without a surrender option.

In the work of Sharma et al.,⁷ sensitivity analysis has been performed between the GLWB fund and the varying risk-free rate, resulting in significant variations in the fund value with a slight variation in the rate of interest. In this direction, Claymore Marshall et al.⁸ had shown that the GMIB value changes significantly with a slight change in interest rate. Moreover, since these guarantees have a long maturity period, assuming a risk-free rate as a constant is far from the real scenario because of the uncertainty of the inflation rate and unexpected shocks over time. The randomness of the interest rate can be observed through the actual data, which motivates us to consider the Stochastic interest rate. Moreover, more realistic insured behavior will be implied after considering surrender and incorporating a dynamic withdrawal strategy. As a result, a more realistic model will emerge, motivating us to expand the current model further.

This article looked at the GJR-GARCH, NA-GARCH, E-GARCH, and T-GARCH models for interest rate and stock volatility modeling. All of the "stylized" facts present in the returns will be captured by the models listed above. Siu et al.⁹ considers the use of GARCH-type models for valuing participating life insurance contracts. Though ARCH and GARCH models can account for the volatility clustering and leptokurtosis, and leptokurtosis, since they are symmetric, they fail to model the leverage effect. To model this, a family of asymmetric GARCH models exists; some of GJR-GARCH Models, E-GARCH Models, NA-GARCH Models, and T-GARCH Models. Therefore, we have used GJR-GARCH, NA-GARCH, E-GARCH, and T-GARCH models to capture this leverage effect. Many standard criteria are used to select the appropriate model for the data. A risk-neutral measure is obtained by following Siu-Hang Li et al.,¹⁰ and Ng et al.¹¹ for the proposed risky asset model. Moreover, we have considered the dynamic withdrawal strategy and surrender benefit. We

then obtained an implicit equation in the charge using the risk-neutral measure. We solved some numerical instances using the implicit equation to find the break-even fee. For the study, we look at three markets: the Japanese market, the US market, and the global market. We have chosen the first two markets based on VA product sales history. We calculated fee values for the three datasets using three distinct models: the asymmetric GARCH model (With and without surrender), the model by Sharma et al.,⁷ and the GBM model in various scenarios.

The rest of the paper is organized as follows. In Section 2, finding of the risk-neutral measure and the model description for the dynamic withdrawal strategy and surrender option have been included. The price model for GLWB valuation is presented in Section 3. Section 4 comprises of choosing asymmetric GARCH models, calculating fees using Monte Carlo simulations, and analyzing the fund's value based on various parameters. The concluding notes in Section 5 propose some potential further work.

2 | MODEL DESCRIPTION AND RISK-NEUTRAL MEASURE

In this paper, the following notations are considered.

Notations

y	Initial insured age (i.e., insured age at period 0)
t	Time in years
R_t	Rate of interest at time t
H_{t-}	Value of fund after fee deduction at the beginning of t^{th} year
H_t	Value of fund before guarantee deduction at the end of t^{th} year
H_{t+}	Value of fund after guarantee deduction at the end of t^{th} year
H_0	Initial value of fund
S_t	Value of stock price at the end of t^{th} year
S_0	Initial value of stock price
γ	Fee charged by the insurance company
G	Fixed yearly withdrawal as $g\%$ of H_0
$s_p(t)$	Surrender penalty for the year t
$l(t)$	Net amount received by the policyholder in the t^{th} year
ω_t	Discretionary withdrawal amount at the year t

2.1 | Dynamic-withdrawal strategy and surrender option

Under the Dynamic-withdrawal strategy and surrender option, the policyholder can withdraw more or less than the guaranteed amount (G) from the fund and surrender the contract whenever he/she prefers. Suppose the surrender option is exercised before maturity. In that case, the insurer imposes a penalty on the policyholder. Similarly, in the case of dynamic withdrawal, a penalty of κ is imposed on the price withdrawn above the guaranteed amount (G). The net amount received by the policyholder in the case of dynamic withdrawal is

$$l(t) = \begin{cases} G, & \text{if } 0 \leq \omega_t \leq G, \\ G + (1 - \kappa)(\omega_t - G), & \text{if } \omega_t > G, \end{cases} \quad (1)$$

where ω_t denotes the discretionary withdrawal amount at the year t , the withdrawal strategy depends on the account value and the information available till time t . Now, since the account value depends on the returns, we get $\omega_t \sim N(G, \sigma_{\omega_t})$.

In case of surrender of the policy, the policyholder gets the positive balance of fund value after the deduction of the surrender penalty by the insurance company. A surrender penalty is a percentage of the fund value and is charged depending upon the time of surrender. Usually, the penalty percentage decreases over time. Let $s_p(t)$ be the surrender penalty percentage at time t then, the surrender value at time t will satisfy the following equation

$$A(t) = \left(1 - \frac{s_p(t)}{100}\right) H_t. \quad (2)$$

2.2 | Risk-neutral Pricing

Risk-neutral pricing is the approach for determining the no-arbitrage price of an investment. A risk-neutral probability measure is a measure under which the expected return from the underlying stock and the risk-free market are the same. Risk-neutral pricing involves determining a risk-neutral measure under which the underlying asset or index price is a martingale to evaluate a no-arbitrage price. In other words, a risk-neutral probability measure is a set of probabilities under which the given market prices of a collection of trades would be equal to the expectations of the winnings or losses of each trade.

In the case of the GLWB guarantee, we don't have any unique risk-neutral measure. Indeed, under a discrete-time GARCH model, the market is incomplete. Under the independence and stationarity of the increments of the returns and the infinite divisibility of their underlying distribution, we have several methods to find the risk-neutral measure. Among these methods, two are the conditional Esscher transform method (Siu et al.¹² and Ng et al.¹¹) and the utility maximization approach (Rubinstein¹³ and Tardelli¹⁴). In our proposed model, both the stock returns and the returns from a risk-free bond are normally distributed, which leads to infinite divisibility with a conditional moment generating function. Conditional Esscher transform approach can be used for a GARCH model for returns having random shocks or innovations that are infinite-divisible distributed and have a finite conditional moment generating function. Since we consider log return series for the index prices and difference return series for the interest rate have stationary and independent increments in our model. Hence, we applied the conditional Esscher transform approach to determine the risk-neutral pricing. The use of the Esscher transform to option pricing was proposed in Gerber and Shiu¹⁵ and that the use of the conditional Esscher transform approach to price options in a GARCH model was proposed in Siu et al.¹⁰

2.2.1 | Construction of risk-neutral measure

Let us consider a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$, where $\{\mathcal{F}_t\}_{t \geq 0}$ is the filtration satisfying the usual conditions and $\mathbb{P}$ is the data generating probability under which the index returns X_t and the interest rate returns r_t are given by Equations (C1) and (C2) [Appendix D] with i.i.d Gaussian innovations. Let $\tau = \{1, 2, \dots, T\}$ be a set of time indices and let us assume that all the financial movements occur at $t \in \tau$. Consider the natural filtration as $\{\mathcal{F}_t\}_{t \in \tau}$ such that for $t \in \tau$, $\mathcal{F}_t$ contains all the information up to and including t and $\mathcal{F}_T = \mathcal{F}$. Using $\mathbb{P}$ as a probability measure $X_t | \mathcal{F}_{t-1}$ and $r_t | \mathcal{F}_{t-1}$ follows a normal distribution (i.e., $X_t | \mathcal{F}_{t-1} \sim N(c_1, \sigma(1)^2_t)$ and $r_t | \mathcal{F}_{t-1} \sim N(c_2, \sigma(2)^2_t)$), Where c_1 and c_2 are constant and $\sigma(i)^2_t$, $i = 1, 2$ the volatility (for more details, see Appendix D).

