



Pricing Variable Annuity Contract with GMAB Guarantee Under a Regime Switching Local Volatility Model

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Abstract

This article investigates the pricing of variable annuity guarantees, with particularly emphasising on the Guaranteed Minimum Accumulation Benefit (GMAB) with several embedded options, like roll-up and ratchet. A continuous time model that integrates regime switching dynamics into the local volatility is proposed for the contract's pricing. As local volatility pioneered by Deelstra and Rayée (2013), on their proposed model by explicitly introducing the regime switching components into the local volatility. Model calibration and valuation are performed through the application of Monte Carlo simulations and Gaussian Process Regression methodologies. For the illustration of the capability of this model and some possible improvements in the model, a numerical and sensitivity analysis is conducted. Uses an empirical test on historical S&P500 Index data to illustrate its practical implications. The results showed that time to maturity, interest rates, and volatilities significantly impact the pricing, and it also illustrates the proposed approach's accuracy and computational efficiency.

Keywords Regime switching · Local volatility · Machine learning · Variable annuity guarantees · Monte Carlo method

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1 Introduction

Variable annuities (VAs) are unit-linked managed funds that offer several guarantees to their beneficiaries, including death benefit protection, living benefit protection, and lifetime income protection. “GMxB” is a common abbreviation for these guarantees, where x represents the kind of benefit. Some of the common guarantees are guaranteed minimum death benefit (GMDB), guaranteed minimum accumulation benefit (GMAB), guaranteed minimum withdrawal benefit (GMWB), and guaranteed minimum income benefit (GMIB). Graphical illustrations and practical examples of each benefit can be found in Ledlie et al. (2008). In the last ten years, variable annuities have become essential to the insurance market worldwide. They have surpassed fixed annuities in popularity as a safe investment option. In particular, the guaranteed minimum accumulation benefits (GMAB), introduced in 1990 (refer Ledlie et al. (2008)), guarantees policyholders a minimum account value at contract maturity.

In most cases, the guaranteed payout equals the account value paid by policyholders at purchase. As a result of intense competition, insurers have increased the scope of these guarantees by adding some extra benefits. There are various ways the benefit base may be reset on anniversary dates (step-up or ratchet). If the market has done well, it can roll up at a defined rate (roll-up rate) regardless of market performance (e.g., refer Bauer et al. (2008)). Some of the technical challenges that these products create are design, pricing, valuation, and risk management.

A fundamental modeling aspect involves capturing the underlying asset's volatility. One of the popular models used for volatility modeling is the local volatility model. Local volatility models were introduced by Dupire (1994) and extended by Deelstra and Rayée (2013) for the pricing of long-dated foreign exchange (FX) derivatives, where they considered three sources of randomness (i.e., the three-factor model): the FX spot rate with local volatility, the domestic interest rate, and the foreign interest rate. These models are more suitable for pricing and hedging strategies. The primary benefit of local volatility models over stochastic volatility models is that the volatility is a deterministic function of the underlying spot price and time, hence avoiding the problem of pricing in incomplete markets (refer Deelstra and Rayée (2013)). Secondly, the local volatility function is defined in terms of implied volatilities or market call prices, and the calibration is performed directly on the surface of implied volatility. Therefore, local volatility models often represent the surface of implied volatilities with more precision than stochastic volatility models. Moreover, the calibration of local volatility models is easy and fast since the underlying price is the only source of randomness, and it can perfectly match any arbitrage-free set of European option prices. Also, the market becomes complete when adopting local volatility models, and thus, any contingent claim can be perfectly replicated with a portfolio consisting of the underlying and bond only. It is worth noting that the work by Deelstra and Rayée (2013) did not consider the regime switching component with the local volatility model. This article aims to fill this gap by incorporating regime switching dynamics in the local volatility, which improves the model's realism and applicability in dynamic market environments.

Considering different economic regimes, such as uptrend and downtrend, is crucial for accurately modeling these long-dated products. The fact that the model's constant parameters don't match market data is another reason to include a regime switching parameter. Several authors in actuarial science have investigated the application of regime switching models within a continuous-time Markov switching framework. While not providing an exhaustive list, notable references include Hardy (2001), Elliott et al. (2005), Siu (2005), Lin et al. (2009), Fan et al. (2015), and Ignatieva et al. (2016). These studies commonly employ diffusion processes influenced by a continuous-time Markov chain to model interest rates and/or the reference risky asset. Elliott et al. (2005) apply the regime switching model for option pricing. They evaluated option price when underlying risky assets are driven by Markov-modulated Geometric Brownian Motion (GBM). In other words, the market characteristics, such as the market interest rate, the appreciation rate, and the volatility of the underlying risky asset, depend on unobservable economic conditions modeled by a continuous-time Hidden Markov chain. As the market provided by the Markov-modulated GBM model is incomplete in general and the martingale measure is thus not unique, they used a regime switching random Esscher transform to establish an equivalent martingale pricing measure.

Fan et al. (2015) investigated the valuation of point-to-point equity-indexed annuities (EIAs), annual ratchet EIAs, and GMDB in the context of variable annuities using a double regime switching model. However, their research work did not consider local volatility with regime switching, a key point we incorporate into this research work. This analysis enhances the clarity of pricing challenges in variable annuity contracts and provides significant insight into the methods used to reduce risk. Economic considerations are one of the significant reasons to utilize Markov chains to accommodate rapid changes in volatility. Many researchers have previously studied the time-dependent model, including Costabile (2017) and others. Additionally, the motivation for incorporating regime changes stems from the observed dynamics in real-world markets, as introduced by Hamilton (1990). However, some evidence supports that asset prices or the volatility process connected with them should reflect a shifting regime, providing better performance in the real market, as outlined in Elliott et al. (2016). These authors used the statistical study of Maghrebi et al. (2014) as an example of why models should contain two regimes as part of a risk-neutral approach.

In this direction, Elliott et al. (2015) obtained the Dupire formula for the price of a European call option over different states of the economy when the price dynamic of the underlying assets follows a regime switching local volatility model. They used a regime switching Esscher transform to determine a price function for valuation under incomplete market conditions. A regime switching version of the Dupire formula is used to obtain a system of coupled forward equations. van der Stoep et al. (2017) provides a Monte Carlo simulation framework for assessing hybrid local volatility models.

To the best of our knowledge, applying a regime switching local volatility model modulated by a continuous-time Markov chain has not been explored in the valuation framework of equity-linked life insurance contracts like VAs. Table 1 compares existing literature with various models and valuation techniques. This model more

Table 1 Comparison with the existing literature

Existing Work	Model	Regime switching	Local Volatility	Machine Learning
Hardy (2001)	Regime switching Lognormal Model	Yes	–	–
Elliott et al. (2005)	Geometric Brownian Motion (GBM)	Yes	–	–
Siu (2005)	Geometric Brownian Motion (GBM)	Yes	–	–
Lin et al. (2009)	Generalized Geometric Brownian Motion	Yes	–	–
Fan et al. (2015)	Double Regime switching Model with Jump	Yes	–	–
Ignatieva et al. (2016)	Geometric Brownian Motion (GBM) with Stochastic Mortality	Yes	–	–
Cui et al. (2017)	Stochastic Volatility Model with Jumps	Yes	–	–
Kang et al. (2022)	Generalised Regime switching (GRS) Models	Yes	–	–
Deelstra and Rayée (2013)	Geometric Brownian Motion (GBM)	–	Yes	–
Proposed Work	Geometric Brownian Motion (GBM)	Yes	Yes	Yes

accurately reflects the dynamic nature of financial markets by accounting for regime switching dynamics and local volatility adjustments. This added realism enhances the model's ability to capture the complexities of market behavior.

