
Steady state analysis of fluid queues driven by birth death processes with rational rates

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Abstract: Birth death processes with rational birth death rates have been studied by Maki (1976). This paper analyses the steady state behaviour of a fluid queue driven by a finite birth death process with rational birth and death rates. Two specific models are considered and closed form solutions are obtained for the equilibrium distribution of the buffer occupancy by finding the explicit eigenvalues of the underlying tridiagonal matrix. Numerical illustration is presented for different values of the size of the state space of the background process and for different values of the parameter involved.

Keywords: fluid queue; stationary distribution; eigenvalues; tridiagonal matrices.

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1 Introduction

A fluid model is a mathematical model used to describe the amount of fluid in a system, for example, a reservoir or dam, of infinite or finite capacity. The rate at which the fluid flows into the system can be analysed as being controlled by an operator which is a continuous time Markov chain and is called the background process.

In Guillemin and Sericola (2012) analyse the transient behaviour of a fluid queue driven by a general ergodic birth and death process using spectral theory in the Laplace transform domain. Many authors have found the steady state solution for various fluid models. Steady state analysis is useful to study a system in long run. A recent study by Guillemin and Sericola (2015) analyses the congestion periods in a finite fluid buffer where net rate depends on a recurrent Markov process. Some latest works for finding the steady state solution is in the area of tandem fluid queues (Van Doorn and Scheinhardt, 1977) and fluid queues with catastrophes (Vijayalakshmi and Thangaraj, 2015). Sericola (2001) also found the closed form solution for finite buffer fluid queues modulated by a finite state Markov chain. By using the differential system, the fluid level distribution is obtained in terms of a series, whose coefficients are computed by means of recurrence relations. A fluid queue driven by infinite state Markov chain was given by Sericola and Tuffin (1999). Guillemin and Sericola (2007) have discussed the stationary distribution of a fluid queue driven by a finite state Markov chain. Van Doorn and Scheinhardt (1977) analysed fluid queues driven by infinite state birth death process. Ramaswami (1999) described the use of matrix-analytic methods in the context of fluid models and provided efficient algorithm for computing the stationary distribution. Lenin and Parthasarathy (2000) have studied numerically the behaviour of fluid buffer driven by truncated birth death processes by giving a methodology to find the eigenvalues and eigenvectors. Then specific models were studied by some authors. Parthasarathy et al. (2002) found the steady state solution of a fluid queue driven by an M/M/1 queue using the continued fraction approach. Parthasarathy and Lenin (2000) found the closed form steady state solution for a fluid queue modulated by an M/M/1/N queue by finding eigenvalues and eigenvectors. Also Parthasarathy and Vijayashree (2010) found the steady state distribution for fluid queues driven by birth and death process with quadratic rates by using the methodology of finding eigenvalues.

As described in the literature above, some of the existing works show the importance of the birth and death rates of the background model and their impact on the level of fluid in the buffer. We have seen constant birth death rates (Parthasarathy et al., 2002), quadratic rates (Parthasarathy and Vijayashree, 2010), state dependent rates (Parthasarathy and Lenin, 2000) and many others. Thus, this paper analyses the steady state behaviour of a fluid queue modulated by a finite state birth death process wherein the birth and death rates are in rational form. Rational birth and death rates for queues was first considered by Maki (1976). He analysed the distribution function of a birth death process with birth and death rates as rational functions of the state of the process, where the numerator and denominator are general polynomials. In this paper we extend the logic to fluid queues, i.e., we assign rational values to the birth and death rates of the background birth death process for which the corresponding determinant of the underlying tridiagonal matrix can be found using known results on tridiagonal matrices.

The rest of the paper is organised as: Section 2 describes the background Markov model in detail. In Section 3 gives the detailed description of the fluid model. In Section 4, two specific fluid models are considered and give the explicit stationary solution for each of them, Section 5 gives the numerical illustration for the proposed work and Section 6 is the conclusion which discusses the unique contributions, limitations of the paper and future work.

2 Background Markov model

In Maki (1976) considers birth death processes $\{X(t), t \geq 0\}$ with rational rates of the form:

$$\begin{aligned}\lambda_j &= \frac{A_p(j)}{B_q(j)} \\ \mu_j &= \frac{C_r(j)}{D_s(j)}, j = 0, 1, 2, \dots\end{aligned}\tag{1}$$

where j represents the state of the birth death process and $A_p(x)$, $B_q(x)$, $C_r(x)$ and $D_s(x)$ are polynomials of degree p , q , r , s respectively. It is also assumed that the polynomials are such that $\frac{A_p(x)}{B_q(x)}$ and $\frac{C_r(x)}{D_s(x)}$ are positive for: $x = 1, 2, 3, \dots$, $\frac{A_p(0)}{B_q(0)} > 0$, and $\frac{C_r(0)}{D_s(0)} \geq 0$.

In Maki (1976) the author discusses various conditions on p , q , r and s , such that a unique solution exist for the distribution of $X(t)$ at any time t .