We will develop a martingale pricing probability measure equivalent to measure $\mathbb{P}$ by using the conditional Esscher transform notion. Following Buhlmann et al.¹⁶ Let us consider a sequence $\{Z_t\}_{t \in \tau}$ with initial value $Z_0 = 1$ and for $t \in \mathbb{Z}^+$ such that

$$Z_t = \prod_{k=1}^t \frac{e^{\mu_k X_k}}{\mathbb{E}[e^{\mu_k X_k} | \mathcal{F}_{k-1}]}, \quad (3)$$

for some $\{\mu_1, \mu_2, \dots, \mu_t\}$ predictable with respect to the filtration $(\mathcal{F}_t)_{t \in \tau}$. Here $\mathbb{E}$ denotes the expectation under the real-world probability measure $\mathbb{P}$. Since $\mathbb{E}(Z_t | \mathcal{F}_{t-1}) = Z_{t-1}$, therefore $\{Z_t\}_{t \in \tau}$ is a martingale. Now let us consider $\mathbb{P}_t$ to be the restriction of $\mathbb{P}$ on the information $\mathcal{F}_t$. Since $\{Z_t\}_{t \in \tau}$ is a martingale, therefore, using this property let us construct a new family of measures in the sample space $(\Omega, \mathcal{F})$ say $\tilde{\mathbb{P}}_t$ such that $d\tilde{\mathbb{P}}_t = Z_t d\mathbb{P}_t$ and $\tilde{\mathbb{P}}_t = \tilde{\mathbb{P}}_{t+1} / \mathcal{F}_t$ and a probability

measure $\tilde{\mathbb{P}} = \tilde{\mathbb{P}}_T$. For any set $A \in \mathcal{B}(\mathbb{R})$ and the indicator function $I(X_t \in A)$, the conditional distribution of $\{X_t \in A\}$

$$\tilde{\mathbb{P}}_t(X_t \in A | \mathcal{F}_{t-1}) = \mathbb{E}_{\mathbb{P}_t} [I(X_t \in A) \frac{e^{\mu_t X_t}}{\mathbb{E}_{\mathbb{P}_t}(e^{\mu_t X_t})} | \mathcal{F}_{t-1}], \quad (4)$$

is known as the conditional Esscher transform. Now, in order to obtain the distribution function of X_t from the above equation under $\tilde{\mathbb{P}}_t$ replace A with $(-\infty, x]$ where $x \in \mathbb{R}$ which lead us to

$$F_{\tilde{\mathbb{P}}_t}(x) = \frac{\int_{-\infty}^x e^{\mu_t y} \cdot dF_{\tilde{\mathbb{P}}_t}(y | \mathcal{F}_{t-1})}{\mathbb{E}_{\tilde{\mathbb{P}}_t}[e^{\mu_t X_t} | \mathcal{F}_{t-1}]}. \quad (5)$$

Now, given $\mathcal{F}_{t-1}$ the MGF (Moment generating function) of X_t under $\tilde{\mathbb{P}}_t$ is given by

$$\mathbb{E}_{\tilde{\mathbb{P}}_t}[e^{X_t z}; \mu_t | \mathcal{F}_{t-1}] = \mathbb{E}_{\mathbb{P}_t} \left[\frac{e^{X_t(z+\mu_t)} | \mathcal{F}_{t-1}}{\mathbb{E}_{\mathbb{P}_t}[e^{X_t(\mu_t)} | \mathcal{F}_{t-1}]} \right], \quad (6)$$

using above equation and the fact that $X_t | \mathcal{F}_{t-1} \sim N(c_1, \sigma(1)_t^2)$ we have

$$\mathbb{E}_{\tilde{\mathbb{P}}_t}[e^{z X_t}, \mu_t | \mathcal{F}_{t-1}] = e^{(c_1 + \sigma(1)_t^2 \mu_t)z + \frac{1}{2} \sigma(1)_t^2 z^2}. \quad (7)$$

Now, to have a risk-neutral measure $\mathbb{Q}$ equivalent to the data generating probability $\mathbb{P}$, let us consider an Esscher parameter's sequence μ'_t such that the following equality holds

$$\mathbb{E}_{\tilde{\mathbb{P}}_t}[e^{X_t}; \mu'_t | \mathcal{F}_{t-1}] = \mathbb{E}_{\tilde{\mathbb{P}}_t}[e^{r_t}; \mu'_t | \mathcal{F}_{t-1}], \quad t \in T/\{0\}, \quad (8)$$

where $r_t | \mathcal{F}_{t-1} \sim N(c_2, \sigma(2)_t^2)$. This equality holds because in a risk-neutral world the total expected returns from a risk-free bond is equal to the total expected return from any stock. Now, since $r_t | \mathcal{F}_{t-1} \sim N(c_2, \sigma(2)_t^2)$, therefore

$$\mathbb{E}_{\tilde{\mathbb{P}}_t}[e^{r_t}; \mu'_t | \mathcal{F}_{t-1}] = e^{(c_2 + \sigma(2)_t^2 \mu'_t) + \frac{1}{2} \sigma(2)_t^2}, \quad (9)$$

therefore by comparing above equation with Equation (7) with $z = 1$ and than substituting $\mu'_t = \frac{c_2 - c_1}{\sigma(2)_t^2} + \frac{\sigma(1)_t^2}{\sigma(2)_t^2}((c_2 - c_1)(\sigma(1)_t^2 - \sigma(2)_t^2) - \frac{1}{2}) + \frac{\sigma(1)_t^2 - \sigma(2)_t^2}{2}$ in Equation (8) we get

$$\mathbb{E}_{\mathbb{Q}_t}[e^{X_t z} | \mathcal{F}_{t-1}] = e^{(c_2 z + \sigma(1)_t^2((c_2 - c_1)(\sigma(1)_t^2 - \sigma(2)_t^2) - \frac{1}{2})z + \frac{1}{2} \sigma(1)_t^2 z^2)}. \quad (10)$$

where $\mathbb{Q}_t$ is the restriction of $\mathbb{Q}$ on the information $\mathcal{F}_t$ such that $\mathbb{Q}_t = \mathbb{Q}$. Using this risk-neutral probability measure $\mathbb{Q}$, the interest rate risk may not be priced since the probability laws of the interest rate process may not change after the measured change from $\mathbb{P}$ to $\mathbb{Q}$. $\mathbb{E}_{\mathbb{Q}}$ is the expected value under the measure $\mathbb{Q}$ and hence under $\mathbb{Q}$, $X_t | \mathcal{F}_{t-1} \sim N(c_2 + \sigma(1)_t^2 \left((c_2 - c_1)(\sigma(1)_t^2 - \sigma(2)_t^2) - \frac{1}{2} \right), \sigma(1)_t^2)$.

3 | PRICING MODEL

Let us consider the notations mentioned at the beginning of “Model Description and risk-neutral measure,” For simplicity purposes, let us make some assumptions that the model's description will follow.

Assumptions

1. We analyze a single index and assume that the premium is a one-time lump sum investment of H_0 made by the insured at the start of the contract, which makes $\frac{H_0}{S_0}$ as the number of initial stocks.

2. The insurance company's fee is deducted from the fund's value when fund units are canceled.
3. We'll assume the fee is charged at the beginning of the year, and the guarantee ($g\%$ of H_0) is deducted at the end. In addition, the policyholder's annual withdrawals are carried out at the end of the year.
4. When the fund value becomes less than the ω_t of that year, then the company will pay only a fixed amount G ($g\%$ of H_0) for the remaining period.
5. Death benefits are paid at the end of the year when the insured dies.