The regime switching local volatility model introduces flexibility to VAs pricing and risk management strategies by recognizing the changing nature of market regimes. Its ability to capture the dynamics of market conditions, such as shifts in volatility, enables VAs providers to optimize their strategies, adjusting the risk mitigation intensity during periods of market instability and adopting a more conservative approach in stable markets. The regime switching local volatility model offers improved accuracy in pricing variable annuity guarantees, reduces the risk of underestimating or overestimating liabilities, and provides the flexibility to choose appropriate risk mitigation instruments based on the prevailing market regime. Previous research has either concentrated on regime switching models or local volatility models independently in the insurance sector. This research work is among the first to expressly incorporate these two approaches to price annuity guarantees, highlighting the novelty of this combination. The integration of local volatility with regime switching creates a model that can handle both continuous market fluctuations and discrete structural changes. This proposed model offers a more robust framework for pricing annuity guarantees, especially in markets characterized by volatility clustering and regime shifts.

The valuation framework for Guaranteed minimum benefits (GMBs) under the regime switching Log-Normal model for fund dynamics and stochastic mortality was developed by Ignatieva et al. (2016). The authors used the Fourier Space Time-stepping (FST) algorithm for pricing and hedging. Cui et al. (2017) introduces an innovative and effective transform-based approach for valuing equity-linked

annuities (ELAs). This method encompasses equity-indexed annuities (EIAs) and cliquet-style payoff structures widely utilized in the insurance market. The pricing framework accommodates a broad range of stochastic volatility models with jumps and regime switching Lévy model. They considered the models, which include exponential Lévy processes, regime switching Lévy processes, and stochastic volatility models with various jump size distributions. However, Cui et al. (2017) did not incorporate the local volatility model with regime switching, which we investigate in this research work.

De Spiegeleer et al. (2018) applied a machine-learning approach to speed up derivative pricing in their research work. The authors construct the training set across a grid and then interpolate at the test points using the Gaussian process regression (GPR). In addition to approximating the implied volatility surface, they illustrate the speed-up of the Gaussian Process compared to the Monte Carlo approaches and the acceptable accuracy loss applied to pricing and Greek estimates using the Heston model. Additionally, GPR shows better results than cubic spline interpolation, a commonly used numerical approximation technique for efficient point estimation. Regarding the sensitivity needed to hedge a VAs portfolio, employing a two-step meta-modeling approach suggested by Gan and Lin (2017) is best. More recently, Kang et al. (2022) provided a flexible valuation method for variable annuity contracts with guaranteed minimum maturity benefit (GMMB) riders built on an underlying fund that develops according to a generalized regime switching framework. In their study, several sensitivity experiments have been provided to evaluate how variable annuity prices react to changes in critical underlying factors such as contract maturity, guarantee level, and rate matrix.

The theoretical foundation of the regime switching local volatility model lies in its integration of regime switching dynamics and local volatility. It offers a comprehensive framework to capture the dynamic behavior of financial markets (e.g., refer Buffington and Elliott (2002)). The model incorporates a continuous-time Markov chain to represent varying market regimes. This integration is mathematically expressed through a stochastic differential equation, allowing for the simultaneous evolution of the underlying asset and its volatility in different regimes. Calibration involves estimating the model parameters from the real-world data S&P500 index, typically done through Monte Carlo (MC) simulations and GPR.

Insurance companies mainly trade the GMAB guarantee, a product more complex in its payoff compared to pure guaranteed annuity products. As a result, finding a closed-form solution for its price under this proposed model may not be possible. Numerical methods such as MC simulations are used to evaluate GMAB guarantees. The Monte Carlo approach for the valuation of variable annuity guarantees used by Bacinello et al. (2011) is considered for pricing contracts and calibration of regime switching local volatility function. Euler's discretization was often employed on a finite time frame in the Monte Carlo simulations to evaluate variable annuities. Euler's discretization method imitates a Gaussian process with a high chance of turning negative at every time grid selection. In practice, this is addressed by imposing a reflective or assimilating barrier on the process by terminating it at zero. We used the same methodology for the Monte Carlo simulation, which explains our numerical findings.

Also, to speed up the calculation, we use Gaussian processes regression, a machine learning technique in pricing and calibration. Williams and Rasmussen (2006) provide a comprehensive introduction to Gaussian process regression or Gaussian processes (GPs). In contrast to other machine learning approaches, such as neural networks and support vector machines, which only give point estimates, GPs measure the uncertainty of their predictions. The availability of an effective training procedure for the model's hyper-parameters is a further reason for using GPs. In addition to a variety of advantageous statistical and mathematical qualities, including universality, the implementation support infrastructure is established and given by open-source packages for machine learning such as *GpyTorch*, *scikit-learn*, *Edward*, and *STAN*.

Integrating Monte Carlo simulation and Gaussian process regression has significant advantages compared to existing methods in the literature. This approach is employed for pricing and calibration within a regime switching local volatility model. Although finite difference approaches are commonly employed, they may face challenges when dealing with the variations that arise from regime switching dynamics. This approach distinguishes itself by obtaining an acceptable balance between accuracy and computational speed, offering a robust solution that efficiently handles the difficulties presented by regime switching dynamics.

Under this framework, a comprehensive sensitivity analysis of VAs contract pricing about different model parameters is also provided. All trends noticed are under financial intuition. In addition, our numerical studies indicate that less frequent structural changes have a more significant effect on VAs pricing than more frequent parameter changes. This is substantial evidence that the regime switching models are better appropriate for VAs pricing.

The remainder of the article is organized as follows: Sect. 2 compares the local volatility model without regime switching and with regime switching. Section 3 presents a GMAB contract and its pricing. In Sect. 4, we describe the specification of the regime switching local volatility model. Section 5 explains the calibration of the model using the GPR and Monte Carlo methods and describes its application in this context. Section 6 presents the numerical and sensitivity analysis. Finally, we discuss concluding remarks and future work in Sect. 7.

2 Local Volatility Without Regime Switching vs. Local Volatility with Regime Switching

In this section, we provided a detailed comparison between the local volatility model without regime switching and local volatility with regime switching. We used the hedging error metric to show graphically how the local volatility without regime switching model failed to capture the various shocks, reducing the product's flexibility and making it less realistic. Hedging error is a key metric in risk management that measures the difference between the actual returns of a hedged portfolio and the expected returns if the hedge were perfect. It essentially quantifies the imperfections in a hedging strategy. Hedging error demonstrates the

model's effectiveness in adapting to sudden market changes, showing how well it adjusts its hedging strategy in response to different market states.