One of Maki's results stated as a conjecture, asserts that if $p = q$ and $r = s$ a unique solution exist for the distribution of $X(t)$ at any time t . Extending Maki's work to fluid queues, we consider a fluid queue model governed by a birth death process in this paper. The birth and death rates given by $\lambda'_j s$ and $u'_j s$ respectively are rational functions of linear polynomials of the form:

$$\begin{aligned}\lambda_j &= \frac{A_1(j)}{B_1(j)} \\ \mu_j &= \frac{C_1(j)}{D_1(j)}, j = 0, 1, 2, \dots\end{aligned}\tag{2}$$

With these above rates $p = q = 1$ and $r = s = 1$.

2.1 Notations and assumptions

In Table 1, we give the notations and assumptions used in our model. An important assumption in the paper is, that the value of α is greater than one, which is required for the stability of the model.

Table 1 Notations

<i>Symbol</i>	<i>Meaning</i>
λ_j	Arrival rate of $X(t)$
μ_j	Departure rate of $X(t)$
α	Constant
r_j	Net input rate of fluid into the buffer when $X(t)$ is in state j

3 Model description

In this paper, we consider a fluid queue driven by a birth death process, $\{X(t), t \geq 0\}$ with state space $S = \{0, 1, \dots, N\}$. The birth and death rates as stated in equation (2) are rational functions of linear polynomials given by:

$$\begin{aligned}\lambda_j &= \frac{j-1+\alpha}{2j+2\alpha}, j=0, 1, \dots, N-1 \\ \mu_j &= \frac{j+\alpha}{2j+2\alpha-2}, j=1, 2, \dots, N\end{aligned}\quad (3)$$

where α is a constant satisfying $\alpha > 1$. As can be seen from the above expression, the rates λ_j and μ_j are constants dependent on the state of the background process. The above values of birth death rates satisfy:

$$\begin{aligned}\lambda_{j-1} + \mu_j &= 1, j=1, 2, \dots, N \\ \lambda_j \mu_j &= \frac{1}{4}, j=1, 2, \dots, N-1.\end{aligned}\quad (4)$$

Let $C(t)$ denote the content of the buffer at time t , then $\{(X(t), C(t)), t \geq 0\}$ constitute a two-dimensional Markov process. When the background process $X(t)$ is in state $j, j=0, 1, \dots, N$, fluid flows into the buffer at a net input rate $r_j, j=0, 1, \dots, N$.

The buffer occupancy distribution is defined as:

$$F_j(t, u) = P\{X(t) = j, C(t) \leq u\}; t \geq 0, u \geq 0, j \in S$$

i.e., the probability of the background birth death process in state j and the content less than equal to u .

The Kolmogorov forward equation for the Markov process $\{(X(t), C(t)), t \geq 0\}$ are given by Tanaka et al. (1995):

$$\frac{\partial F_0(t, u)}{\partial t} + r_0 \frac{\partial F_0(t, u)}{\partial u} = -\lambda_0 F_0(t, u) + \mu_1 F_1(t, u) \quad (5)$$

for $j=1, 2, \dots, N$:

$$\frac{\partial F_j(t, u)}{\partial t} + r_j \frac{\partial F_j(t, u)}{\partial u} = \lambda_{j-1} F_{j-1}(t, u) - (\lambda_j + \mu_j) F_j(t, u) + \mu_{j+1} F_{j+1}(t, u) \quad (6)$$

as $t \rightarrow \infty$ we let:

$$F_j(u) = \lim_{t \rightarrow \infty} F_j(t, u)$$

and the equations (5) and (6) become respectively:

$$r_0 \frac{dF_0(u)}{du} = -\lambda_0 F_0(u) + \mu_1 F_1(u) \quad (7)$$

and for $j = 1, 2, \dots, N$:

$$r_j \frac{dF_j(u)}{du} = \lambda_{j-1} F_{j-1}(u) - (\lambda_j + \mu_j) F_j(u) + \mu_{j+1} F_{j+1}(u). \quad (8)$$

The boundary conditions are given by:

$$F_j(0) = 0, \quad j = 1, 2, \dots, N$$

as, if the net input rate is positive, the buffer cannot stay empty. It should also satisfy:

$$\lim_{u \rightarrow \infty} F_j(u) = p_j, \quad j \in S$$

where $p_j, j \in S$ are the stationary state probabilities of the background process given by:

$$p_j = \frac{\pi_j}{\sum_{i \in S} \pi_i} \quad j = 0, 1, \dots, N$$

with:

$$\pi_j = \frac{\lambda_0 \cdots \lambda_{j-1}}{\mu_1 \cdots \mu_j}, \quad j = 1, 2, \dots, N \text{ and } \pi_0 = 1.$$