After deduction of the charges, H_{t-} is the value of the fund at the start of the t^{th} year. For the year t , the H_{t-} sum is invested in the market and rises to H_t at the t^{th} year-end. Amount G is withdrawn from H_t , leaving H_t^+ value remaining in the fund. The process will go in the same way for $(t+1)^{\text{th}}$ year and so on till there is a positive fund value. Let T be a random variable denoting the maximum number of years lived by an individual, and the time t is the number of the years from policy inception ranging from $t = 1, 2, \dots, T$. An individual's maximum age is believed to be ω years old. So, the parameter T equals $\omega - y$. By considering H_{0+} to be H_0 , the fund value dynamics are as follows

$$H_{t-} = H_{(t-1)+}(1 - \gamma - s_p(t)), \quad (11)$$

$$H_t = H_{t-} \frac{S_t}{S_{t-1}}, \quad (12)$$

$$H_{t+} = \begin{cases} H_t - l(t), & \text{if } H_t > l(t), \\ 0, & \text{otherwise,} \end{cases} \quad (13)$$

where $t \in \{1, 2, \dots, T\}$. Equation (11) represents the t^{th} year fund value after the fee deduction γ and surrender penalty $s_p(t)$. Equation (12) represents the growth fund value parallel to the changes in the stock prices, and Equation (13) represents the fund value after the annual guarantee deduction. Now we will have the following cases:

Case 1: Until the insured dies, the fund value is always positive.

Case 2: When fund value has become zero or $H_t - l(t) < 0$ and insured is still alive.

In the case of 1, the insured will receive the guaranteed amount from the account till death and the remaining balance as a lump sum payment, which his nominee will receive. In this case, the insurer is not liable to pay anything. In the second scenario, no death benefit will be given, but the insurer will be responsible for paying living benefits for the remainder of the insurer's life. It is to be noted that if the fund is insufficient to pay the guarantee, its value becomes zero, which will occur only at the end of the year.

Now let us consider the case 2 when $H_n < l(n)$ (i.e., the fund value becomes less than $l(n)$ for the first time at the end of the n^{th} year) in this case $H_{n+} = 0$. The insurer is obligated to pay the annuitant a lifetime annuity of amount G in addition to $l(n) - H_n$, hence of which the company's cost or obligation at the end of n^{th} year is given by the following Equation

$$(l(n) - H_n)p_y^{(n,s)} + \sum_{k=1}^{\omega-y-n} (p_y^{(k+n,s)} \cdot G)e^{-\sum_{i=1}^k r_i}, \quad (14)$$

where $e^{-\sum_{i=1}^k r_i}$ is the discounting factor and $p_y^{(k,s)}$ is the likelihood of a life aged y surviving until age k without any surrender till k . In the Equation (14) $(l(n) - H_n)$ is the extra amount that needs to be paid by the insurer to the insured to accomplish the guaranteed amount for the n^{th} year. In contrast, the second term signifies the value of the annuity for the remainder of the annuitant's life at time n . This leads us to the insurer's initial value of obligation as

$$\left[(l(n) - H_n)p_y^{(n,s)} + \sum_{k=1}^{\omega-y-n} (p_y^{(k+n,s)} \cdot G)e^{-\sum_{i=1}^k r_i} \right] e^{-\sum_{j=1}^n r_j}, \quad (15)$$

now in order to calculate the break-even fee (γ) we have two possible ways:

Case 1: Equate the expected value of the cost mentioned above in Equation (15) to the expected value of income to the insurer from this fund under the risk-neutral measure $\mathbb{Q}$.

Case 2: Equate the expected present value $E(PV)$ of the policyholder's inflows and outflows. Where $B_L(0) + B_D(0)$ is the expected present value of the inflows and H_0 is the expected present values of the outflows under the risk-neutral measure $\mathbb{Q}$.

This paper will consider the case 2 to obtain the break-even fee. In the case 2, there are two different types of benefits, namely death benefit and living benefit, which will generate the insured's income from this fund. The living benefit is a lifetime guaranteed income of the amount of $l(t)$. In contrast, death benefit includes positive fund value(if any) provided to the beneficiary in the event of the insured's death. Following the case 2, to calculate the break-even fee, equate the anticipated present value of inflows to the expected present value of the outflows, which gives

$$H_0 = B_L(0) + B_D(0), \quad (16)$$

where $B_L(0)$ and $B_D(0)$ represent the expected present value of the living benefit and death benefit, respectively, under the risk-neutral measure $\mathbb{Q}$.

This research will evaluate two scenarios for calculating the expected present value of the policyholder's inflows: no surrender till death and surrender before death.

3.1 | Without surrender

In this scenario, there is no surrender by the policyholder before death. Now, the values for $B_L(0)$ and $B_D(0)$ are given as follows:

$$B_L(0) = \sum_{k=1}^{\omega-y} (p_y^{(k,s)} \cdot l(k)) \cdot e^{-\sum_{i=1}^k r_i}, \quad (17)$$

for the expected present value of the death benefit, we will consider the present value of the m^{th} year death benefit, which is given by:

$$B_D(m) = \mathbb{E}_{\mathbb{Q}}[H_m], \quad (18)$$

$$= \frac{H_0}{S_0} \mathbb{E}_{\mathbb{Q}} \left[\left(S_m \prod_{i=1}^m (1 - \gamma - s(i)) - \sum_{i=1}^{m-1} \frac{gS_0}{100} \left(\prod_{k=0}^{i-1} (1 - \gamma - s(i)) \right) \frac{S_m}{S_i} \right)^+ \right], \quad (19)$$

where $\mathbb{E}_{\mathbb{Q}}$ is the expected value under risk-neutral measure $\mathbb{Q}$ and $(f)^+ = \max(f, 0)$. Now its present value is given by

$$B_D(0) = \sum_{j=1}^{\omega-y} e^{-\sum_{i=1}^j r_i} \cdot B_D(j) \cdot p_y^{(j-1,s)} \cdot q_{y+j-1}^1, \quad (20)$$

where q_y^k represents the likelihood of a person of age dying in the following k years. It's worth noting that the condition in Equation (19) has been substituted with the chance of dying between the ages of $m - 1$ and m years, and hence we get

$$B_L(0) + B_D(0) = \sum_{k=1}^{\omega-y} (p_y^{(k,s)} \cdot l(k)) \cdot e^{-\sum_{i=1}^k r_i} + \sum_{j=1}^{\omega-y} e^{-\sum_{i=1}^j r_i} \cdot B_D(j) \cdot p_y^{(j-1,s)} \cdot q_{y+j-1}^1, \quad (21)$$

which leads us to the final Equation as

$$H_0 = \sum_{k=1}^{\omega-y} (p_y^{(k,s)} \cdot l(k)) \cdot e^{-\sum_{i=1}^k r_i} + \sum_{j=1}^{\omega-y} e^{-\sum_{i=1}^j r_i} \cdot B_D(j) \cdot p_y^{(j-1,s)} \cdot q_{y+j-1}^1. \quad (22)$$

The above implicit Equation (22) in γ is solved to obtain the break-even value for the fee (γ) charged by the insurer.