As illustrated in Fig. 1, when regime switching is involved, the hedging error exhibits greater variability but responds more dynamically, leading to a reduced overall error when compared to the local volatility model alone. On the other hand, due to its incapacity to adjust to abrupt changes in the market, the hedging error rises gradually over time in the case of local volatility without regime switching. The regime switching local volatility model dramatically decreased hedging errors, as seen in the graph. Regime switching enables the model to account for both long-term economic trends and volatility changes, making it more capable of adapting to shifting market conditions. As a result, the regime switching local volatility model produces a more precise and dependable hedging strategy, making it a superior choice for pricing annuity guarantees.

By combining local volatility with regime switching, this proposed model captures both short-term volatility changes and long-term economic shifts. Local volatility allows for more precise modelling of the asset price path while regime switching accounts for changes in economic conditions. This combined approach enhances the accuracy and robustness of pricing annuity guarantees. Thus, combining a local volatility model and a regime switching model to formulate a hybrid model, referred to as the so-called local regime switching model in literature, is an elegant idea, trying to take advantage of some nice features associated with each model standing alone (refer Elliott et al. (2015)).

Our proposed model, in comparison to the one presented in Deelstra and Rayée (2013), exhibits significantly lower hedging errors owing to its ability to adapt to different market regimes. This capability ensures a more reliable risk management strategy, highlighting this proposed model's robustness and practical relevance compared to the earlier model discussed in the literature (refer Cui et al. (2017), and Fan et al. (2015)).

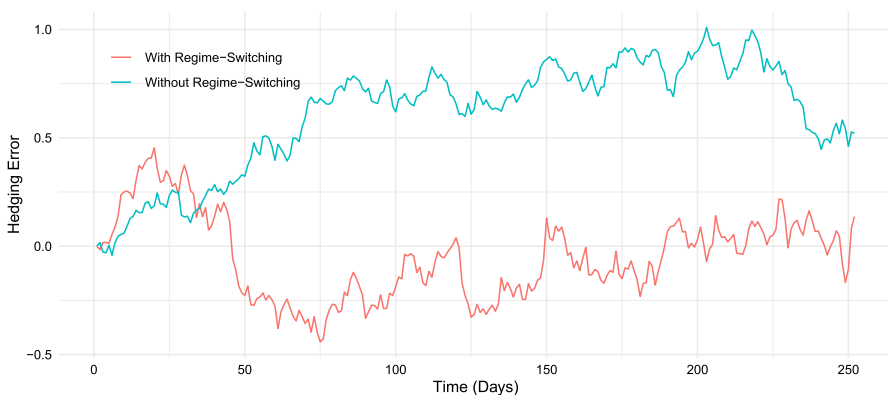


Fig. 1 Hedging Error: Local volatility model without regime switching vs. Local volatility with regime switching

3 The GMAB Contract and its Pricing

In this section, we provide a detailed discussion of the GMAB contract. We follow the contract presented in Krayzler et al. (2016), adding additional benefits like roll-up, ratchet, and death benefits.

Let us consider the below-mentioned notations, which will be followed in the contract and model description, with the time horizon $T = [0, \infty)$, and set of discrete-time points t_i , for $i = 1, 2, \dots, N$, which are referred as contract anniversary time points.

Notations

$F(t)$	the insured's fund value at time $t \in [0, T]$
$G(t)$	guaranteed amount at time $t \in [0, T]$
$S(t)$	risky asset price at time $t \in [0, T]$
$r(t)$	the instantaneous continuously compounded interest rate at time $t \in [0, T]$
$\sigma(t, S(t))$	regime switching local volatility of the risky share at time $t \in [0, T]$
x_0	insured's age at the start of the policy
q_i	insured's probability of death at time t_i
p_i	insured's alive probability at t_i
τ	time of the policyholder's death

A policyholder who is x_0 years old at the time of inception and has a GMAB contract is considered to outline our problem. To initiate the policy at time $t = 0$, the insured pays an initial payment, which is then invested in assets associated with risk, most often mutual funds. Assume that the insurer continually deducts a proportion of $g > 0$ from the account balance for the guarantees. Then, the insured's fund value $F(t)$ at time $t \in [0, T]$ is given by:

$$\frac{F(t)}{F_0} = e^{-gt} \frac{S(t)}{S_0}, \quad (1)$$

where $F_0 = F(0)$ and $S_0 = S(0)$ are the fund value and risky asset price at time $t = 0$ respectively. Because the VAs contract ensures the minimal accumulated benefit, the insurance company will pay to the policyholder's $\max(F(T), G(T))$ at maturity time T .

Assuming a set of discrete-time points t_i , for $i = 1, 2, \dots, N$, which are referred to as contract anniversary time points. Contract renewal, roll-up, and ratchets may occur at these time points. Let us denote $(F(t_i)^-, G(t_i)^-, t_i)$ as the policy state of the system just before i^{th} anniversary happens at time t_i , and $(F(t_i)^+, G(t_i)^+, t_i)$ the same state values just after the anniversary happens. Furthermore, let us consider that the renewal of the contract occurs at time t_i , if $F(t_i) > G(t_i)$, the guarantee amount is then adjusted to equal the account's value at t_i . On the contrary, if $G(t_i) > F(t_i)$, then the difference is paid into the fund by the insurance company so that the account value and guaranteed amount are equal at the beginning of the next period. This procedure is repeated at each time step t_i .

If the guarantee surpasses the fund's value at the period T , the insurer must pay the difference at contract maturity T . After the renewals, the contract fund amount is revised as follows:

$$F(t_i)^+ = F(t_i)^- + H(t_i), \quad (2)$$

where $H(t_i)$ is the payout at the event time t_i . Furthermore, if the insured does not intend to renew the contract before its expiration date T , then at maturity, the policyholder will get one payout of $\max(G(T) - F(T), 0)$.

If the contract has a roll-up benefit, the guaranteed amount G is raised by the predefined roll-up rate c percentage. The guaranteed amount G will never decrease unless the contract is partially or fully withdrawn. Then, the contract guaranteed amount at time $t_N = T$ is as follows:

$$G(T) = F_0 e^{cT}, \quad (3)$$

where F_0 denotes the initial premium of the contract. If the contract has a ratchet option, then the guaranteed amount G at time point $t_N = T$ will be the maximum of fund values at all the previous time points, which is given by the following equation:

$$G(T) = \max_{0 < t_1 < \dots < t_N = T} F(t_i). \quad (4)$$

For the death benefit, the guarantee amount that will be paid out at death is a roll-up of the initial investment, which is accumulated annually at a rate of c percentage each year. If the policyholder dies at event times t_i , the death benefit is paid to the death benefit account and compounded at the risk-free rate until maturity T :

$$DB(t_i)^+ = DB(t_i)^- + \max(G(t_i)^-, F(t_i)^-). \quad (5)$$

Since after death, no future benefits are possible, therefore $F(t_i)^+ = 0$ and $G(t_i)^+ = 0$. If the insurer is still alive at the time of maturity T , he/she will get the final payoff (FP) determined by:

$$FP = \max(F(T), G(T)). \quad (6)$$

It is common for insurers to analyze and replicate their policy pool using deterministic mortality rates. This article also takes advantage of this assumption by modeling future death rates as a determinate curve, assuming independence between mortality and the financial component.