In matrix notation the system of differential equations (7) and (8) can be written as:

$$\frac{d\mathbf{F}(u)}{du} = \mathbf{R}^{-1} \mathbf{Q}^T \mathbf{F}(u), \quad u \geq 0 \quad (9)$$

where $\mathbf{F}(u) = [F_0(u), F_1(u), \dots, F_N(u)]^T$, $\mathbf{R} = [r_0, r_1, \dots, r_N]$ and $\mathbf{Q}$ is the infinitesimal generator matrix of the process $\{X(t), t \geq 0\}$:

$$\mathbf{Q} = \begin{bmatrix} -\lambda_0 & \lambda_0 & 0 & & \\ \mu_1 & -(\lambda_1 + \mu_1) & \lambda_1 & \cdots & \\ & \ddots & \ddots & \ddots & \lambda_{N-1} \\ & & & \mu_N & -\mu_N \end{bmatrix}_{(N+1) \times (N+1)}$$

4 Stationary distribution

We introduce the following two sets:

$$S^+ \{j : r_j > 0\}$$

and:

$$S^- \{j : r_j > 0\}$$

where, S^+ consists of all states of the background queue for which the net input rate into the buffer is positive and S^- consists of all states with negative input rate.

It has already been shown (Mitra, 1988; Stern and Elwalid, 1991) that $\mathbf{R}^{-1}Q^T$ has exactly N_+ negative eigenvalues and $N_- - 1$ positive eigenvalues and one zero eigenvalue, where N_+ is the cardinality of the set S^+ and N_- is that of S^- .

Let $\xi_j, j = 0, 1, \dots, N$ be the eigenvalues of $\mathbf{R}^{-1}Q^T$ where, $\xi_j < 0, j = 0, 1, \dots, N_+ - 1, \xi_{N_+} = 0, \xi_j > 0, j = N_+ + 1, \dots, N$.

Lenin and Parthasarathy (2000) have studied numerically the behaviour of fluid buffer driven by truncated birth death processes by giving a methodology to find the eigenvalues and eigenvectors. Using the same methodology, the solution for our model is given by:

$$F_j(u) = p_j + \sum_{l=0}^{N_+-1} \beta_j^l \exp(\xi_l u), j \in S, u \geq 0 \quad (10)$$

where:

$$\beta_j^l = \frac{c_{Nj} B_j(\xi_l) \sum_{m \in S^-} d_m c_{mN} B_m(\xi_l)}{B_N(\xi_l) B'_{N+1}(\xi_l)}, l, j \in S \quad (11)$$

where $\xi_0, \xi_1, \dots, \xi_N$ are the eigenvalues of the matrix $\mathbf{R}^{-1}Q^T$, d_m are constants given by $d_m = F_m(0)$, $m \in S$ and:

$$c_{mj} = \begin{cases} \frac{\lambda_m \lambda_{m+1} \dots \lambda_{j-1}}{r_{m+1} r_{m+2} \dots r_j}, & m < j \\ 1, & m = j. \\ \frac{\mu_{j+1} \mu_{j+2} \dots \mu_m}{r_j r_{j+1} \dots r_{m-1}}, & m > j \end{cases} \quad (12)$$

The polynomials $B_n(s)$ are defined recursively as follows:

$$\begin{aligned} B_0(s) &= 1 \\ B_1(s) - \left(s + \frac{\lambda_0}{r_0}\right) B_0(s) &= 0 \\ B_n(s) - \left(s + \frac{\lambda_{n-1} + \mu_{n-1}}{r_{n-1}}\right) B_{n-1}(s) + \frac{\lambda_{n-2} \mu_{n-1}}{r_{n-1} r_{n-2}} B_{n-2}(s) &= 0, n = 2, \dots, N \\ B_{N+1}(s) - \left(s + \frac{\mu_N}{r_N}\right) B_N(s) + \frac{\lambda_{N-1} \mu_N}{r_{N-1} r_N} B_{N-1}(s) &= 0. \end{aligned} \quad (13)$$

The constants $d_m, m \in S^-$ can be obtained by solving the system of equations:

$$\sum_{m \in S^-} d_m c_{mN} B_m(\xi_l) = 0, l = N_+ + 1, \dots, N$$

$$\sum_{m \in S^-} r_m d_m = p_0 \sum_{j=0}^N r_j \pi_j.$$

It can be seen that the recursive polynomials $B_n(s)$, $n = 1, 2, \dots, N$ can be written in matrix form as:

$$B_n(s) = \begin{vmatrix} s + \frac{\lambda_0}{r_0} & \frac{-\mu_1}{r_0} & & & \\ \frac{-\lambda_0}{r_1} & s + \frac{(\lambda_1 + \mu_1)}{r_1} & \frac{-\mu_2}{r_1} & & \\ & \frac{-\lambda_1}{r_2} & s + \frac{(\lambda_2 + \mu_2)}{r_2} & \frac{-\mu_3}{r_2} & \\ & & \ddots & \ddots & \ddots \\ & & \frac{-\lambda_{n-3}}{r_{n-2}} & s + \frac{(\lambda_{n-2} + \mu_{n-2})}{r_{n-2}} & \frac{-\mu_{n-1}}{r_{n-2}} \\ & & & \frac{-\lambda_{n-2}}{r_{n-1}} & s + \frac{(\lambda_{n-1} + \mu_{n-1})}{r_{n-1}} \end{vmatrix}_{n \times n}$$