3.2 | With surrender

There is a surrender by the policyholder before his death; in this case, let us define $q_{x+t}^{1,s}$ be the probability of surrendering (not death) the policy between the time points t to $t + 1$. Then, the values of $B_L(0)$ and $B_D(0)$ are given by:

$$B_L(0) = \sum_{k=1}^{\omega-y} (p_y^{(k,s)} \cdot l(k)) \cdot e^{-\sum_{i=1}^k r_i}, \quad (23)$$

and

$$B_D(0) = \sum_{j=1}^{\omega-y-1} e^{-\sum_{\lambda=1}^j r_\lambda} \cdot A(j) \cdot p_y^{(j-1,s)} \cdot q_{y+j-1}^{(1,s)}, \quad (24)$$

which gives the final equation

$$H_0 = \sum_{k=1}^{\omega-y} (p_y^{(k,s)} \cdot l(k)) \cdot e^{-\sum_{i=1}^k r_i} + \sum_{j=1}^{\omega-y} e^{-\sum_{\lambda=1}^j r_\lambda} \cdot A(j) \cdot p_y^{(j-1,s)} \cdot q_{y+j-1}^{(1,s)}. \quad (25)$$

4 | NUMERICAL RESULTS AND SENSITIVITY ANALYSIS

4.1 | Data description

Last 50 years, indices were chosen based on data availability and popularity among the people of the respective country. The S&P 500 composite Index from the US market, Japan, the Nikkei average price 225 Index, and the MSCI World Index from the global market are among the datasets. We took into account information from January 1, 1970, until October 31, 2019. The data for all three indices are taken from Thomson and Reuters Datastream. Furthermore, US and Japan's monthly interest rate data is taken from <https://fred.stlouisfed.org> for the previous 50 years.

Modeling the five datasets mentioned above, simulating returns using the fitted model, determining the fee, and accounting for the effect of the surrender penalty on the GLWB value are all included in this section. We used various approaches to find the best-fitting asymmetric GARCH models for the partitioned datasets. Standard criteria include the Akaike Information Criterion (AIC), the Bayesian Information Criterion (BIC), also known as the Schwartz Information Criterion (SIC), and LogLikelihood (LLK) values. Table 1 displays the asymmetric models fitted to the partitioned datasets. Please see “Appendix A” for further information on fitting asymmetric GARCH models to partitioned datasets. These models were then used to establish the GLWB guarantee pricing. Finally, we carried out a sensitivity analysis of the pricing parameters.

To determine the appropriate fee value for the GLWB contract, we simulate daily log returns and index and interest rate returns using the fitted asymmetric GARCH model, standardized GARCH model, and GBM model under the risk-neutral measure. These returns are used to determine the values of S_t , H_t , $B_L(t)$, and $B_D(t)$. For determining the GLWB contract's charged fee(γ) following algorithm is considered:

1. Under measure $\mathbb{Q}$ Simulate 10,000 sample paths of X_t and ω_t , on the basis of $\omega_t \sim \mathbb{N}(G, \sigma_{\omega_t})$ and $X_t \sim \mathbb{N}\left(c_2 + \sigma(1)_t^2 \left((c_2 - c_1)(\sigma(1)_t^2 - \sigma(2)_t^2) - \frac{1}{2}\right), \sigma(1)_t^2\right)$ (see “Construction of risk-neutral measure”). $\sigma_{\omega}(t)$ is constant for each path, and $\sigma(1)_t$ and $\sigma(2)_t$ firstly generated from the fitted models (see Table 1 for fitted models) for each path.
2. Get $l(t)$ first, then the fund value (H_t) as a function of the breakeven fee (γ) for each sample path.
3. To acquire the death benefit values $B_D(t)$ in the case where the surrender option is not exercised before death, and $A(t)$ in the event where the surrender option is executed before death, average simulated values of H_t to produce $E_{\mathbb{Q}}[H_t]$.
4. Calculate γ using Equation (22) (without surrender) and Equation (25) (with surrender).

For the numerical analysis, consider the following assumptions:

TABLE 1 Fitted asymmetric models to the partitioned data sets of S&P 500, Nikkei 225, MSCI world index, and US and Japan interest rate

	Fitted model
S&P 500	
Data 1	GJR-GARCH (1,1)
Data 2	GJR-GARCH (2,1)
Data 3	E-GARCH (2,1)
Data 4	E-GARCH (2,1)
Data 5	T-GARCH (2,2)
Nikkei 225	
Data 1	T-GARCH (2,1)
Data 2	T-GARCH (2,1)
Data 3	GJR-GARCH (2,1)
Data 4	T-GARCH (2,1)
Data 5	GJR-GARCH (1,1)
Data 6	T-GARCH (2,1)
MSCI world	
Data 1	GJR-GARCH (1,2)
Data 2	E-GARCH (2,1)
Data 3	GJR-GARCH (1,2)
Data 4	E-GARCH (2,1)
Data 5	T-GARCH (2,1)
US interest rate	
Data 1	E-GARCH (2,2)
Data 2	E-GARCH (2,2)
Data 3	E-GARCH (2,1)
Data 4	E-GARCH (1,1)
Data 5	E-GARCH (2,2)
Japan interest rate	
Data 1	T-GARCH (1,1)
Data 2	GJR-GARCH (1,1)
Data 3	GJR-GARCH (1,2)
Data 4	E-GARCH (2,1)
Data 5	T-GARCH (1,2)

1. The Nikkei 225 index's independent mortality rates are based on the standard life tables of Japan, which combine death statistics for the year 2017. The S&P 500 and MSCI international indexes, on the other hand, are based on 2017 US combined mortality rates. (The Human Mortality Database provided this information.)
2. Japan market follows Japan's rate of interest while the world and US market follows the US's rate of interest.
3. For numerical analysis, the probability of surrendering before death and the probability of death without surrender are considered the same as the mortality rate of the corresponding markets. If not mentioned then the surrender penalty $s_p(t) = 4\%$, withdrawal penalty $\kappa = 4\%$ and $\sigma_{\omega_t} = 4$ are the default values.
4. Premium paid is a lump sum amount of 100, that is, $H_0 = 100$.
5. The range of break-even fees is considered to be 0 to 1000 basis points (bp).
6. Only two states of decrement, that is, death and surrender.

TABLE 2 Break even fee values for asymmetric GARCH (without surrender: in basis points)

Assymmetric GARCH (without surrender)					
Age	g (in %)	S&P	MSCI	g (in %)	Nikkei
60	8.245	748.3972	726.5925	7.585	435.1016
	8.255	803.1677	781.9723	7.595	539.092
	8.265	860.0088	838.9129	7.605	659.3807
	8.275	919.1959	898.7363	7.615	794.3579
	8.285	986.4752	968.3659	7.625	958.5882
65	12.300	110.0921	86.37588	10.910	22.0943
	12.310	135.393	111.9963	10.920	80.5503
	12.320	161.5637	138.2839	10.930	150.2773
	12.330	189.2329	168.8779	10.940	225.365
	12.340	222.2329	201.3673	10.950	302.4554

TABLE 3 Break even fee values for asymmetric GARCH (with surrender: in basis points)

Assymmetric GARCH (without surrender)					
Age	g (in %)	S&P	MSCI	g (in %)	Nikkei
60	8.245	747.5993	725.9618	7.585	434.1539
	8.255	802.3301	781.1658	7.595	537.9828
	8.265	859.0423	838.4596	7.605	658.3952
	8.275	918.0754	898.0175	7.615	793.3947
	8.285	985.9367	967.6521	7.625	957.6487
65	12.300	109.3374	85.5697	10.91	21.3178
	12.310	134.6727	111.3996	10.92	79.8296
	12.320	161.0702	137.6431	10.93	149.3392
	12.330	189.208	167.9787	10.94	223.9331
	12.340	221.657	200.834	10.95	301.5757

7. Dynamic withdrawal strategy is considered.
8. There is no correlation between the stock returns and rate of interest return.