4 Model Specification

This section presents the model specification for analyzing the GMAB price with regime switching local volatility. Consider a financial model consisting of a bond B and a share S with the time horizon $T = [0, \infty)$. These assets are tradeable continuously over time T . Suppose $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ be a filtered probability space satisfying the standard filtration conditions of completeness and right-continuity,

with $\mathcal{F}_0 = \{\emptyset, \Omega\}$, and P is a real-world probability measure. Using boldface letters for vectors (or matrices) and scalars to emphasize the distinction between vectors (or matrices). Suppose $\mathbf{X} = \{\mathbf{X}(t), t \in T\}$ is a continuous-time, finite-state, irreducible homogeneous Markov chain on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ with a finite state space $\mathcal{S} = \{s_1, s_2, \dots, s_N\}$. Without loss of generality, we can identify the state space of the chain $\mathbf{X}$ with the finite set of unit vectors $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_N\}$, where $\mathbf{e}_i = (0, \dots, 1, \dots, 0)' \in \mathbb{R}^N$. In this research work, the economic states are represented by the process $\mathbf{X}$. For example, with $N = 2$ states, s_1 corresponds to a good economy, and s_2 represents a bad one.

Write $Q = [q_{ij}]_{i,j=1,2,\dots,N}$ for the rate matrix of the chain $\mathbf{X}$. Elliott et al. (1995) obtained the semi-martingale representation for the chain $\mathbf{X}$ given by the following Eq. (7):

$$\mathbf{X}(t) = \mathbf{X}(0) + \mathbf{M}(t) + \int_0^t \mathbf{Q}' \mathbf{X}(u) du, \quad t \in T. \quad (7)$$

Here $\mathbf{M} = \{\mathbf{M}(t), t \in T\}$ is a martingale with respect to the right continuous, complete, filtration generated by the chain $\mathbf{X}$ and under the measure P , and $\mathbf{Q}'$ is the transpose of Q . The instantaneous continuously compounded interest rate $\{r_t\}_{t \in T}$ of the bank account is given by:

$$r(t) = \langle \mathbf{r}, \mathbf{X}(t) \rangle, \quad (8)$$

where $\mathbf{r} = (r_1, r_2, \dots, r_N)' \in \mathbb{R}^N$ with $r_i > 0$ for each $i = 1, 2, \dots, N$ and $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^N$. Then the dynamics $\{B(t), t \in T\}$ of the bank account evolve over time as:

$$dB(t) = r(t)B(t)dt, \quad B(0) = 1. \quad (9)$$

Suppose $\mu(t)$ denotes the appreciation rate of the risky share at time t . Then, the Markov chain $\mathbf{X}$ determines $\mu(t)$ as:

$$\mu(t) = \langle \boldsymbol{\mu}, \mathbf{X}(t) \rangle, \quad t \in T.$$

Here, $\boldsymbol{\mu} = (\mu_1, \mu_2, \dots, \mu_N)' \in \mathbb{R}^N$, for each $i = 1, 2, \dots, N$; μ_i is the appreciation rate of the risky share corresponding to the i^{th} state of the economy. Let $\sigma(t, S(t), \mathbf{X}(t))$ denote the regime switching local volatility of the risky share at time t . The chain then determines $\sigma(t, S(t), \mathbf{X}(t))$ as:

$$\sigma(t, S(t), \mathbf{X}(t)) = \langle \boldsymbol{\sigma}(t, S(t)), \mathbf{X}(t) \rangle, \quad t \in T.$$

Here, $\boldsymbol{\sigma}(t, S(t)) = (\sigma(t, S(t), \mathbf{e}_1), \sigma(t, S(t), \mathbf{e}_2), \dots, \sigma(t, S(t), \mathbf{e}_N))' \in \mathbb{R}^N$; $\sigma(t, s, \mathbf{x}) > 0$, for each $(t, s, \mathbf{x}) \in T \times \mathbb{R}^+ \times \mathcal{E}$.

Moreover, the transition between the two states follows an exponential distribution with parameter q_{ij} as:

$$P(t_{ij} > t) = e^{-q_{ij}t}, \quad i, j = 1, 2 \text{ and } i \neq j,$$

where q_{ij} is the transition rate from State i to j , and t_{ij} is the time spent in State i before going to State j .

Let $W = \{W(t), t \in T\}$ denote a standard Brownian motion defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ with respect to the $\mathbb{P}$ -augmentation of its natural filtration. Suppose that the chain $\mathbf{X}$ and the Brownian motion W are stochastically independent. Therefore, under $\mathbb{P}$ the evolution of the price of the risky share over time is given by the following Eq.:

$$dS(t) = \mu(t)S(t)dt + \sigma(t, S(t), \mathbf{X}(t))S(t)dW(t), \quad t \in T. \quad (10)$$

Write $F^W = \{F^W(t), t \in T\}$ and $F^{\mathbf{X}} = \{F^{\mathbf{X}}(t), t \in T\}$ for the right-continuous complete filtrations generated by the Brownian motion W and the chain $\mathbf{X}$, respectively.

Due to the extra source of uncertainty given by the Markov chain, the market under the regime switching model is incomplete. Consequently, there are several price kernels for appraisal. We use a regime switching version of the Esscher transform proposed by Elliott et al. (2005) to establish a price kernel for valuation and the risk-neutral price dynamics of the underlying risky share. Consider that $\tilde{\mathbb{P}}$ is the risk-free martingale measure chosen by the regime switching Esscher transform in Elliott et al. (2005). Given as:

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \frac{e^{\int_0^t \alpha(u)dW(u)}}{\mathbb{E}_{\mathbb{P}} \left[e^{\int_0^T \alpha(u)dW(u)} \mid \mathcal{F}^{\mathbf{X}}(T) \right]},$$

for each $t \in T$, and $\mathbb{E}_{\mathbb{P}}$ is an expectation under $\mathbb{P}$. Then under $\tilde{\mathbb{P}}$, the dynamic of underlying is given by the following Eq.:

$$dS(t) = r(t)S(t)dt + \sigma(t, S(t), \mathbf{X}(t))S(t)dW(t), \quad t \in T \quad (11)$$

with initial value

$$S(0) = S_0.$$

Since we assume that there are no jumps in stock price during a regime switch, the rate matrix of the chain $\mathbf{X}$ remains unchanged under $\tilde{\mathbb{P}}$, that is, the rate matrix of the chain $\mathbf{X}$ under $\tilde{\mathbb{P}}$ is $\mathbf{Q}$.

Let $\mathcal{P}(F, G, \sigma_i, t)$ be the value of the contract at time t , which is the average market price of the several policies provided to clients assuming F to be the fund value, G is the guarantee account value, and σ_i is the underlying fund's local volatility in regime i .

GMAB's fair price at time $t = 0$ can be calculated using the risk-neutral measure $\tilde{\mathbb{P}}$ as an expected value of the discounted future payoff, i.e.

$$\mathcal{P}(0) = \mathbb{E}_{\tilde{\mathbb{P}}} \left[e^{-\int_0^T r(u)du} \max(F(T), G(T)) \right] \mathbb{I}_{\{\tau > T\}}. \quad (12)$$

In the preceding expression, the indicator function $\mathbb{I}_{\{\tau > T\}}$ ensures that the insured is still alive at time T and $\mathbb{E}_{\tilde{\mathbb{P}}}$ denotes an expectation under the risk-neutral Esscher measure $\tilde{\mathbb{P}}$.

