

Optimal investment-disinvestment choices in health-dependent variable annuity

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ABSTRACT

This paper exploits the influence of the policyholder's health status on the optimal time at which the policyholder decides to stop paying health-dependent premiums and starts withdrawing health-dependent benefits from a variable annuity (VA) contract accompanied by a guaranteed lifelong withdrawal benefit (GLWB). A mixed continuous-discrete time model is developed to find the optimal time for withdrawal regime initiation. The model determines the investment and disinvestment triggers according to the market conditions for both dynamic and static cases. In the static case, the optimal time is computed at the policy's inception time. In contrast, in the dynamic case, the optimal initiation time is achieved by recursive calculation of the exercise frontier of a real deferral option. Another finding is the sensitivity analysis of the contract concerning the insurance fee and the age of the policyholder.

1. Introduction

People always look for strategies to protect their financial future while minimizing personal finance and retirement planning risks. Annuities have long been understood to provide a solid income source for retirees. An annuity is a financial instrument that guarantees a consistent stream of payments over a specified period or for the duration of an individual's lifetime. One type of this instrument is the variable annuity, which is a contractual arrangement characterized by a variable value that is dependent on the performance of a portfolio consisting of sub-accounts (see Brown et al., 2001). The policyholder can avail of a variable annuity contract by making either a single payment or a series of purchase payments (see Bernard et al., 2017). This policyholder's premium is then invested in one or more mutual funds that invest in stocks, securities, currency market instruments, or a combination of the three. Hence, the value of the policyholder's investment as a variable annuity owner can fluctuate according to the performance of funds.

Variable annuity consists of various embedded options/guarantees provided by the insurer for which the policyholder has to pay extra fees to avail them. These include the Guaranteed Minimum Death Benefits (GMDB) and the Guaranteed Minimum Living Benefits (GMLB); see Hardy (2003) and Ledlie et al. (2008). Four types of guarantees fall into the category of GMLB:

- Guaranteed Minimum Accumulation Benefits (GMAB)
- Guaranteed Minimum Income Benefits (GMIB)
- Guaranteed Minimum Withdrawal Benefits (GMWB)
- Guaranteed Lifetime Withdrawal Benefits (GLWB)

These additional guarantees provide stability and safety as the policyholder approaches retirement. GLWB was first introduced in 2004 and became the most popular guarantee in the VA market [Drinkwater et al., 2014]. GLWB is an extended version of the GMWB option, in which policyholders can withdraw a predetermined guaranteed amount at fixed intervals throughout their lifetime. In the event that the account value reaches zero over the insured individual's lifespan, they retain the ability to make withdrawals of the guaranteed sum until their demise. Numerous methods have been proposed to assess the valuation of variable annuity (VA) products that have embedded options. Sharma et al. (2020) developed a time-series

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framework to obtain a fair price for the variable annuity embedded with GLWB. They consider a lump sum payment at the start. Extending their work, Singh et al. (2023) investigated the pricing problem of a variable annuity (VA) contract embedded with a GLWB rider. They considered a non-constant interest rate with a dynamic withdrawal strategy and surrender benefit. Hsieh et al. (2018) proposed a Monte Carlo valuation algorithm that uses the variance reduction technique to evaluate the long-term care annuities with GLWB. The proposed algorithm provided a faster way to evaluate similar types of products.

Traditional annuities, however, frequently lack flexibility and fail to consider the varying needs of individuals, particularly when it comes to costs associated with one's health. A model was developed by Moosavi and Payandeh (2021) to assess the value of variable long-term care annuities. These annuities feature static guaranteed lifetime withdrawal benefits and limited hospitalization coverage benefits. In 2022, D'Amico et al. (2023) proposed and evaluated a new insurance product, i.e., the variable annuity product, accompanied by the health status and the GLWB. In literature, the variable annuity product doesn't have the benefit of investment and disinvestment according to the policyholder's will. Investment criteria related to the policyholder's health status are not considered in annuity products.