Break-even fee values for the asymmetric GARCH, Model with a constant interest rate, and GBM model are calculated and compared. The break-even fee is the value of γ for which the E(PV) of the outflows equals the premium H_0 . The break-even fee for ages 60 and 65 with and without surrender options are shown in Tables 2 and 3, which correspond to different guarantee amounts g percent of H_0 . As previously mentioned, the charge is calculated as a percentage of the fund value until the fund value turns non-negative. Tables 2 and 3 illustrate that a greater cost is associated with a higher guaranteed withdrawal amount. Even if the cost is 0 bp, a very low guarantee will not allow the contract's present value to match the premium. Similarly, a very high guarantee cannot make the contract's current value equal to the premium, even if the cost is 3000 bp. Furthermore, there is no need for a break-even charge corresponding to each guarantee percentage. Therefore the value of g has been selected such that the value of the break-even fee exists between 0 and 1000 basis points.

Additionally, in Figure C1, plots of fee versus value of g show the fee behavior with the change in the value of g . Increases in g enhance the guaranteed withdrawal amounts while decreasing the death benefit value. The increase in the value of the life benefit overcomes the reduction in the value of the death benefit as g rises. As a result,

as the value of g rises, the value of GLWB as a whole rises. The effect is greater for younger policyholders, as the difference in GLWB values corresponding to different g values decreases and eventually vanishes as the policyholder's age at inception rises. Senior policyholders might thus benefit from a larger guarantee value without paying a higher charge.

We obtained pricing results by comparing constant and variable volatility of returns using stock prices modeled as GBM (see Table A2). Furthermore, we acquired pricing data with the previous model⁷ to compare the constant and variable interest rates (see Table A1). Tables 2 and 3 show the suggested model's break-even fee, and Tables A1 and A2 that of the fee achieved by the model⁷ and GBM model. These models observed that the fee was over-estimated, and somewhere, it was underestimated. Furthermore, the fund's behavior is influenced by the guaranteed size and the price charged. In the case of a large guarantee, the fund will deplete sooner than projected, resulting in a liability to the VA provider and, as a result, significant losses. Furthermore, a low fee will jeopardize the insurance company's long-term financial stability.

We examine the influence of the death and living benefits of a change in age at inception y on the GLWB value. Hence, the insurer should have sufficient cash to pay living benefits if the fund is depleted. Further, we saw that the initial insured age also impacts the fee, that is, with the increment of initial insured age, $\gamma(\text{fee})$ decreases. As the initial insured age increases, contract length reduces, resulting in fewer withdrawals and a lower living benefit amount. Furthermore, when T decreases, the time it takes to discount the death benefit value is reduced, resulting in a more considerable death benefit. An increase in initial insured age lowers the overall GLWB value since the loss in living benefit value is larger than the gain in death benefit value for younger policyholders. However, for senior policyholders, the increase in death benefit value is greater than the decrease in living benefit value (age 80 and above).

In the case of GLWB with the surrender option, in comparison to the behavior of surrender penalty, that is, $s_p(t)$ on GLWB value, a change in $s_p(t)$ affects the surrender value only. The higher the value of the surrender penalty, the lesser the surrender value, and vice versa. Figure B1 also supports this conclusion. The GLWB value falls with the increase in the surrender penalty. Furthermore, we can observe that age does not have that much impact on the value of the contract with the change of surrender penalty.

5 | CONCLUSION AND FUTURE WORK

This paper focused on describing the practice of the various forms of retirement benefits and lifetime guarantees. These guarantees also allow policyholders to participate in the stock market while protecting them from its downward tendency. On the other hand, these assurances should be set so that insurers do not incur long-term losses or charge an exorbitant fee, reducing demand for the product. Hedging and pricing of these guarantees are critical for financial organizations. The product may not appeal to investors if the insurer charges a high price. If the insurer charges a small amount, the insurance company may be unable to pay the lifelong commitments. Obtaining a GLWB contract's fair fee is an essential issue to handle in this regard.

For a fair fee calculation, the fund value must be simulated so that all the stylized elements of the underlying assets are appropriately captured. The well-known GBM model fails to capture the common stylistic aspects of the underlying assets. Hence we employed the asymmetric GARCH models to account for the volatility of returns of the index and the rate of interest returns with surrender benefit and dynamic withdrawal approach. Future work in this area will make the situation more realistic by removing assumptions such as a constant surrender charge and replacing discretionary withdrawal amounts with appropriate processes. The sensitivity analysis of the GLWB(with surrender) fund value corresponding to modifying the surrender penalty (see Figure B1) reveals that even slight changes in the surrender penalty result in considerable changes in the GLWB value. Hence, the assumptions of a constant surrender penalty can be replaced with an appropriate model and by broadening the discretionary withdrawal amount. Moreover, the situation where the drifts for the index returns and the interest rate returns depend on the conditional variances can be considered for a more realistic scenario.

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DATA AVAILABILITY STATEMENT

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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REFERENCES

1. Piscopo G, Haberman S. The valuation of guaranteed lifelong withdrawal benefit options in variable annuity contracts and the impact of mortality risk. *N Am Actuar J*. 2011;15(1):59-76.
2. Condrón CM. Variable annuities and the new retirement realities. *Geneva Pap Risk Insur: Issues Pract*. 2008;33(1):12-32.
3. Drinkwater M, Iqbal I, Montiminy J. Variable annuity guaranteed living benefits utilization: 2012 experience. A joint study sponsored by the Society of Actuaries and LIMRA: Schamburg; 2014.
4. Dai M, Kuen Kwok Y, Zong J. Guaranteed minimum withdrawal benefit in variable annuities. *Math Financ: Int J Math Stat Financ Econ*. 2008;18(4):595-611.
5. Peng J, Leung KS, Kwok YK. Pricing guaranteed minimum withdrawal benefits under stochastic interest rates. *Quant Financ*. 2012;12(6):933-941.
6. Dai TS, Yang SS, Liu LC. Pricing guaranteed minimum/lifetime withdrawal benefits with various provisions under investment, interest rate and mortality risks. *Insur: Math Econ*. 2015;64:364-379.
7. Sharma N, Dharmaraja S, Arunachalam V. A time series framework for pricing guaranteed lifelong withdrawal benefit. *Comput Econ*. 2020;58(4):1225-1261.
8. Marshall C, Hardy M, Saunders D. Measuring the effectiveness of static hedging strategies for a guaranteed minimum income benefit. *N Am Actuar J*. 2012;16(2):143-182.
9. Siu TK, Lau JW, Yang H. On valuing participating life insurance contracts with conditional heteroscedasticity. *Asia-Pac Financ Mark*. 2007;14(3):255-275.
10. Siu-Hang Li J, Hardy MR, Tan KS. On pricing and hedging the no-negative-equity guarantee in equity release mechanisms. *J Risk Insur*. 2010;77(2):499-522.
11. Ng ACY, Li JSH, Chan WS. Modeling investment guarantees in Japan: a risk-neutral GARCH approach. *Int Rev Financ Anal*. 2011;20(1):20-26.
12. Siu TK, Tong H, Yang H. On pricing derivatives under GARCH models: a dynamic Gerber-Shiu approach. *N Am Actuar J*. 2004;8:17-31.
13. Rubinstein M. The valuation of uncertain income streams and the pricing of options. *Bell J Econ*. 1976;7(2):407-425.
14. Tardelli P. Utility maximization in a pure jump model with partial observation. *Probab Eng Inf Sci*. 2011;25(1):29-54.
15. Gerber H, Shiu E. Option Pricing by Esscher Transforms. *Trans Soc Actuar*. 1994;46(3):99-191.
16. Bühlmann H, Delbaen F, Embrechts P, Shiryayev AN. No-arbitrage, change of measure and conditional Esscher transforms. *CWI Quart*. 1996;9(4):291-317.
17. Zeileis A, Leisch F, Hornik K, Kleiber C. strucchange: an R package for testing for structural change in linear regression models. *J Stat Softw*. 2002;7:1-38.
18. Engle RF. Autoregressive conditional heteroscedasticity with estimates of the variance of United Kingdom inflation. *Econ: J Econ Soc*. 1982;50(4):987-1007.
19. Bollerslev T. Generalized autoregressive conditional heteroskedasticity. *J Econ*. 1986;31(3):307-327.