Certain basic findings on the occupancy time inside states for Markov chains are required for the zero-coupon bond. Following the same method as Ignatieva et al. (2016), apply for the valuation of the zero-coupon bond price. The chain $\mathbf{X}$ illustrates the two regimes that characterize the economy, as $\mathbf{X} = (e_1)'$ for regime 1 and $\mathbf{X} = (e_2)'$ for regime 2. Let us define $\mathbb{I}(t) = 0$ if $\mathbf{X}(t) = (e_1)'$ and $\mathbb{I}(t) = 1$ if $\mathbf{X}(t) = (e_2)'$. Then, the occupation time in regime 2 is $\mathbf{T}_{t,T}$ for the time period $[t, T]$ is given by:

$$\mathbf{T}_{t,T} = \int_t^T \mathbb{I}(s) ds,$$

and it holds that the time- t value of the zero-coupon bond is given by:

$$\begin{aligned} B(t, T) &= \mathbb{E}_{\mathbb{P}} \left[e^{-\int_t^T r(s) ds} \mid \mathcal{G}(t) \right] = \mathbb{E}_{\mathbb{P}} \left[e^{-(r_1 T_{t,T} + r_2 (T - T_{t,T}))} \mid \mathcal{G}(t) \right] \\ &= e^{-r_2 (T-t)} \mathbb{E}_{\mathbb{P}} \left[e^{-(r_1 - r_2) T_{t,T}} \mid \mathcal{G}(t) \right]. \end{aligned}$$

Darroch and Morris (1968) provide the moment-generating function of $\mathbf{T}_{t,T}$ as:

$$\phi_{t,T}(u) = \mathbb{E}_{\mathbb{P}} (e^{u T_{t,T}} \mid \mathcal{G}(t)) = \pi_t' e^{(T-t)(Q+uV)} \cdot \mathbf{v},$$

where π_t is the time- t vector of probabilities over states 1 and 2, $Q = \begin{pmatrix} -q_{12} & q_{12} \\ q_{21} & -q_{21} \end{pmatrix}$ is the rate matrix of chain $\mathbf{X}$, $V = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Then, the zero-coupon bond's time- t value may be expressed as:

$$\begin{aligned} B(t, T) &= e^{-r_2 (T-t)} \phi_{t,T}(-(r_1 - r_2)) \\ &= e^{-r_2 (T-t)} \pi_t' e^{(T-t)(Q-(r_1-r_2)V)} \cdot \mathbf{v} \\ &= e^{-r_2 (T-t)} \pi_t' \cdot \exp(\tilde{M}), \end{aligned}$$

where, $\tilde{M} = \begin{pmatrix} (-q_{12} - r_1 + r_2)(T-t) & q_{12}(T-t) \\ q_{21}(T-t) & -q_{21}(T-t) \end{pmatrix} \cdot \mathbf{v}$.

Having described the pricing structure for GMAB, we next present the machine learning approach for calibrating the local volatility surface.

5 Model Calibration

Practitioners are used to calibrating a model before they use it to price a derivative. The calibration is setting all of the model's parameters so that the prices of all European options calculated by the model are as close as possible to the prices on the market for those options. More specifically, traders need a model that, after calibrating, can price vanilla options so that the implied volatilities they come up with match the ones quoted on the market. He and Zhu (2017) presents theoretical findings about the calibration of local regime switching models. Jourdain and

Zhou (2020) presented the existence of a calibrated regime switching local volatility model. They demonstrate the existence of a solution to the Fokker-Planck equation, provided that the range of the squared stochastic volatility component is within certain limits. Subsequently, they infer the existence of the calibrated model. For calibration, we used one of the machine learning methods called GPR (refer Williams and Rasmussen (2006)).

First, consider a European call option on S with a strike price of K and an expiration date of T . At time T , the payoff function for a call option is $(S(T) - K)^+$. Using the pricing kernel, given the current information $\mathcal{G}(t)$, the price of the call option at the current time t is:

$$C(t) = E_{\tilde{P}} \left[\exp \left(- \int_t^T r(u) du \right) (S(T) - K)^+ \mid \mathcal{G}(t) \right].$$

Now $\{(S(t), \mathbf{X}(t)), t \in T\}$ is jointly Markovian with respect to the filtration $\mathcal{G}$. So, if $S(t) = s$ and $\mathbf{X}(t) = \mathbf{x}$,

$$\begin{aligned} C(t) &= E_{\tilde{P}} \left[\exp \left(- \int_t^T r(u) du \right) (S(T) - K)^+ \mid \mathcal{G}(t) \right], \\ &= E_{\tilde{P}} \left[\exp \left(- \int_t^T r(u) du \right) (S(T) - K)^+ \mid S(t) = s, \mathbf{X}(t) = \mathbf{x} \right], \\ &= \tilde{C}(t, s, \mathbf{x}). \end{aligned}$$

For each $i = 1, 2, \dots, N$, let $C_i = C(t, s, \mathbf{e}_i)$ and write $\mathbf{C} = (C_1, C_2, \dots, C_N)' \in \mathfrak{R}^N$. Then as shown in Buffington and Elliott (2002), the $C_i, i = 1, 2, \dots, N$, satisfy the following system of coupled partial differential equations (PDEs):

$$-r_i C_i + \frac{\partial C_i}{\partial t} + r_i s \frac{\partial C_i}{\partial s} + \frac{1}{2} \sigma_i^2 s^2 \frac{\partial^2 C_i}{\partial s^2} + \langle \mathbf{C}, \mathbf{Q} \mathbf{e}_i \rangle = 0, \quad t \in T, \quad (13)$$

with the terminal condition:

$$C(T, s, \mathbf{e}_i) = (s - K)^+.$$

Proposition 1 For each $i = 1, 2, \dots, N$, $C_i^{K,T} = C(t, s, \mathbf{e}_i, K, T)$ satisfies Eq. (13) with terminal condition $C(T, s, \mathbf{e}_i) = (s - K)^+$. Then for each $(t, s) \in T \times (0, \infty)$ and each $i = 1, 2, \dots, N$ the function $h_i = H(t, K, \mathbf{e}_i) = C_i^{K,T}$ satisfies the following initial value problem:

$$\frac{\partial h_i}{\partial T} = \frac{1}{2} \sigma_i^2 K^2 \frac{\partial^2 h_i}{\partial K^2} - r_i K \frac{\partial h_i}{\partial K} - \langle \mathbf{H}, \mathbf{Q}' \mathbf{e}_i \rangle, \quad T > t$$

with an initial condition:

$$H(t, K, \mathbf{e}_i) = (s - K)^+,$$

where $\mathbf{H} = (h_1, h_2, \dots, h_N)' \in \mathfrak{R}^N$. In particular, if $\frac{\partial^2 h_i}{\partial K^2} \neq 0$,

$$\sigma_i = \sigma(K, T, \mathbf{e}_i) = \sqrt{2 \left(\frac{\frac{\partial h_i}{\partial T} + r_i K \frac{\partial h_i}{\partial K} + \langle \mathbf{H}, \mathbf{Q}' \mathbf{e}_i \rangle}{K^2 \frac{\partial^2 h_i}{\partial K^2}} \right)}. \quad (14)$$

Proof For proof, see Elliott et al. (2015)).