and similarly:

$$B_{N+1}(s) = \begin{vmatrix} s + \frac{\lambda_0}{r_0} & \frac{-\mu_1}{r_0} & & & \\ \frac{-\lambda_0}{r_1} & s + \frac{(\lambda_1 + \mu_1)}{r_1} & \frac{-\mu_2}{r_1} & & \\ & \frac{-\lambda_1}{r_2} & s + \frac{(\lambda_2 + \mu_2)}{r_2} & \frac{-\mu_3}{r_2} & \\ & & \ddots & \ddots & \ddots \\ & & \frac{-\lambda_{N-2}}{r_{N-1}} & s + \frac{(\lambda_{N-1} + \mu_{N-1})}{r_{N-1}} & \frac{-\mu_N}{r_{N-1}} \\ & & & \frac{-\lambda_{N-1}}{r_N} & s + \frac{\mu_N}{r_N} \end{vmatrix}_{(N+1) \times (N+1)}$$

Now, we consider two fluid models driven by birth death processes with rates as in (1), whose net input rates are given as follows:

Model 1 The input rates are $r_0 = (1 - \alpha) r < 0$ and $r_j = r > 0, j = 1, 2, \dots, N$.

Model 2 The input rates are $r_0 = (1 - \alpha)r < 0$, $r_j = r > 0$, $j = 1, 2, \dots, N - 1$ and

$$r_N = \frac{(N + \alpha)r}{(2N + 2\alpha - 1)}.$$

For the above models, as $r_0 < 0$ and $r_i > 0$, $i = 1, 2, \dots, N$, thus the cardinality of the set S^- , i.e., $N_- = 1$ and that of S^+ is $N_+ = N$. Thus there will be one zero eigenvalue and N negative eigenvalues.

For both the models the stationary state probabilities of the background BDP are given by:

$$p_j = \frac{\left(\prod_{i=0}^{j-1} \lambda_i\right)}{\left(\prod_{i=1}^j \mu_i\right)} p_0 \text{ for } j = 1, \dots, aN$$

where:

$$p_0 = \frac{\pi_0}{\sum_{j \in S} \pi_j} = \frac{1}{\sum_{j \in S} \pi_j}.$$

for $j = 1, 2, \dots, N$:

$$\begin{aligned} \pi_j &= \frac{\left(\prod_{i=0}^{j-1} \lambda_i\right)}{\left(\prod_{i=1}^j \mu_i\right)} \\ &= \frac{1}{2^j} \left(\frac{\alpha - 1}{\alpha + j - 1} \right) \\ &= \frac{1}{2^j} \left(\frac{\alpha + j}{\alpha} \right) \\ &= \frac{\alpha(\alpha - 1)}{(\alpha + j - 1)(\alpha + 1)}. \end{aligned}$$

thus:

$$p_0 = \frac{1}{\alpha(\alpha - 1) \sum_{j \in S} \frac{1}{(\alpha + j - 1)(\alpha + j)}}.$$

and hence we get:

$$p_j = \frac{1}{(\alpha + j - 1)(\alpha + j) \sum_{j \in S} \frac{1}{(\alpha + i - 1)(\alpha + i)}}. \quad (14)$$

It has been shown in Lenin and Parthasarathy (2000) that the characteristic polynomial of the matrix $\mathbf{R}^{-1}Q^T$ is given by $B_{N+1}(s)$ which can be written as:

$$B_{N+1}(s) = s^* \begin{vmatrix} s + \frac{\lambda_0}{r_0} + \frac{\mu_1}{r_1} & \frac{\sqrt{\lambda_1 \mu_1}}{r_1} & 0 & & & \\ \frac{\sqrt{\lambda_1 \mu_1}}{r_1} & s + \frac{\lambda_1}{r_1} + \frac{\mu_2}{r_2} & \frac{\sqrt{\lambda_2 \mu_2}}{r_2} & & & \\ & \ddots & \ddots & \ddots & & \\ & & & & \frac{\sqrt{\lambda_{N-1} \mu_{N-1}}}{r_{N-1}} & \\ & & & \frac{\sqrt{\lambda_{N-1} \mu_{N-1}}}{r_{N-1}} & s + \frac{\lambda_{N-1}}{r_{N-1}} + \frac{\mu_N}{r_N} \end{vmatrix}_{N \times N} \quad (15)$$

Now we find the eigenvalues in explicit form and the corresponding steady state buffer content distributions for Model 1 and 2.