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APPENDIX A. ASYMMETRIC GARCH MODEL SELECTION

We compared the AIC, BIC, and Log-Likelihood values to find the best-fitting model. The difference between AIC and BIC is that BIC penalizes free parameters more harshly than AIC. The best-fitted model can be discovered by decreasing AIC and BIC and maximizing the log-likelihood function values. If two or more models have the same AIC, BIC, and log-likelihood values, the model with the fewest parameters is picked. In addition, if no one model has the highest AIC, BIC, or log-likelihood values, the models are only compared using BIC and log-likelihood values (Tables A1 and A2).

TABLE A1 Break even fee values for model with constant interest rate (in basis points)

Assymetric GARCH (without surrender)					
Age	g (in %)	S&P	MSCI	g (in %)	Nikkei
60	8.245	78.3601	102.8454	7.585	43.3290
	8.255	85.0746	109.6214	7.595	57.1566
	8.265	91.7769	116.4826	7.605	71.7121
	8.275	98.4431	123.3731	7.615	87.2884
	8.285	105.6779	131.0205	7.625	103.1826
65	12.300	118.5616	150.3054	10.91	294.8658
	12.310	125.0107	157.0747	10.92	315.7676
	12.320	131.317	163.8367	10.93	338.967
	12.330	138.27	170.6188	10.94	363.6338
	12.340	145.0381	177.7576	10.95	388.5918

TABLE A2 Break even fee values for GBM model (in basis points)

Assymetric GARCH (without surrender)					
Age	g (in %)	S&P	MSCI	g (in %)	Nikkei
60	8.245	122.8001	122.7904	7.585	251.3454
	8.255	127.4173	127.4519	7.595	264.1108
	8.265	132.0813	132.0826	7.605	278.2909
	8.275	136.7881	136.795	7.615	293.4397
	8.285	141.7878	141.7891	7.625	308.9133
65	12.3	224.2124	224.2196	10.91	494.3991
	12.31	230.7736	230.7679	10.92	514.4345
	12.320	237.3823	237.381	10.93	535.3255
	12.330	244.0496	244.0468	10.94	559.5356
	12.340	250.7659	250.7677	10.95	584.2721

APPENDIX B. GEOMETRIC BROWNIAN MOTION MODEL

We also investigated returns modelling with the GBM model to demonstrate the importance of variable volatility. The conservative GBM model assumes that return volatility remains constant throughout time. Under a risk-neutral metric, the stock price dynamics for a GBM are:

$$S_t = S_0 e^{\left(r - \frac{1}{2}\sigma^2\right)t + \sigma B_t} \quad (\text{B1})$$

where r is the continuously compounded risk-free rate, σ^2 is the variance and B_t is a Wiener process (Figure B1).

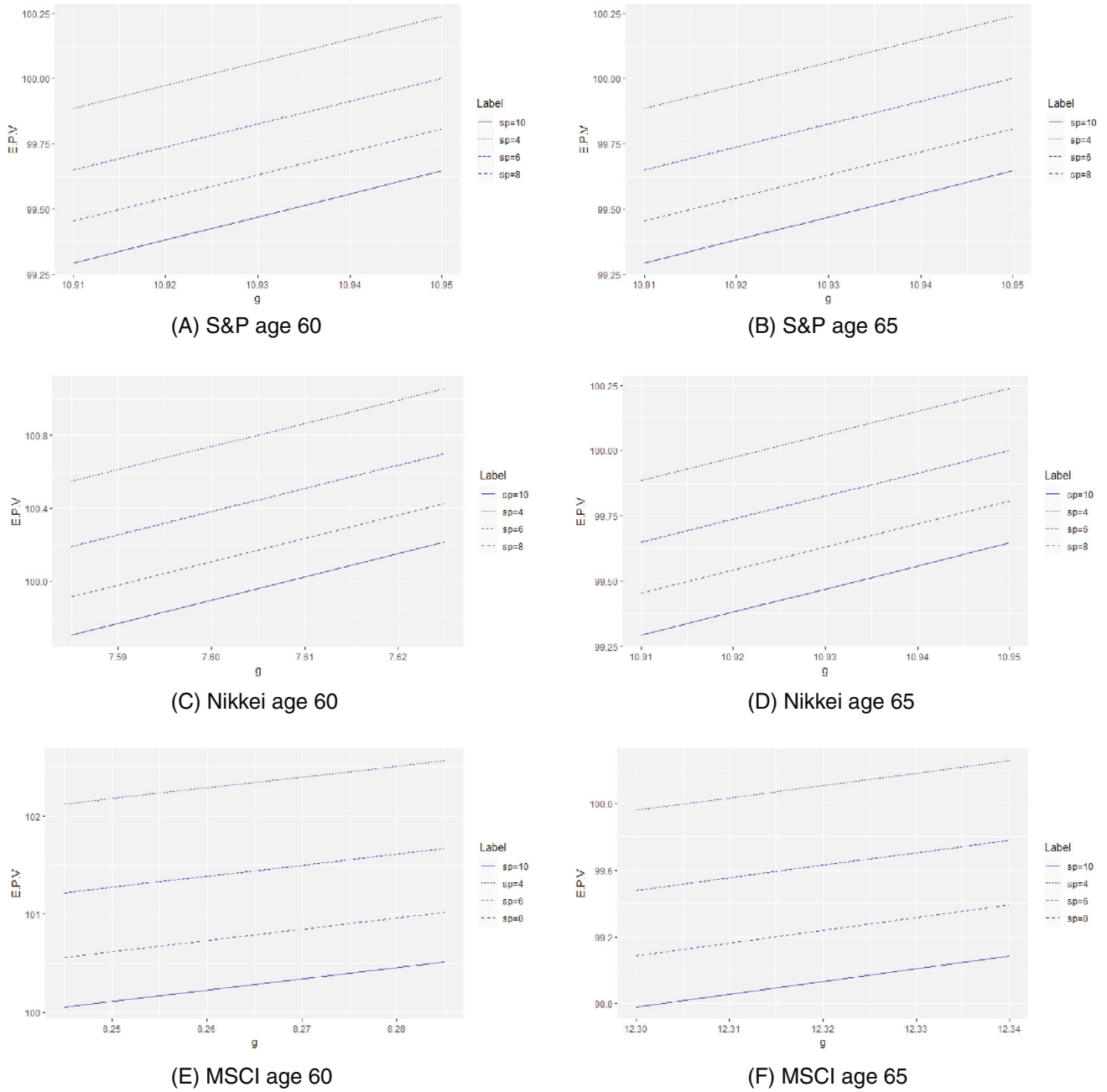


FIGURE B1 Plot of EPV versus surrender charges

APPENDIX C. RETURNS PROPERTIES AND BEHAVIOR

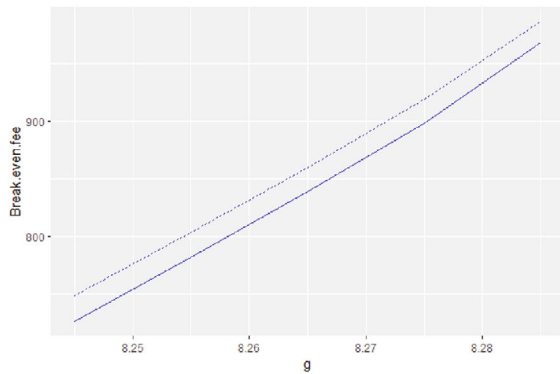
Stylized features in return and log-return series:

This section will discuss the stylistic elements of the log-return and return series. A few of them are stationarity, volatility clustering, leverage impact, and conditional heteroscedasticity.

