Transient solution of fluid queue modulated by two independent birth-death processes

Shruti Kapoor and Selvamuthu Dharmaraja*

Department of Mathematics,
Indian Institute of Technology Delhi,
Hauz Khas, New Delhi 110016, India
Email: shruti85kapoor@gmail.com
Email: dharmar@maths.iitd.ac.in
*Corresponding author

Viswanathan Arunachalam

Department of Statistics,
Universidad Nacional de Colombia,
Bogota, Colombia
Email: varunachalam@unal.edu.co

Abstract: The objective of this paper is to study the transient distribution of the buffer content in any intermediate node of a wireless network based on IEEE 802.11 standards. The steady state solution of the discussed model has already been given in Arunachalam et al. (2010). The methodology used, maps the underlying model to a fluid queue model driven by two independent finite state birth-death processes with the aim to simplify the solution which is obtained in closed form with numerical illustration. Along with the buffer occupancy distribution, other performance measures: throughput, server utilisation and expected buffer content are also obtained numerically.

Keywords: IEEE 802.11 wireless networks; fluid queue; buffer occupancy distribution.


Biographical notes: Shruti Kapoor is currently doing her PhD in Mathematics from Indian Institute of Technology Delhi, India. She is a post-graduate in Mathematics from Indian Institute of Technology Delhi, India. Her current research interests include queuing theory, fluid queues and Markov modelling.

Selvamuthu Dharmaraja received his PhD in Mathematics from Indian Institute of Technology Madras, India in 1999. He is currently an Associate Professor in the Department of Mathematics and joint Faculty of Bharti School of Telecommunication Technology and Management at Indian Institute of Technology Delhi, India. His current research interests include queuing models, and performance issues of wireless networks and dependability analysis of communication systems.

Copyright © 2019 Inderscience Enterprises Ltd.
Viswanathan Arunachalam earned his PhD degree in Mathematics from Indian Institute of Technology Madras, India in 1996. From 1996 to 1997, he was a Post-Doctoral Fellow in the Department of Cell Research and Immunology at Tel Aviv University, Tel Aviv, Israel. He is currently an Associate Professor in Department of Statistics at the Universidad Nacional de Colombia, Bogota, Colombia. His research interests include stochastic processes and its applications in biology, reliability and queuing theory, mathematics of financial derivatives, and statistics of financial markets.

1 Introduction and related work

A fluid model is a mathematical model used to describe the amount of fluid in a reservoir or dam, of infinite or finite capacity. The rate at which the fluid flows into the system can be analysed as being controlled by an operator, which is a continuous time Markov chain and is called the background process.

Steady state analysis of the buffer content distribution has been studied by various authors and different methodologies have been used to obtain the exact solution in many cases. Sericola (2001) and Sericola and Tuffin (1999) derived the solution for finite and infinite buffer fluid queues respectively, with input from $M/M/1$ queue using recurrence relations. Van Doorn and Scheinhardt (1997) gave the methodology to obtain the buffer content distribution using orthogonal polynomials. Parthasarathy et al. (2002) analysed the steady state behaviour of a fluid queue driven by an $M/M/1$ queue using continued fraction approach. Ramaswami (1999) described the use of matrix-analytic methods in the context of fluid models and provided efficient algorithm for computing the stationary distribution. In Arunachalam and Dharmarja (2014), differential equation techniques have been used to obtain the steady state solution of a fluid queue driven by two independent birth and death processes. In Virtamo and Norros (1994), steady state solution of $M/M/1$ queue is studied using the spectral decomposition. This solution is further studied in detail by Adan and Resing (1996).

Steady state behaviour gives us important information of a system in long run, but to study the dynamical nature of a system, transient analysis plays an important role. Sericola (1998) gave the transient solution of fluid queues driven by a Markov process using recurrence relations. Also, Parthasarathy et al. (2005) gave the exact transient solution of fluid queues driven by $M/M/1$ queue representing the solution in terms of functions satisfying recurrence relations. Nabli (2004) analysed transient and asymptotic behaviour of general Markov fluid models where the input and output rates are assumed to be modulated by a finite state irreducible Markov process. Fluid queues driven by $M/M/1/N$ queue was analysed by Parthasarathy and Lenin (2000), and for specific values of net input rates, closed form transient solution was obtained. Ren and Kobayashi (1995) found the transient buffer content distribution for a fluid model with background birth-death process having linear state dependent rates.