Theorem 1: For the fluid model, Model 1, the eigenvalues ζ_l , $l = 0, 1, \dots, N$ of the matrix $\mathbf{R}^{-1}\mathbf{Q}^T$ are given by:

$$\zeta_0 = 0, \zeta_l = -\frac{1}{r} + \frac{1}{r} \cos\left(\frac{(2l-1)\pi}{2N+1}\right), l = 1, 2, \dots, N.$$

Further the stationary distribution of the buffer content is:

$$F_j(u) = p_j + \sum_{l=1}^N \frac{c_{Nj} B_j(\zeta_l) d_0 c_{0N}}{B_N(\zeta_l) \zeta_l \prod_{i=1, i \neq l}^N (\zeta_l - \zeta_i)} \exp(\zeta_l u), j \in S, u \geq 0$$

where:

$$p_j = \frac{1}{(\alpha + j - 1)(\alpha + j) \sum_{i \in S} \frac{1}{(\alpha + i - 1)(\alpha + i)}}$$

$$c_{Nj} = \begin{cases} \left(\frac{1}{2}\right)^N \left(\frac{N + \alpha}{\alpha(1 - \alpha)}\right), & j = 0 \\ \left(\frac{1}{2}\right)^{N-j} \frac{1}{r^{N-1}} \left(\frac{N + \alpha}{j + \alpha}\right), & j = 1, 2, \dots, N. \end{cases}$$

$$c_{0N} = \left(\frac{1}{2r}\right)^N \left(\frac{\alpha - 1}{\alpha + N - 1}\right).$$

$$d_0 = \frac{1}{\alpha(\alpha - 1) \sum_{j \in S} \frac{1}{(\alpha + j - 1)(\alpha + j)}} \left(1 - \sum_{i=1}^N \frac{\alpha}{(\alpha + i - 1)(\alpha + i)}\right)$$

Proof: Substituting the values of λ_j , μ_j , r'_j s for Model 1. In equation (15) and using (4) we get:

$$B_{N+1}(s) = s^* \begin{vmatrix} s + \frac{1}{2r} & \frac{1}{2r} & 0 & & \\ \frac{1}{2r} & s + \frac{1}{r} & \frac{1}{2r} & & \\ & \frac{1}{2r} & s + \frac{1}{r} & \frac{1}{2r} & \\ & & \ddots & \ddots & \ddots \\ & & \frac{1}{2r} & s + \frac{1}{r} & \frac{1}{2r} \\ & & & \frac{1}{2r} & s + \frac{1}{r} \end{vmatrix}_{N \times N}$$

which can be written as:

$$= s^* \begin{vmatrix} s + \frac{1}{r} - \sqrt{\frac{1}{2r^* 2r}} & \frac{1}{2r} & & & \\ \frac{1}{2r} & s + \frac{1}{r} & \frac{1}{2r} & & \\ & \frac{1}{2r} & s + \frac{1}{r} & \frac{1}{2r} & \\ & & \ddots & \ddots & \ddots \\ & & \frac{1}{2r} & s + \frac{1}{r} & \frac{1}{2r} \\ & & & \frac{1}{2r} & s + \frac{1}{r} \end{vmatrix}_{N \times N}$$

Now using the following identity given in Losonczi (1992):

$$\begin{vmatrix} v - \sqrt{pq} & p & 0 & & \\ q & v & p & \dots & \\ & q & v & p & \\ & & \ddots & \ddots & \ddots \\ & & & q & v \end{vmatrix}_{N \times N} = \prod_{l=1}^N \left(v - 2\sqrt{pq} \cos \left(\frac{(2l-1)\pi}{2N+1} \right) \right)$$

we get:

$$B_{N+1}(s) = s^* \prod_{l=1}^N \left(s + \frac{1}{r} - \frac{1}{r} \cos \left(\frac{(2l-1)\pi}{2N+1} \right) \right).$$

Thus the eigenvalues of $\mathbf{R}^{-1}Q^T$ are given by:

$$\zeta_0 = 0, \zeta_l = -\frac{1}{r} + \frac{1}{r} \cos \left(\frac{(2l-1)\pi}{2N+1} \right) l = 1, 2, \dots, N.$$

From equations (10) and (11) it can be seen that to compute the cumulative distribution function $F_j(u)$, we need to compute p'_j s and β_j^l , where β_j^l further requires computation of various constants.

From equation (12) we get:

$$c_{Nj} = \begin{cases} \left(\frac{1}{2}\right)^N \left(\frac{N+\alpha}{\alpha(1-\alpha)}\right), & j=0 \\ \left(\frac{1}{2}\right)^{N-j} \frac{1}{r^{N-1}} \left(\frac{N+\alpha}{j+\alpha}\right), & j=1, 2, \dots, N. \end{cases}$$

$$c_{0N} = \frac{\lambda_0 \lambda_1 \dots \lambda_{N-1}}{r_1 r_2 \dots r_N} = \left(\frac{1}{2r}\right)^N \left(\frac{\alpha-1}{\alpha+N-1}\right).$$

$$d_0 = p_0 \frac{\sum_{i=0}^N \pi_i r_i}{r_0} = \frac{1}{\alpha(\alpha-1) \sum_{j \in S} \frac{1}{(\alpha+j-1)(\alpha+j)}} \left(1 - \sum_{i=1}^N \frac{\alpha}{(\alpha+i-1)(\alpha+i)}\right)$$

The background stationary state probabilities p_j , $j = 1, 2, \dots, N$ can be obtained from equation (14) and thus we get our required solution.