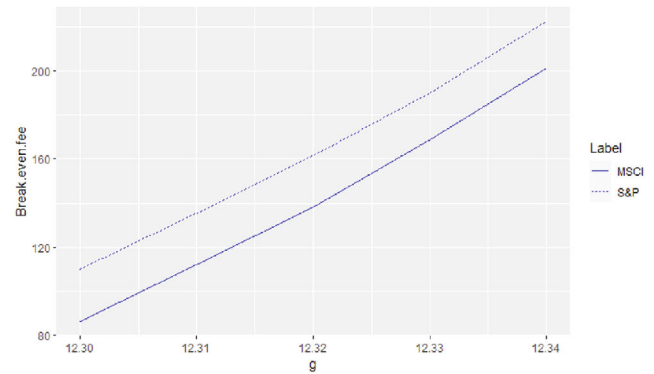
Non-stationarity: Financial price series data is not usually stationary. Differencing, taking logarithms, and log-differencing are the most frequent strategies for stationary time-series data. We looked at log-returns and returns in this paper, which is the differencing and log-differencing of the stock price series S_t and the interest rate series R_t .

$$X_t = \log(S_t) - \log(S_{t-1}), \quad (C1)$$

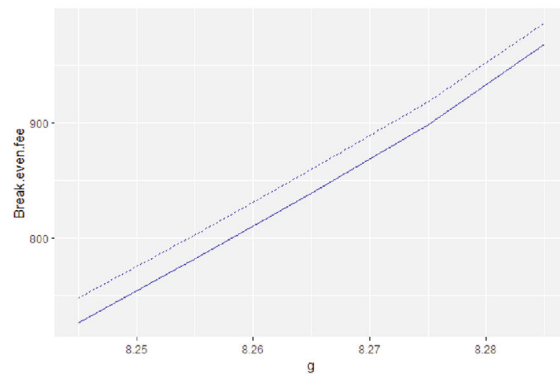
$$r_t = R_t - R_{t-1}. \quad (C2)$$



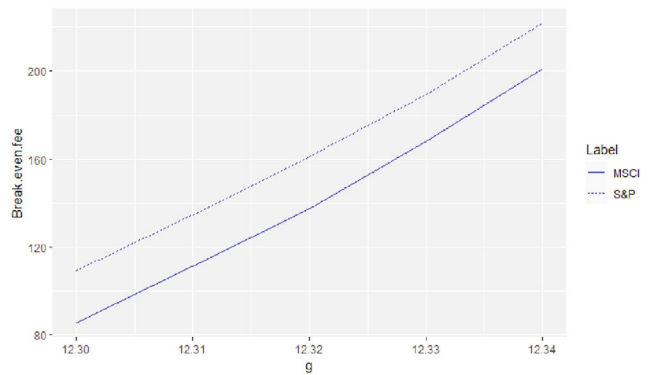
(A) S&P and MSCI age 60 without surrender



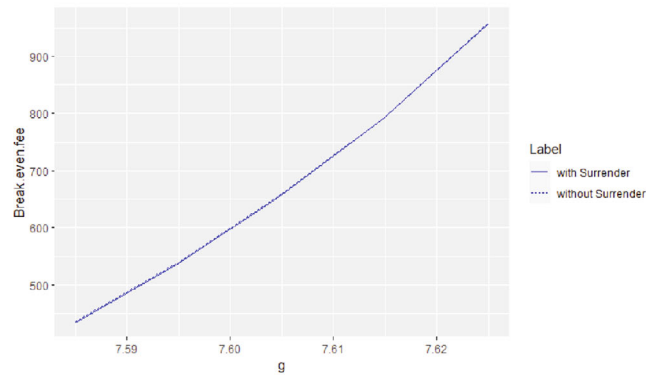
(B) S&P and MSCI age 65 without surrender



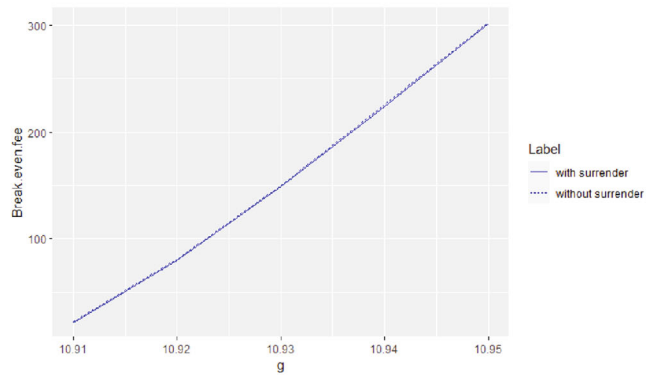
(C) S&P and MSCI age 60 with surrender



(D) S&P and MSCI age 65 with surrender



(E) nikkei age 60



(F) nikkei age 65

FIGURE C1 Break even fee versus g

Conversion of non-stationary data into stationary data takes place through this process. The time series plots of the historical returns of the five datasets are shown in Figures C1, C2. Positive returns are accompanied by higher positive returns, while negative returns are followed by higher negative returns, as shown in Figure C2. Volatility clustering is the term for this occurrence. The observations are supported by the series stationarity as determined by the Augmented-Dicky-Fuller (ADF) test. The ADF test's alternate hypothesis is that the series is stationary. Table C1 demonstrates that the ADF test has a p-value of less than equal to 0.01 (less than 1%, 5%, and 10%), indicating the rejection of the null hypothesis at the 99 percent level of significance. As a result, evidence for the five datasets stationarity is provided.

Non-normality: Under the GBM assumption of constant volatility, daily log-returns and returns are considered independent and identically distributed. As indicated in Table C1 by kurtosis and skewness, empirical distributions exhibit