5.1 GPR Explained

We present a concise overview of Gaussian Process Regression, with a more in-depth treatment available in Williams and Rasmussen (2006). Let $(X, \mathbf{y}) = \{(\mathbf{x}_i, y_i) \mid i = 1, \dots, n\}$ denote a training set of observations, where each $\mathbf{x}_i$ is an input vector of dimension d and y_i is the corresponding output. The objective is to establish the relationship between inputs and outputs. In GPR, this is achieved by assuming $y_i = f(\mathbf{x}_i) + \varepsilon_i$, where $f(\mathbf{x})$ is a Gaussian process and $\varepsilon_i \sim \mathcal{N}(0, \sigma_n^2)$ are i.i.d. random variables representing the noise in the data.

A Gaussian process $f(\mathbf{x})$ is a collection of random variables, potentially infinite, with any finite subset having a joint Gaussian distribution. The process $f(\mathbf{x})$ is defined by its mean function $m(\mathbf{x})$ and covariance or kernel function $\rho(\mathbf{x}, \mathbf{x}')$. Assuming a zero mean function, for a sample $(X, \mathbf{f}) = \{(\mathbf{x}_i, f_i) \mid i = 1, \dots, n\}$ generated from $f(\mathbf{x})$, we have $\mathbf{f} \sim \mathcal{N}(\mathbf{0}, \rho(X, X))$, where $\rho(X, X)$ is the covariance matrix.

As a Bayesian method, GPR initiates from a prior Gaussian process to encode initial knowledge about the function f (e.g., preferring smooth functions). It then integrates this prior with observed data points $(X, \mathbf{y})$ to derive a posterior distribution of functions.

5.2 GPR Kernel Function

The performance of GPR is sensitive to the choice of kernel function. Selecting an appropriate kernel that effectively captures the underlying patterns in the data requires domain knowledge and experimentation. The model's robustness can be compromised if an inappropriate kernel is chosen. The goal is to construct, by GPR, an arbitrage-free and continuous call price surface $C : \mathfrak{R}^+ \times \mathfrak{R}^+ \rightarrow \mathfrak{R}^+$. Let $C^*(T, K)$ the time-0 market price of the call option with maturity T and strike K on S , which is observed for a finite number of pairs (T, K) , interpolating C^* up to some error term. The corresponding local volatility surface $\sigma_i(\cdot, \cdot)$ for $i = 1, 2$, can be retrieved by the Dupire formula Eq. (14).

Consider a zero-mean Gaussian process prior on the mapping $c = c(\mathbf{x})$ with kernel function ρ given for any $\mathbf{x} = (T, k)$, $\mathbf{x}' = (T', k')$ by:

$$\rho(\mathbf{x}, \mathbf{x}') = \sigma^2 \omega_T(T - T', \theta_T) \omega_k(k - k', \theta_k). \quad (15)$$

Here $\theta_T = \theta$ and $\theta_k = \sigma^2$ correspond to length scale and variance hyperparameters of the kernel function ρ , whereas the functions ω_T and ω_k are kernel correlation functions. Prediction and uncertainty quantification are made using the conditional distribution given by:

$$c \mid c(\mathbf{x}) + \epsilon = y,$$

where $y = [y_1, \dots, y_n]^\top$ are n noisy observations of the function c at input points $\mathbf{x} = [x_1, \dots, x_n]^\top$, corresponding to observed maturities and strikes $\mathbf{x}_i = (T_i, k_i)$. The additive noise term $\epsilon = [\epsilon_1, \dots, \epsilon_n]^\top$ is assumed to be a zero-mean Gaussian vector, independent from $c(\mathbf{x})$, and with a homoscedastic covariance matrix given as $\Sigma^2 I_n$, where I_n is the n -dimension identity matrix.

5.3 Application of GPR

In a nutshell, the whole procedure works as follows. For a large set of input parameters (reflecting the contract and the market situation), contract price and model parameters are stored in a repository. These values can come from market data or be estimated/calculated under the given model using existing techniques. A dedicated kernel function ρ is deduced from this data structure. Once this kernel function ρ is calculated, the training of the model is completed. After applying GPR related computation to this kernel function, any new calculation regarding the contract price or local volatility structure is executed extremely fast. We have applied the standard GPR fitting algorithms.

5.4 Monte Carlo Technique

The Monte Carlo technique is used to calibrate the local volatility given by Eq. (14). More exactly, we compute an estimate of the expectation $E_{\tilde{P}} \left[\exp \left(- \int_t^T r(u) du \right) (S(T) - K)^+ \mid S(t) = s, \mathbf{X}(t) = \mathbf{x} \right]$ using Monte Carlo simulations. Therefore, we must simulate the $S(t)$ up to time T in each regime, beginning with the initial $S(0)$. Following the Monte Carlo principle, we simulate n times the stochastic variable $S(t)$ up to time T , assuming that $\Delta t = t_{k+1} - t_k$, for $k = 0, \dots, n-1$ as the discretization time step. In the Monte Carlo simulation approach, Eq. (11) is discretized using the Euler discretization method.

$$S(t_{k+1}) = S(t_k) + S(t_k) (r(t_k) + \sigma(t_k, S(t_k), \mathbf{X}(t_k)) \sqrt{\Delta t} Z), \quad (16)$$

where Z is the standard normal random variable. As shown in Eq. (16), the local volatility function must be known to simulate the trajectory for $S(t)$. Therefore, the only feasible method of operation is forward in time. Assume market Call prices are known for n maturities $t_1, t_2, \dots, t_n$. In addition, we assume that the initial local volatility $\sigma_i(t_0, K)$ for $i = 1, 2$ regimes is known for every strike K . The first stage involves determining the local volatility function at time step t_1 for each strike K . Knowing the initial local volatility function $\sigma_i(t_0, K)$, it is possible to simulate $S(t_1)$. Then, the expectation may be easily calculated for all K . Using Eq. (14), we can now construct the local volatility function at time t_1 for all strikes K . Using the local volatility computed at time t_1 and the simulated pathways up to time t_2 , we can calibrate the local volatility at time t_2 using the same process. Using this method, we may construct the local volatility equation up to a maturity date $T = t_n$ for both regimes Fig. 2.

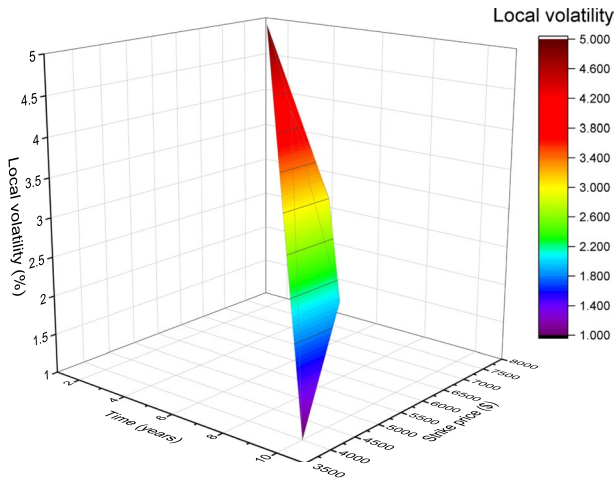


Fig. 2 Local volatility surface with respect to strike price and time to maturity

6 Numerical Results and Sensitivity Analysis

This section describes the data taken for the numerical calculation, along with the numerical results and sensitivity analysis. In sensitivity analysis, we investigate the impact of various parameters on the expected present value of the contract. Notably, the fair pricing of GMAB is influenced by factors such as risk-free interest rates, volatility, transition rates, and initial probabilities. This investigation means to shed light on the relationship between GMAB pricing and the parameters specified in the regime switching model.