Theorem 2: For the fluid model, Model 2, the eigenvalues ξ_l , $l = 0, 1, \dots, N$ of the matrix $\mathbf{R}^{-1} \mathbf{Q}^T$ are given by:

$$\xi_0 = 0, \xi_l = -\frac{1}{r} + \frac{1}{r} \cos\left(\frac{(2l-1)\pi}{2N}\right), l = 1, 2, \dots, N.$$

Further the stationary distribution of the buffer content is:

$$F_j(u) = p_j + \sum_{l=1}^N \frac{c_{Nj} B_j(\xi_l) d_0 c_{0N}}{B_N(\xi_l) \xi_l \prod_{i=1, i \neq l}^N (\xi_l - \xi_i)} \exp(\xi_l u), j \in S, u \geq 0$$

where:

$$p_j = \frac{1}{(\alpha+j-1)(\alpha+j) \sum_{i \in S} \frac{1}{(\alpha+i-1)(\alpha+i)}}$$

$$c_{Nj} = \begin{cases} \left(\frac{1}{2}\right)^N \left(\frac{N+\alpha}{\alpha(1-\alpha)}\right), & j=0 \\ \left(\frac{1}{2}\right)^{N-j} \frac{1}{r^{N-1}} \left(\frac{N+\alpha}{j+\alpha}\right), & j=1, 2, \dots, N. \end{cases}$$

$$c_{0N} = \left(\frac{1}{2r}\right)^N \left(\frac{(\alpha-1)(2\alpha+2N-1)}{(\alpha+N)(\alpha+N-1)}\right).$$

$$d_0 = \frac{1}{\alpha(\alpha-1) \sum_{j \in S} \frac{1}{(\alpha+j-1)(\alpha+j)}} \\ \times \left(1 - \sum_{i=1}^{N-1} \frac{\alpha}{(\alpha+i-1)(\alpha+i)} - \frac{\alpha}{(\alpha+N-1)(2\alpha+2N-1)} \right)$$

Proof: Again substituting the values of λ_j, μ_j, r'_j s for Model 2. In equation (15) and using (4) we get:

$$B_{N+1}(s) = s^* \begin{vmatrix} s + \frac{1}{2r} & \frac{1}{2r} & 0 & & \\ \frac{1}{2r} & s + \frac{1}{r} & \frac{1}{2r} & & \\ & \frac{1}{2r} & s + \frac{1}{r} & \frac{1}{2r} & \\ & & \ddots & \ddots & \ddots \\ & & & \frac{1}{2r} & s + \frac{1}{r} & \frac{1}{2r} \\ & & & & \frac{1}{2r} & s + \frac{3}{2r} \end{vmatrix}_{N \times N}$$

Now using the following identity given in Losonczi (1992):

$$\begin{vmatrix} v - \sqrt{pq} & p & 0 & & \\ q & v & p & \dots & \\ & q & v & p & \\ & & \ddots & \ddots & \ddots \\ & & & q & v + \sqrt{(pq)} \end{vmatrix}_{N \times N} = \prod_{l=1}^N \left(v - 2\sqrt{pq} \cos \left(\frac{(2l-1)\pi}{2N} \right) \right)$$

we get:

$$B_{N+1}(s) = s^* \prod_{l=1}^N \left(s + \frac{1}{r} - \frac{1}{r} \cos \left(\frac{(2l-1)\pi}{2N} \right) \right).$$

Thus the eigenvalues of $\mathbf{R}^{-1}Q^T$ for Model 2. Are given by:

$$\xi_0 = 0, \xi_l = -\frac{1}{r} + \frac{1}{r} \cos \left(\frac{(2l-1)\pi}{2N} \right) l = 1, 2, \dots, N.$$

also:

$$c_{0N} = \left(\frac{1}{2r} \right)^N \left(\frac{(\alpha-l)(2\alpha+2N-1)}{(\alpha+N)(\alpha+N-1)} \right).$$

and:

$$d_0 = \frac{1}{\alpha(\alpha-1) \sum_{j \in S} \frac{1}{(\alpha+j-1)(\alpha+j)}} \\ \times \left(1 - \sum_{i=1}^{N-1} \frac{\alpha}{(\alpha+i-1)(\alpha+i)} - \frac{\alpha}{(\alpha+N-1)(2\alpha+2N-1)} \right)$$

Thus the solution has been obtained in the required form.

4.1 Observations

- 1 The condition of $\alpha > 1$ has been assumed to maintain positive birth and death rates.
- 2 In this paper we have not made any comparisons with known results as most of the papers in literature have worked on constant and linear birth and death rates.