The aim of this paper is to find the buffer content in any intermediate node of a wireless network based on IEEE 802.11 standards, which can be modelled as a fluid queue driven by two independent birth death processes. In Arunachalam et al. (2010), the steady state distribution of a fluid queue driven by two independent birth death processes
Transient solution of fluid queue

have been discussed using eigenvalue approach. De Souza et al. (1995) gave a methodology to find the transient distribution of cumulative reward based on probability. In this paper, we present a new methodology of finding the transient distribution for the model discussed in Arunachalam et al. (2010) using the approach given in de Souza et al. (1995). The solution obtained is in terms of recurrence relations where the recurrence is explained using probability concepts.

The rest of the paper is organised as follows: Section 2 gives a description of the fluid model. Section 3 discusses the steps required to obtain the transient solution and Section 4 illustrates the solution obtained and other performance measures numerically.

2 Model description

The IEEE 802.11 wireless LAN (WLAN) is the most widely used WLAN standard nowadays (LAN MAN, 1999). Hence, the flow of information from one node to another (via any intermediate node) in a network based on IEEE 802.11 protocol is modelled using the fluid queue approach. We consider a Markov modulated fluid queue with infinite buffer capacity. The information is buffered at the intermediate node for service where the server typically is a communication channel. The rate at which the information arrives to an intermediate node fluctuates randomly, often with a high degree of correlation in time (Stern and Elwalid, 1991). We assume that the buffer is building up and getting depleted with variable rates. The application of this model in the context of IEEE 802.11 WLAN has been discussed in Arunachalam et al. (2010). As described we take the input process as \( X(t), \ t \geq 0 \) with finite state space \{1, 2, \ldots, N\} with birth and death rates as \( \lambda_i, i = 1, 2, \ldots, N-1 \) and \( \mu_i, i = 2, 3, \ldots, N \). The inflow rate into the buffer is given by \( c_i \), which can take any real value.

The release rate of fluid from the buffer depends on the transmission rate of the serving communication channel. The IEEE 802.11 WLAN standard supports multiple transmission rates that can maximise the system throughput in the face of adverse conditions. The IEEE 802.11b physical layer (PHY) specifies four different data rates, 1, 2, 5.5, and 11 Mbps. Thus for our model, we let the outflow rate from the buffer be determined by another independent BDP \{Y(t), \ t \geq 0\} with four states 1, 2, 3, 4 evolving in the background. These four states represent the four different transmission rates supported by the IEEE 802.11b protocol. Let \( \alpha_i, i = 1, 2, 3 \) be the birth rates and \( \beta_i, i = 2, 3, 4 \) be the death rates of this BDP. When \( Y(t) \) is in some state \( i, i \in \{1, 2, 3, 4\} \), the outflow rate from the buffer is given by \( h_i \). On combining the BDPs \( X(t) \) and \( Y(t) \), we obtain a CTMC \{Z(t), \ t \geq 0\} with state space \( S = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4), \ldots, (N, 1), (N, 2), (N, 3), (N, 4)\} \). This CTMC has \( 4N \) states and the state transition diagram for this stochastic process is shown in Figure 1. Whenever, \( Z(t) = (i, j), i \in \{1, 2, \ldots, N\}, j \in \{1, 2, 3, 4\} \), the outflow rate from the buffer is \( h_j \). The described CTMC model with state dependent input rates and four different output rates is shown in Figure 2.

For simplification, we enumerate the state \( (i, j) \) under a single index \( \{1, 2, \ldots, 4N\} \). Hence, corresponding to the new indexing of the states of \{Z(t), \ t \geq 0\}, we define a new CTMC, \{K(t), \ t \geq 0\} with finite state space \( S = \{1, 2, \ldots, 4N\} \). Hence, we have a fluid queue driven by a CTMC (which is not a BDP).
3 Transient solution

In this section, we obtain the transient distribution of the buffer occupancy. The buffer content at any time $t$ is denoted by $C(t)$ and we assume $C(0) = 0$. Hence, we have a two dimensional stochastic process $\{C(t), K(t), t \geq 0\}$. The net flow rate into the buffer, denoted by $r_n$, is given by

For $n = 0, 1, 2, \ldots, N-1; j = n + 1$

$$
\eta_j \begin{cases} 
  c_j - h_1, i = 4n + 1 \\
  c_j - h_2, i = 4n + 2 \\
  c_j - h_3, i = 4n + 3 \\
  c_j - h_4, i = 4n + 4 
\end{cases}
$$  

(1)
Transient solution of fluid queue

We have the following differential equation
\[
\frac{dC(t)}{dt} = \begin{cases} 
  r_{K(t)}, & \text{if } C(t) > 0 \text{ or } r_{K(t)} > 0 \\
  0, & \text{otherwise}
\end{cases}
\]