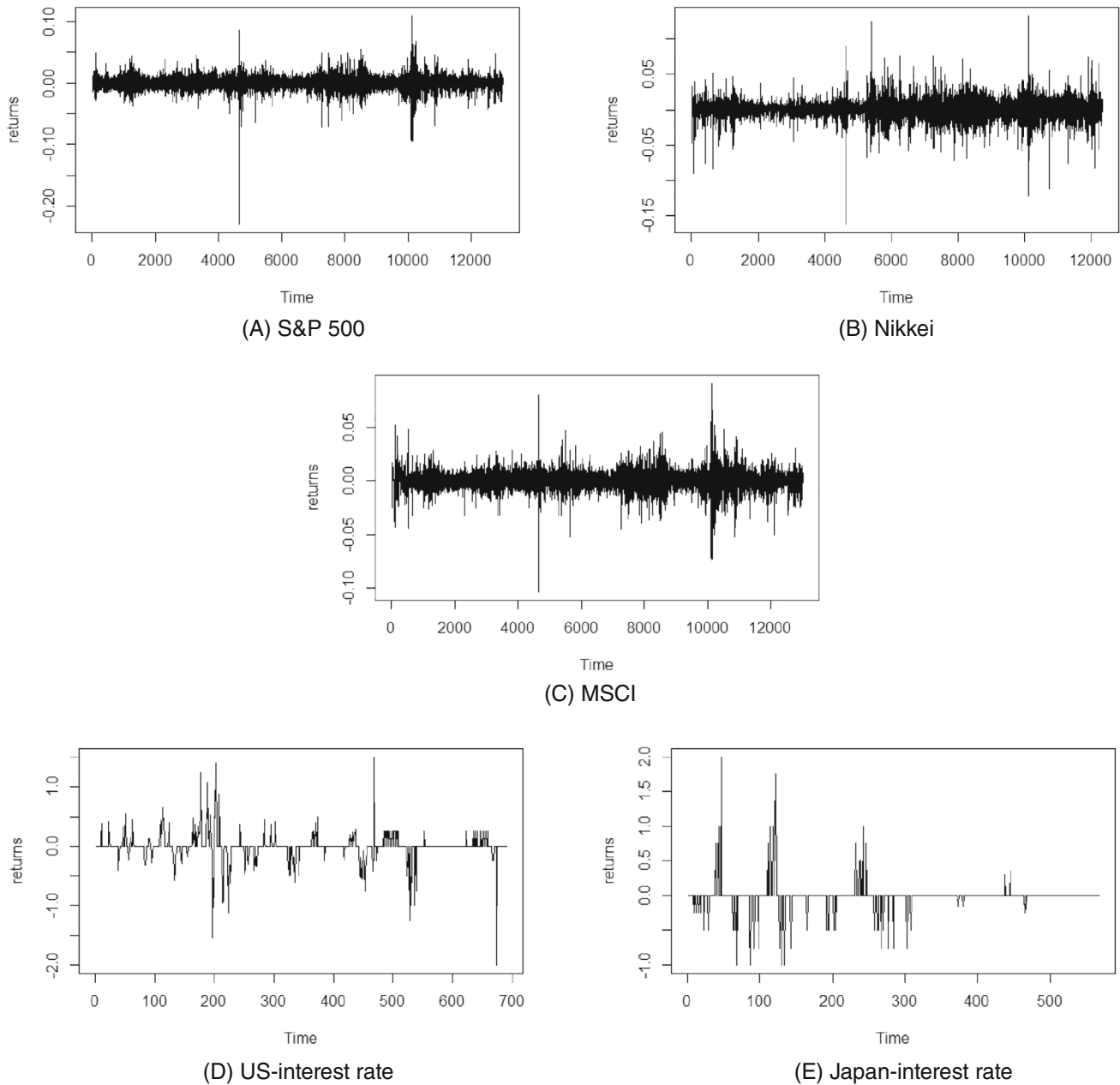


FIGURE C2 Returns Plot

heavy tails and large peaks near the center compared to the normal distribution. We used the Jarque-Bera (JB) test to rule out the possibility that the returns are non-normally distributed. The JB test's resultant p-values are zero, conclusively rejecting the null hypothesis, as shown in Table C1. As a result, there is no normal distribution in the datasets.

Auto-correlation: Auto-correlation is the relationship between a time series and a delayed version of itself. When the auto-correlation of a series is more significant than zero, it signifies that the data values are not independent of preceding information.

The stock returns are not unaffected by past advances and have considerable auto-correlation. Engle's ARCH test is used to confirm the presence of the ARCH effect, with the null hypothesis being "there is no ARCH effect." Table D1 shows the results of Engle's ARCH test on the five datasets. Engle's ARCH test provides p-values of 0 for all analyzed lags. So, the null hypothesis of no ARCH effect is rejected, and the ARCH effect is present in all three datasets.

TABLE C1 Data statistics

	US-IR	Japan-IR	S&P 500	Nikkei	MSCI
Mean	-0.004703	-0.010494	0.000271	0.000175	0.000240
Minimum	-2	-1	-0.228997	-0.161354	-0.103633
Maximum	1.5	2	0.109572	0.132346	0.090967
Standard deviation	0.242408	0.22488	0.010385	0.012573	0.008291
Skewness	-1.00386	1.84959	-1.013450	-0.427521	-0.490187
Kurtosis	16.76163	26.6016	26.1116752	10.109685	-0.490187
<i>p</i> value (ADF test)	0.01	0.01	< 0.01	< 0.01	< 0.01
<i>p</i> value (JB test)	0	0	0	0	0

Leverage effect: Compared to a similar gain, the leverage effect causes future volatility to rise, followed by a loss. As demonstrated in the returns plots of Figure C2, stock volatility tends to be higher, corresponding to a negative shock in the returns instead of a positive shock of the same magnitude. Hence, asset returns and changing volatility negatively affect the datasets, leading to the leverage effect. As a result, conditional heteroscedasticity models must be examined to account for the leverage effect.

Structural breakpoints: Breakpoints are significant data trends that cause the fitted model's parameters to shift. There is a high possibility of breakpoints because the daily data of the returns spans such an extended period. The breakpoints function in the Strucchange package was used to find breakpoints in the five datasets (Zeileis et al.¹⁷), followed by SC-test to confirm the presence of structural breaks in the data. Sctest is a generic function used to perform/extract structural change tests on various objects. We conducted structural change tests based on F statistics with the null hypothesis of no structural breakpoint. Table D2 shows the *p*-value for each of the five datasets and the breakpoints. The ADF test has a *p*-value of less than 0.01, suggesting that the null hypothesis is rejected at the 99% significance level, as shown in Table D2. As a result, evidence supporting the presence of structural breakpoints is offered in five datasets.

APPENDIX D. VOLATILITY MODELLING

There are several stylized features, namely: Volatility clustering, leverage effect, auto-correlation among the squared and absolute values, and so forth, which are often shown by the log-returns of share prices, money market, interest rates, and stock indices. A typical model among the various proposed models to capture these features is

$$X_t = \mu + \xi_t, \quad (D1)$$

$$\xi_t = \sigma_t z_t, \quad (D2)$$

here z_t are i.i.d random variables following normal distribution (i.e., $\{z_t\} \sim \text{i.i.d } N(0, 1)$) and $\{\sigma_t\}$ is a non-negative process also known as volatility process.

z_t and σ_t are independent for fixed t . The stationarity of X_t and σ_t is assumed to be strict, and μ is a constant which can be estimated from the data. This volatility plays a crucial role in asset pricing models that are often considered constant.

TABLE D1 Engle's ARCH test *p* values

Lag	S&P 500	Nikkei	MSCI	US-IR	Japan-IR
5	0	0	0	0.005	0
7	0	0	0	0.003	0
12	0	0	0	0.001	0
15	0	0	0	0.003	0
25	0	0	0	0.008	0.0001

TABLE D2 Structural breakpoints and Structural change test p values

S&P 500	Nikkei	MSCI	US-IR	Japan-IR		p -value
3290	1950	2653	168	87	S&P	0
6035	3265	4222	271	192	Nikkei	0
7449	5216	7887	434	284	MSCI	0
9008	7019	9108	537	369	US-IR	0
—	8129	—	—	—	Japan-IR	0

However, this is not true in real life. In the year 1982, Engle¹⁸ proposed a model for $\{\sigma_t\}$, According to which volatility process is given by

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^p \alpha_i \xi_{t-i}^2. \quad (\text{D3})$$

This model is known as Auto-Regressive Conditional Heteroskedasticity (ARCH process). Later on, the generalized version of the ARCH model was given by Bollerslev,¹⁹ Known as the Generalised ARCH model (GARCH). The volatility process under GARCH(p,q) is given by

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^p \alpha_i \xi_{t-i}^2 + \sum_{j=1}^q \beta_j \sigma_{t-j}^2. \quad (\text{D4})$$

There are two types of GARCH models: symmetric and asymmetric. Because conditional volatility in symmetric models is believed to be a function of the size of past values rather than the sign (Tables D1 and D2).