5 Numerical illustration

In this section we present numerical results to illustrate the steady state distribution of the two models. To visualise the foregoing discussions, we plot the graphs of the buffer content distribution for the two models under consideration by assuming certain values for the parameters. The variation in the buffer content distribution is well brought out by evaluating them numerically. The steady state distribution for buffer occupancy is defined as:

$$F(u) = \lim_{t \rightarrow \infty} P[\text{buffer content } C(t) \leq u], u \geq 0$$

Hence, we have:

$$F(u) = \sum_{j=0}^N F_j(u)$$

where $F_j(u), j \in S$ is obtained by Theorem 1.

Figures 1 and 2 are the graphical representation of the steady state distribution of Model 1 and 2 respectively for different values of the parameter α . The values of N and r are taken to be 5 and 1 respectively and α takes values 2, 3, 4 and 5. It is observed that with fixed number of states of the background process and fixed net input rates, the buffer content distribution $F(x) \rightarrow 1$ as $x \rightarrow \infty$ for all values of $\alpha > 1$.

In Figures 2 and 3, the steady state distribution for the buffer content is presented for various values of N for Model 1 and 2 respectively. We assume $r = 1$ and $\alpha = 2$ and for illustration we let N take different values 3, 5, 7 and 10. It is observed that for different values of N , the buffer content distribution $F(x) \rightarrow 1$ as $x \rightarrow \infty$.

Figure 1 Buffer content distribution for different values of the parameter α for Model 1
(see online version for colours)

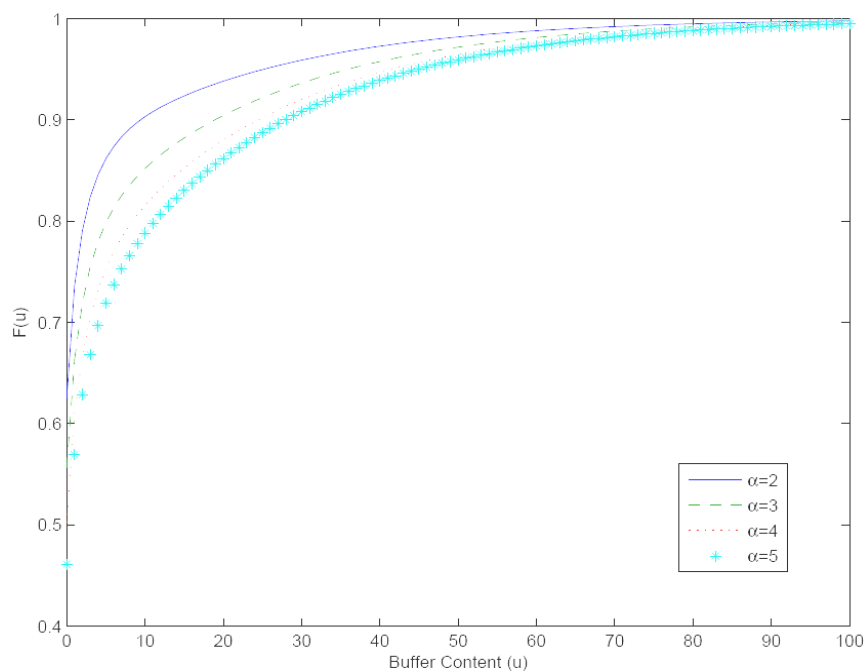


Figure 2 Buffer content distribution for different values of the parameter α for Model 2
(see online version for colours)

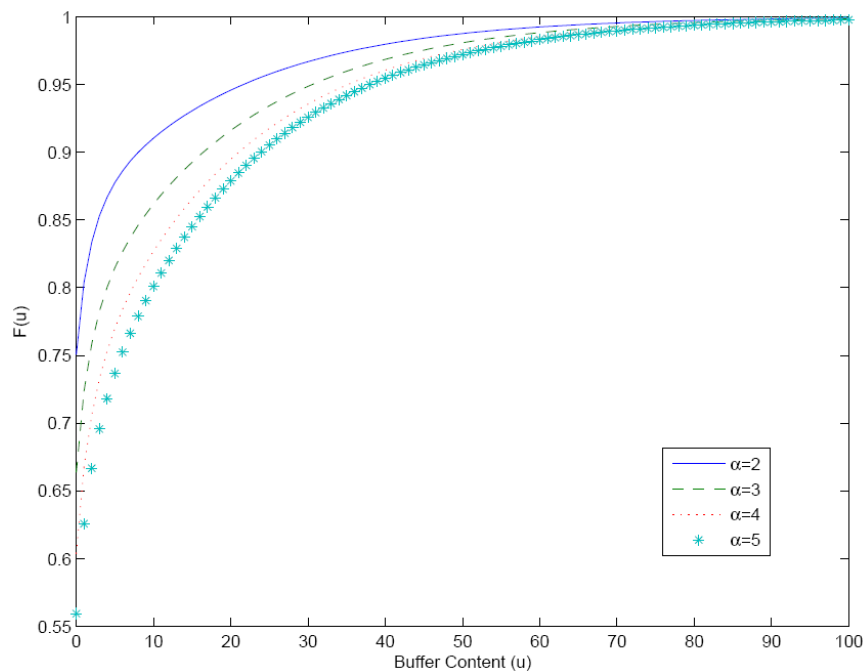


Figure 3 Buffer content distribution for different values of N for Model 1 (see online version for colours)