Now we define,
\[
F_j(t, x) = P[K(t) = i, C(t) \leq x], \quad i = 1, 2, \ldots, 4N, \quad t, x \geq 0
\]
as the two-dimensional distribution function of the process \{(K(t), C(t)), t \geq 0\}. The Kolmogorov forward equation for the Markov process \{(K(t), C(t)), t \geq 0\} are given by (Tanaka et al., 1995),
\[
\frac{\partial F_j(t, x)}{\partial t} = -\eta_j \frac{\partial F_j(t, x)}{\partial x} + \sum_{j \in S} F_j(t, x)\mathcal{Q}(j, i), i \in S.
\] (2)

Let \(Q\) be the infinitesimal generator matrix of the CTMC \{K(t), t \geq 0\} given by,
\[
\begin{pmatrix}
\beta_i - \alpha_i & \alpha_i & 0 & 0 & \lambda_i & \ldots \\
0 & -\beta_2 - \beta_3 - \lambda_i & \alpha_2 & 0 & 0 & \ldots \\
0 & \beta_3 & -\alpha_3 - \beta_3 - \lambda_i & \alpha_3 & 0 & \ldots \\
0 & 0 & \beta_4 & -\beta_4 - \lambda_i & 0 & \ldots \\
\mu_2 & 0 & 0 & 0 & -\alpha_4 - \mu_2 - \lambda_2 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\] (3)

For numerical simplification, we assume that
\[
\lambda_i = \lambda, \quad \text{for } i = 1, 2, \ldots, N - 1
\]
\[
\mu_i = \mu, \quad \text{for } i = 2, \ldots, N
\]
\[
\alpha_i = \alpha, \quad \text{for } i = 1, 2, 3
\]
\[
\beta_i = \beta, \quad \text{for } i = 2, 3, 4
\]

Using the process of uniformisation, we define \(\ell = \lambda + \mu + \alpha + \beta\) and let \(P = I - \frac{Q}{\ell}\).

Define \(Z = \{Z_n : n = 0, 1, \ldots\}\) be a discrete time Markov chain with finite state space \(S\) and transition probability matrix \(P = I - \frac{Q}{\ell}\) and \(\{N(t), t \geq 0\}\) be a Poisson process with rate \(\ell\), that is independent of \(Z\). Then \(X(t) = ZN(t)\) for \(t \geq 0\).

We wish to obtain \(P(C(t) > x)\), i.e., the probability that the workload accumulated in the interval \((0, t)\) is greater than some value \(x\). We assume that the background process has been uniformised and thus have \(n\) transitions during the period \((0, t)\), i.e., \(N(t) = n\), at times
\[
0 < \tau_1 < \tau_2 < \ldots < \tau_n < t.
\]

These events split \((0, t)\) into \(n + 1\) intervals with lengths
Thus, each interval is associated with a net rate, which is based on the state of the process
during the interval. Define $V'_k = \sum_{j=0}^{k-n} l_{t_{k-j}}$, where $V'_k$ represents the number of
intervals during which the background process is in state $j$, where $j$ takes the value
$1, 2, \ldots, 4N$, depending on the background process being in some state for which the net
input rate is $r_j$. Let $k_1, k_2, \ldots, k_n$ denote the number of intervals associated with rate
$r_{k_1}, r_{k_2}, \ldots, r_{k_n}$ respectively. We call $k = (k_1, k_2, k_3, \ldots, k_n)$ as a partition of $n + 1$, i.e.,
$|k| = k_1 + k_2 + \ldots + k_n = n + 1$. Thus, conditioning on the number of transitions $n$ and
$V_n = (V'_1, V'_2, \ldots, V'_{4N}) = (k_1, k_2, \ldots, k_n) = k$, we obtain

$$P[C(t) > x] = \sum_{j=0}^{\infty} e^{-j} \frac{(j)^{\alpha}}{n!} \sum_{|k|=n+1} G(n, k) M(t, x, n, k)$$

where $M(t, x, n, k) = P[C(t) > x|N(t) = n, V_N = k]$ and $G(n, k) = \sum_{i \in S} G_i[n, k]$. $G_i[n, k]$ is
the probability that the state visited after the last transition is $i$ given $n$ transitions and
partition $k$. If $i$ and $j$ are the states visited after the last $(n - 1)^{th}$ and $n$th transitions, then $k$
is equal to the previous partition + 1 at the entry corresponding to the net rate associated
with state $j$. Thus for the model described above we have

$$G_i[n, k] = G_2[n-1, k-1, i] \frac{\beta}{\ell} + G_3[n-1, k-1, i] \frac{\mu}{\ell}$$