6.1 Data Description

This research work considers $S[NONSPACE]$ &P 500 composite index prices from the US market for the numerical analysis. The index was chosen based on data availability and popularity among the people of the respective country. We considered information from January 1, 2000, until January 1, 2022. The data for the $S[NONSPACE]$ &P 500 index is taken from Bloomberg Terminal Figs. 3 and 4.

The S&P 500 index calculates the value of its constituent stocks based on their market capitalization. The calculation involves determining the adjusted market capitalization of all companies in the index and subsequently dividing it by a divisor. The divisor is modified to account for stock splits, spinoffs, and other market fluctuations. An inherent limitation of the S&P 500 is its tendency to assign greater significance to companies with larger market capitalization. For example stock prices of Apple and Microsoft exert a significantly larger impact on the index compared to a company with a lesser market capitalization. The S&P 500 index lacks exposure to small-cap companies, which have historically produced better returns. When assessing model performance across different datasets, concerns include generalization issues, data quality, and the stability of model parameters. Caution in adapting

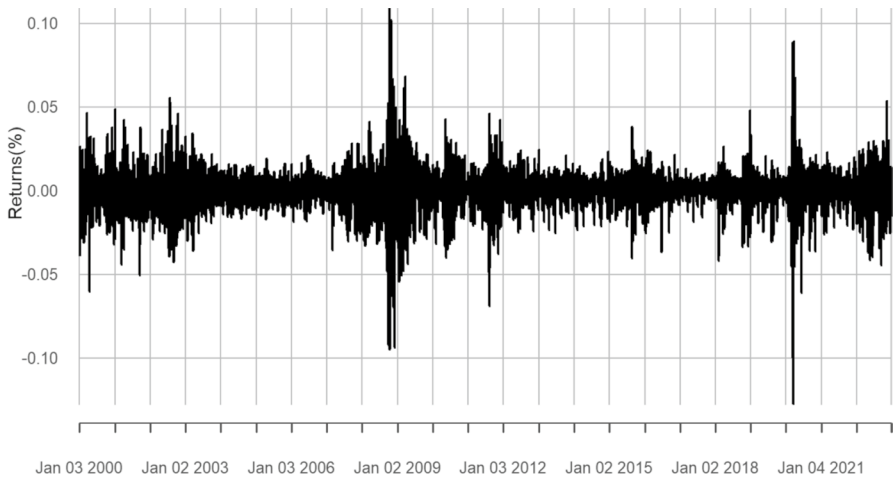


Fig. 3 Returns series of S & P500 Index data from Jan. 2000 to Jan. 2022

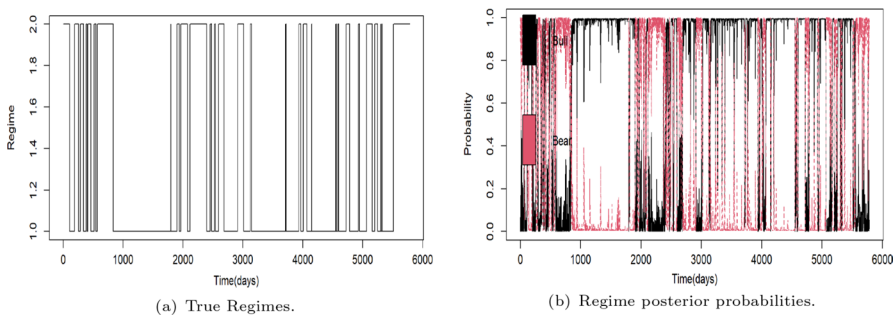


Fig. 4 Regime detection in S&P 500 index return data from Jan. 2000 to Jan. 2022 using hidden Markov model

models to changing market conditions is essential for managing the complexities of financial data and sophisticated modeling approaches.

The pricing performance of GMAB is analyzed using the approach stated before. The objective is to demonstrate how various market conditions can affect the pricing of variable annuity guarantees. This facilitates further investigation into the model's sensitivity to different parameters, enhancing the comprehension of variable annuity pricing patterns in different economic regimes. Unless otherwise stated, the parameter values in Table 2 present the calibrated parameters of the local volatility model with regime switching rates. The interest rate parameters are chosen to illustrate the influence of two distinct regimes on pricing performance. The mortality parameters are taken from US mortality rate data. Since the Markov chain X modulating the regime change is observable, we can almost surely determine which regime we are in at any given moment based on the starting probability p , which is assumed to be near either 0 or 1. Specifically, the equity components of the evaluated variable

Table 2 Calibrated parameters of the regime switching local volatility model and few assumed values for numerical experiments

Initial fund value	$F(0)$	100(\$)	
Initial probability of being in regime 1	p	1	
Maturity	T	15 Yrs	
Age at inception	x_0	60 Yrs	
Regime 1			
	Volatility	σ_1	0.20
	Rate of interest	r_1	0.05
	Transition rate	q_{12}	0.40
Regime 2			
	Volatility	σ_2	0.40
	Rate of interest	r_2	0.02
	Transition rate	q_{21}	0.30

annuity guarantees are based on the $S[NONSPACE] \& P500$ index data(US) from Jan. 2000 to Jan. 2022. We provide a price analysis for variable annuity guarantees, particularly GMAB with several embedded benefits.

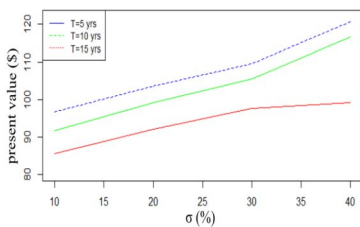
6.2 Sensitivity Analysis

Now, we study the change in the expected present value of the contract for the different parameters. The fair pricing of GMAB is sensitive to risk-free interest rates, volatility, transition rates, and initial probability. We analyze how GMAB prices change based on the parameters set in the regime switching model. Figure 5b depicts the sensitivity of the price with respect to local volatility at the different parameters of the risk-free interest rate for a policyholder of age 60 years with a contract maturity of $T = 15$ years and changing volatility of regime 1 in case of roll-up benefit. Figure 5a and 6a depicts the sensitivity of the price with respect to local volatility at $r = 5\%$ for a policyholder of age 60 years with a contract maturity of $T = 5, 10$ or 15 years for roll-up and ratchet benefit respectively. We note that the price increases as volatility increases and decreases with respect to interest rate, which the basic risk-return argument can explain: the value of a minimal investment return increases as the future becomes more uncertain. In addition, the price is inversely proportional to the contract maturity T in case of roll-up and ratchet benefits. Increasing mortality, which substantially discounts the benefit, causes a price decline for maturities beyond 10 years (as in this case, GMAB only pays benefits if the insured is still alive). The longer the duration of the contract, the more substantial the influence of mortality on pricing. Since high-interest discount rates, the insured benefits more strongly, reducing its present price. Figure 6b shows the sensitivity of GMAB price for the death benefit in different regimes. For both regimes, the prices of death benefit increase with respect to T with the same increasing trend. Additionally, the price in Regime 2 is higher than in Regime 1, which can be attributed to the higher local volatility in Regime 2. In general, in Figures 5–6, the GMAB prices in regime 2 are