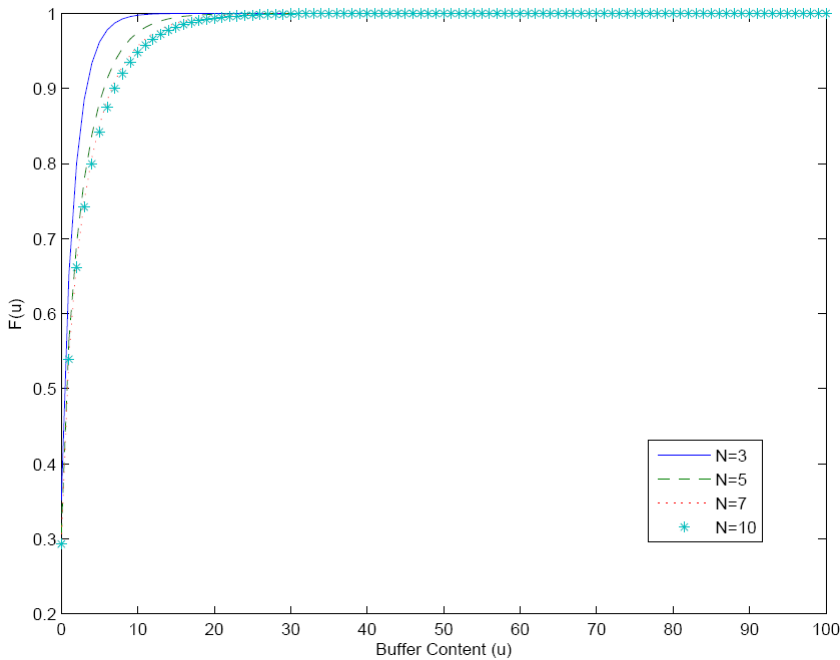
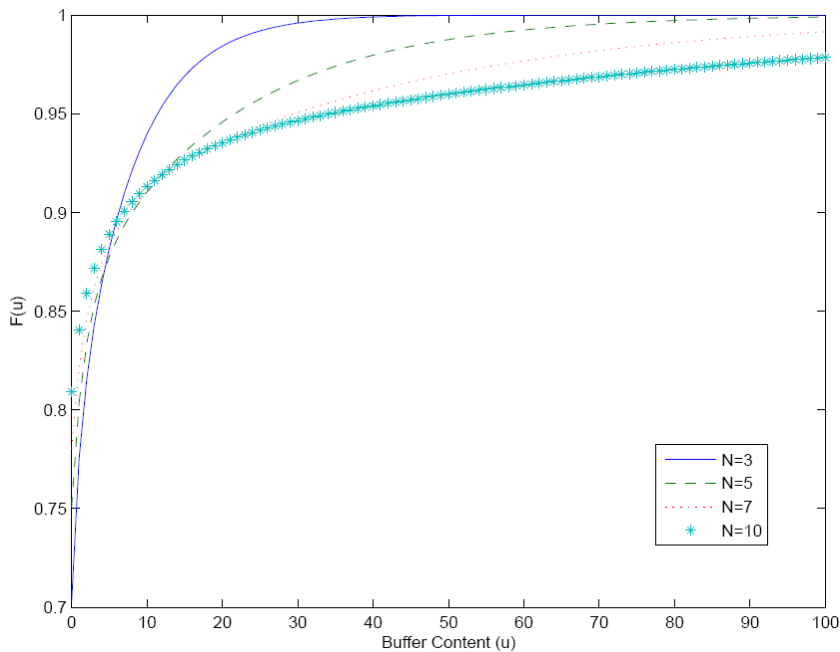


Figure 4 Buffer content distribution for different values of N for Model 2 (see online version for colours)



6 Conclusions and future work

6.1 Unique contributions of the paper

- This paper discusses a new fluid model with background model having rational birth and death rates.
- The solution is obtained in closed form.
- Numerical illustrations have been presented to validate the solution obtained.

6.2 Limitations

- Steady state distribution of the buffer content has been discussed but transient solution helps analyse the behaviour of the system at any time and can be used to improve the efficiency.
- The paper has theoretical relevance, but a direct application could not be found.

6.3 Future work

In this paper, the steady state buffer occupancy distribution for the fluid model is presented. Fluid models can be used to study the bursty nature of the traffic carried in high speed networks. But where steady state analysis is useful to study the congestion in networks, transient analysis is useful in studying the dynamical behaviour of the system to control the congestion. Hence, as future work, we plan to study the time dependent behaviour of the buffer content for the two models considered.

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