$$G_2[n, k] = G_1[n-1, k-1, i] \frac{\alpha}{\ell} + G_3[n-1, k-1, i] \frac{\beta}{\ell} + G_4[n-1, k-1, i] \frac{\mu}{\ell}$$

$$G_3[n, k] = G_2[n-1, k-1, i] \frac{\alpha}{\ell} + G_4[n-1, k-1, i] \frac{\beta}{\ell} + G_4[n-1, k-1, i] \frac{\mu}{\ell}$$

$$G_4[n, k] = G_3[n-1, k-1, i] \frac{\alpha}{\ell} + G_4[n-1, k-1, i] \frac{\mu}{\ell}$$

for $i = 4j + 1, j = 1, 2, \ldots, N - 2$

$$G_i[n, k] = G_{i-4}[n-1, k-1, i] \frac{\lambda}{\ell} + G_{i+4}[n-1, k-1, i] \frac{\mu}{\ell}$$

$$+ G_{i+1}[n-1, k-1, i] \frac{\beta}{\ell}$$

for $i = 4j, j = 2, \ldots, N - 1$

$$G_i[n, k] = G_{i-4}[n-1, k-1, i] \frac{\alpha}{\ell} + G_{i+4}[n-1, k-1, i] \frac{\mu}{\ell}$$

$$+ G_{i-4}[n-1, k-1, i] \frac{\lambda}{\ell}$$
Transient solution of fluid queue

for \( i = 4j + 1, 4j + 2 \) \( j = 1, 2, \ldots, N - 2 \)

\[
G_i[n, k] = G_{i-1}[n-1, k-1] + \frac{\alpha}{\ell} G_{i+4}[n-1, k-1] + \frac{\beta}{\ell} G_{i+1}[n-1, k-1] \\
+ G_{i+4}[n-1, k-1] \frac{\gamma}{\ell} G_{i+1}[n-1, k-1]
\]

for \( i = 4N - 3, 4N - 2, 4N - 1, 4N \)

\[
G_{4N-3}[n, k] = G_{4N-7}[n-1, k-1] + G_{4N-2}[n-1, k-1] \frac{\beta}{\ell} + G_{4N-1}[n-1, k-1] \frac{\gamma}{\ell}
\]

\[
G_{4N-2}[n, k] = G_{4N-6}[n-1, k-1] + G_{4N-3}[n-1, k-1] \frac{\alpha}{\ell} + G_{4N-1}[n-1, k-1] \frac{\gamma}{\ell}
\]

\[
G_{4N-1}[n, k] = G_{4N-5}[n-1, k-1] + G_{4N-4}[n-1, k-1] \frac{\alpha}{\ell} + G_{4N-1}[n-1, k-1] \frac{\gamma}{\ell}
\]

\[
G_{4N}[n, k] = G_{4N-1}[n-1, k-1] \frac{\alpha}{\ell} + G_{4N-4}[n-1, k-1] \frac{\alpha}{\ell} + G_{4N}[n-1, k-1] \frac{\gamma}{\ell}
\]

The recursive function \( G_i[n, k] \) satisfy the initial conditions

\[
G_i[1, (1, 0)] = \pi_{i(0)}^0
\]

\[
G_i[0, (0, 1)] = \pi_{i(0)}^1 \text{ for } i \in S \setminus \{1\}.
\]

Now, we compute \( M(t, x, n, k) \). Assume that the first \( n \) transitions yield the partition \( k \). Let \( U_1, U_2, \ldots, U_n \) be independent and identically distributed random variables having uniform distribution in (0, 1) and \( U(1), U(2), \ldots, U(n) \) be their order statistics with \( U(0) = 0 \) and \( U(n + 1) = 1 \). Then, \( t_i \), i.e., the time of the \( i^{th} \) transition has the same distribution as \( tU_i \) (Rubino and Sericola, 2014). Thus, we have

\[
Y_i \equiv tU_i, Y_2 \equiv t(U_z - U_{z-1}), \ldots, Y_{n+1} \equiv (1 - U_n).
\]

We assume that \( L \) is the number of distinct rates used during the \( n \) transitions. Since \( Y_i \)'s are exchangeable random variables, so rearranging the intervals, we let first \( k_1 \) intervals be associated with the net rate \( r_1 \), next \( k_2 \) intervals be associated with the net rate \( r_2 \), \( k_3 \) intervals be associated with \( r_3 \) and so on.