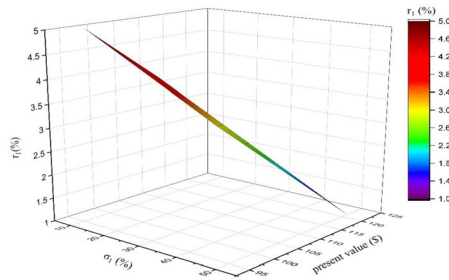
systematically higher than those in regime 1. These make intuitive sense. Regime 2 has a higher local volatility compared with Regime 1, which leads to higher prices in Regime 2. Most of the reasons are due to higher compensation caused by the risk premium in a low economic environment. Table 3 compares the GMAB price with different benefits using the MC method and the price predicted using the GPR technique under the proposed model. The numerical findings, as shown in Table 3, clearly indicate an increasing trend in price in Regime 2 compared to Regime 1. This can be due to the higher volatility and low-interest rate of Regime 2. Moreover, to compare roll-up and yearly ratchet benefits, we apply the Monte Carlo method to

Table 3 GMAB prices (in \$) with two regimes for different benefits, computed by using MC and GPR techniques, as per parameters specified in Table 2

Benefits	Regime 1		Regime 2	
	MC	GPR	MC	GPR
Roll-up	90.61	90.02	96.43	96.50
Ratchets	100.22	100.34	105.02	105.46
Death	99.32	100.04	100.80	100.06

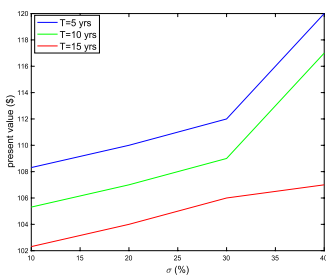


(a) Sensitivity of the price for GMAB with respect to local volatility for maturities $T = 5, 10$ or 15 years for roll-up benefit.

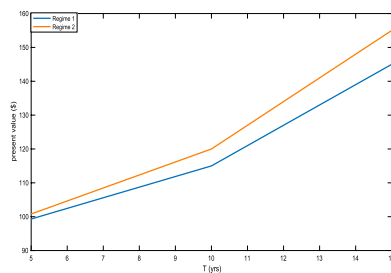


(b) Sensitivity of GMAB prices with respect to the defining parameters of Regime 1 considering roll-up benefits.

Fig. 5 Sensitivity of the price for GMAB with respect to local volatility, interest rate, and contract maturity, $T = 5, 10$ or 15 years for a contract with the roll-up rate $c = 2\%$



(a) Sensitivity of the price for GMAB with respect to local volatility for maturities $T = 5, 10$ or 15 years for ratchet benefit.



(b) Sensitivity of the price for GMAB in different regimes with respect to contract maturity T for death benefit.

Fig. 6 Sensitivity of the price for GMAB with respect to local volatility and contract maturity, $T = 5, 10$ or 15 years for different benefits in different regimes

calculate the fair price of the GMAB contracts for a given maturity. The results in Table 3 show that the yearly ratchet benefit base has higher prices than the roll-up benefit base due to the additional options in the ratchet benefits base. Furthermore, it is also observed that the price evaluated using the GPR method is quite accurate, and the computational time savings are significant. Feng et al. (2022) extensively examined three fundamental elements of VAs: pricing, evaluation, and risk management, through a comprehensive review spanning the past three decades.

7 Conclusion and Future Work

In this article, we aim to address a gap observed in previous research by incorporating regime switching dynamics into the local volatility model for pricing long-term life insurance policies, specifically VAs with GMAB, considering several options, like roll-up and ratchet. While previous studies, such as Deelstra and Rayée (2013), Fan et al. (2015), and Cui et al. (2017), investigated the valuation of various annuity guarantees, they did not consider local volatility with regime switching, which is a key aspect of this research work. This integration makes the model more realistic and useful in changing market conditions. Secondly, this research employs a rapid and efficient Gaussian process regression method to determine prices. It also introduces the use of this method to calibrate the local volatility surface and determine prices for GMAB guarantees in a regime switching scenario. The GPR technique offers substantial time savings compared to other methods, as the majority of computing work is concentrated in the training phase. This step only needs to be done once before the model can be utilized to make predictions. The calibration procedure employs Monte Carlo and GPR algorithms, which have been validated using real-world data, specifically the S&P500 Index. The duration of training for GPR can be substantial, particularly when working with large datasets. The time needed to train the model may pose challenges for real-time applications.

This study investigates the significant influence of non-constant volatility, specifically local volatility with regime switching, on the valuation of variable annuity guarantees. This statement highlights the importance of pricing being directly influenced by the implied volatility surface of the underlying asset. It emphasizes the need for model calibration that considers the complete implied volatility surface rather than focusing solely on implied volatility specific to maturity. This study emphasizes the significance of practitioner precaution in selecting models, choosing market data, and calibrating due to the sensitivity of variable annuity guarantee prices to both model parameters and market data. Moreover, this research emphasizes substantial pricing disparities across different types of benefits, indicating that the price of death benefits is lower than that of ratchet benefits and higher than that of roll-up benefits with the same maturity. Furthermore, the study highlights the significant impact of interest rates and volatility on pricing, demonstrating that prices tend to increase as volatility rises and fall as interest rates increase.

The conducted numerical experiments in this article provide a concrete illustration of the practical applicability of the regime switching local volatility model, particularly in the context of variable annuity pricing with GMAB and its implications for hedging strategies. Consider a scenario where an investor manages a portfolio of variable

annuities, encountering abrupt market uncertainties due to geopolitical events. This model, equipped with its regime switching feature, adeptly captures the transition to a high-volatility regime. This proves invaluable for investors in pricing VAs and dynamically adjusting their hedging strategies to navigate heightened market uncertainty. Furthermore, this framework provides an outline for further research. Future studies will focus on the valuation under stochastic local volatility with regime switching dynamics and assess the impact of stochastic interest rates within a regime switching framework. Additionally, the model's effectiveness can be assessed across various datasets.

This study faced several challenges that require careful consideration in future research works. A significant challenge observed was the time-consuming process of training the Gaussian Process regression model, especially when working with large datasets. This poses a significant hindrance for real-time applications when the requirement for prompt model responses is vital. To tackle this difficulty, future research could investigate optimization techniques, parallel computing, or alternative machine learning algorithms that provide similar accuracy levels while requiring less training time. However, there may be difficulties in expanding the model to incorporate stochastic local volatility within a regime switching framework. To tackle these issues, it may be necessary to enhance numerical approaches, investigate sophisticated mathematical procedures, or utilize advancements in computational resources.

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Declarations

Conflict of interest statement The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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