D'Amico et al. (2023) proposed a modeling framework that gives the policyholder the flexibility to withdraw different amounts according to his/her health status. A one-time lump sum is invested in a risky fund that follows a Geometric Brownian Motion (GBM). The expected life and death benefits are calculated, and the fair rate of the fee charged by the insurance company is determined as the solution of the equilibrium equation between expected benefits and premiums.

Another important aspect of the study of variable annuity contracts is the timing of the starting withdrawal phase. Huang et al. (2014) provides a model to decide when to initiate the withdrawal phase in Variable Annuity with a GLWB rider and begin lifetime income payments. They considered variable annuity products that solely had GLWB riders. The impact of the policyholder's health on the withdrawal rates wasn't taken into account. Habib et al. (2020) provided us with a model in order to find an optimal Deferred Income Annuity (DIA) product. Villani (2014) provided a real option methodology to value a pioneer's R&D investment opportunity and analyzed the optimal timing to realize the Development investment. To price multidimensional American options, Ibanez and Zapatero (2004) developed a Monte Carlo simulation method based on the estimation of the optimal exercise frontiers. Bernard et al. (2017) investigated the fair fee of a flexible premium variable annuity with recurring premiums instead of a single starting premium during the accumulation phase and offered a simple model to calculate a precise approximation for the fair fee.

Motivated by the above literature, this research paper aims to study the influence of the health-status dynamic on the investment-disinvestment problem in health-dependent variable annuities with GLWB guarantee and their implications for retirees. By analyzing S&P 500 composite index prices and interest rates for the previous 50 years from the US market, using Continuous-Time Markov Chains (CTMC) for modeling the health status of the policyholder (see, e.g. Pitacco, 1995 and Christiansen, 2012) and generalized time-varying GBM for the fund behavior, we seek to provide the most suitable time to initiate the withdrawal phase that can be employed to enhance the financial outcomes of retirees using health-dependent variable annuities with GLWB guarantee. We compute the expected present values of discounted total premiums payment, discounted total withdrawals, and Death benefits to write down the wealth balance equation to evaluate the contract for the policyholder. Then, using the real options approach, we compute the optimal exercise frontier using dynamic programming.

The remainder of the paper is as follows: Section 2 describes the continuous-discrete time model and the mathematical formulas needed to determine the optimal time to initiate the withdrawal phase. Section 3 provides a solution to the optimal time problem and actuarial valuation described in section 2 and describes exercise frontiers. Section 4 includes data description, numerical results of optimal time and option value, and the effect of age and insurance fee on the optimal time. Section 5 summarizes our contribution and results with some potential further work.

2. Problem description and main assumptions

To explore the dynamics of an individual's financial decisions within an insurance framework, we consider the wealth evolution of an individual who buys an insurance policy that makes provision for a series of payments. The series will continue until the policyholder decides to stop and start with withdrawals until his/her death or the depletion of the capital, whichever occurs before. The decision variable is the optimal stopping of investment payments and the consequential switch to the withdrawal regime. The policyholder wants to maximize his/her overall benefit. For simplicity, we do not consider issues related to the utility raised by the money in time but only consider a discount factor to compare different amounts available at different times. Let us describe the considered framework.

This paper uses capital letters to denote random variables and the following notations are considered.

Notations	
y	Initial insured age, i.e., insured age at time period 0
t	Time in years
$r(t)$	Rate of interest at time t
$S(t)$	Value of the stock price at time t
s_0	Initial value of the stock price
γ	Rate of the fee charged by the insurance company
t_r	Withdrawal phase initiation time
$\psi(t_i, t_j) = e^{-\int_{t_i}^{t_j} r(s)ds}$	Time t_i price of the default-free discount bond maturing at t_j

The primary goal is to comprehend how the health-status dynamic affects investment and withdrawal decisions, as well as to identify the Expected wealth function and determine the ideal moment to start the withdrawal phase. The following presumptions will be considered when formulating our product mathematically.

Assumptions.