Since \( C(t) \) represents the accumulated buffer in the interval (0, \( t \)), the event \( \{ C(t) > x | N(t) = n, V_n = k \} \), can be written as

\[
\{ C(t) > x | N(t) = n, V_n = k \} = \sum_{j=1}^{n+1} d_j Y_j
\]
where \(d_1, d_2, \ldots, d_{n+1}\) are the rewards associated with the subintervals of \((0, t)\). Suppose that the vector \(k\) has \(L + 1\) non-zero entries, that is only \(L + 1\) distinct rates out of the possible \(4N\) rates are used. Define \(n_j = \sum_{i=1}^{j} k_{i(i)}\) for \(j = 1, 2, \ldots, L\). Thus, we get

\[
P[C(t) > x|N(t) = n, V_a = k] = P\left[ \sum_{j=1}^{L} (R_{j(i)} - R_{j(i+1)}) U(n_j + R_{j(L+1)}) > \frac{x}{t} \right].
\]

Thus, in order to find \(M(t, x, n, k)\), we need to find the distribution of a linear combination of uniform order statistics on \((0, 1)\). A solution for this was obtained in Weisberg (1971). Using the result in Weisberg (1971), we obtain

\[
M(t, x, n, k) = \sum_{i \geq k} f_i^{(k-1)}(r_i, k) \frac{(k_i - 1)!}{(k_i - 1)!}
\]

where \(f_i^{(k-1)}\) is the \((k_i - 1)\)st derivative of the function

\[
f_i(y, k) = (y-x)^{k} / \prod_{j \neq i} (y - r_j)^{k_j}.
\]

Here, the time parameter \(t\) is implicitly presented in the right-hand side expression of \(M(t, x, n, k)\) in the form of summation limits.

4 Numerical illustration

In this section, we present a numerical illustration for the results presented in Section 3. We have used MATLAB to obtain the results numerically for server utilisation, expected buffer content and mean delay. We have considered the same parameter values presented in Arunachalam et al. (2010) to get the numerical results. The values of parameters are given in Table 1 and \(N = 3\). The parameters values are chosen for the purpose of numerical illustration of the closed form results.

<table>
<thead>
<tr>
<th>Rates</th>
<th>Meaning</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\lambda)</td>
<td>Arrival rate of (X(t))</td>
<td>0.02 to 0.06</td>
</tr>
<tr>
<td>(\mu)</td>
<td>Departure rate of (X(t))</td>
<td>0.06 to 0.09</td>
</tr>
<tr>
<td>(\alpha)</td>
<td>Forward rate of (Y(t))</td>
<td>0.03 to 0.05</td>
</tr>
<tr>
<td>(\beta)</td>
<td>Backward rate of (Y(t))</td>
<td>0.03 to 0.05</td>
</tr>
<tr>
<td>(c_i)</td>
<td>Inflow rate of fluid into the buffer when (X(t)) is in state (i)</td>
<td>(c_1 = 1.5 \text{ Mbps}, c_2 = 2.75 \text{ Mbps}, c_3 = 3 \text{ Mbps}, c_4 = 4.5 \text{ Mbps}, c_5 = 5 \text{ Mbps}, c_6 = 6.5 \text{ Mbps})</td>
</tr>
<tr>
<td>(h_j)</td>
<td>Outflow rate of fluid into the buffer when (Y(t)) is in state (j)</td>
<td>(h_1 = 11 \text{ Mbps}, h_2 = 5.5 \text{ Mbps}, h_3 = 2 \text{ Mbps}, h_4 = 1 \text{ Mbps})</td>
</tr>
</tbody>
</table>
Figure 3 shows the variation for buffer content complementary cumulative distribution function with respect to time for three different values of the buffer content 100, 1,000 and 10,000. It can be shown that for different values of \( x \), as \( x \) increases \( 1 - F(x) \) decreases and \( 1 - F(x) \to 0 \) as \( t \to \infty \). The server utilisation over time is shown in Figure 4 for \( \lambda = 0.03, \mu = 0.07, \alpha = 0.04 \) and \( \beta = 0.03 \). It can be seen that, the server utilisation increases with time. It is also observed that, the server utilisation reaches the steady state for large values of \( t \) (\( t = 3,000 \)).

**Figure 3** Buffer content complementary cumulative distribution function

![Graph](image1.png)

**Figure 4** Utilisation vs. time

![Graph](image2.png)
Figure 5 represents change in expected buffer content with time. From Figure 5, it can be seen that expected buffer content increases with time, as predicted. Further, it is observed that, as $\lambda$ increases, the buffer content increases. This is because as more arrivals take place, the expected buffer content also increases.

**Figure 5** Expected buffer content over time

Acknowledgements

The author (SD) would like to thank the National Board for Higher Mathematics, India, for financial support given to them during the preparation of the paper.

References


Transient solution of fluid queue