A1: The health status of the policyholder is described with a continuous-time Markov process $\{X(t), t \geq 0\}$ with state-space $E = \{1, 2, \dots, D\}$. The elements $1, 2, \dots$ represent different states of the policyholder's health and D is the death of the insured.

For $t > s$, let $p_{ij}(t-s) := \mathbb{P}(X(t) = j \mid X(s) = i)$ be the transition probability from state i at time s to state j at time t and let $\mathbf{Q}$ be the generator matrix for $\{X(t), t \geq 0\}$ such that

$$q_{ij} = \lim_{t \rightarrow 0} \frac{p_{ij}(t)}{t} \quad \forall i \neq j \quad \text{and} \quad q_{ii} = - \sum_{j \neq i} q_{ij}.$$

Then, the following formula holds true

$$p_{ij}(t-s) = \mathbb{P}(X(t) = j \mid X(s) = i) = (e^{\mathbf{Q}(t-s)})_{(i,j)} = \left(\sum_{l=0}^{\infty} \frac{(t-s)^l \mathbf{Q}^l}{l!} \right)_{(i,j)}. \quad (1)$$

The Markov chain hypothesis is supported by a large literature on life insurance mathematics, see, e.g. (Milbrodt and Stracke, 1997, Pitacco, 1999 and Henriksen et al., 2014).

Remark. In principle, the model could be generalized considering the evolution of the health state to be dependent on the policyholder's age. This could be done by considering a time-inhomogeneous Markov process instead of a time-homogeneous one; see, e.g., Hoem (1969). The relations obtained can be managed by considering a time-dependent generator matrix $\mathbf{Q}(t)$ and time-varying transition probabilities $p_{ij}(s, t)$. The complexity brought by relaxing the homogeneity assumption comes out when producing a real-life application where the statistical estimation problem should be considered based on a reliable and large dataset.

A2: The financial market consists of:

(a) One risk-free asset $B = \{B(t), t \geq 0\}$ given by

$$dB(t) = r(t)B(t)dt, \quad B(0) = b_0 > 0,$$

where $r = r(t)$ is the time-varying deterministic risk-free interest rate, which is assumed to be an integrable function.

(b) A risky asset $S = \{S(t), t \geq 0\}$ whose price is driven by:

$$dS(t) = \mu(t)S(t)dt + \sigma(t)S(t)dW_t, \quad S(0) = s_0 > 0,$$

where $\mu = \mu(t)$ is the expected return of the risky asset and $\sigma = \sigma(t)$ is the time-varying volatility. We further assume that $\mu = \mu(t)$ is an integrable function and $\sigma = \sigma(t)$ is a square integrable function. The notation $\mu_{[0,t]} := \{\mu(s), 0 \leq s \leq t\}$ and $\sigma_{[0,t]} := \{\sigma(s), 0 \leq s \leq t\}$ will be used later as required. The process $W = \{W_t, t \geq 0\}$ is a standard Brownian motion.

A3: We assume that the health status of the policyholder is stochastically independent of the financial market, i.e. the σ -algebras

$$\mathcal{H}_t := \sigma\{X(s), s \leq t\} \quad \text{and} \quad \mathcal{F}_t := \sigma\{W_s, s \leq t\},$$

are independent.

This third assumption allows the calculation of the actuarial functions involved in the contract.

2.1. Investment and withdrawal regime

The policyholder makes periodic payments depending on her/his health status, and the premiums are invested by the insurance company in a risky fund (in our case, for simplicity, the risky asset $S(t)$). The insurer charges a fee of rate γ as compensation for the investment activity. Consequently, the fund H_t evolves in time randomly according to the risky asset $S(t)$, the rate of the fee charged, the periodic premiums paid, and, after the withdrawal phase initiation time t_τ , to the periodic random withdrawals. The latter depends on the policyholder's health status and the fund's residual value. The described setting is formalized as follows:

1. Premiums/withdrawals are planned over a discrete time set. Let T_a be the set of times when the transactions are made (premiums/withdrawal), i.e.,

$$T_a = \{t \in \mathbb{R} : t = ak \text{ with } a \in \mathbb{R}^+ \text{ and } k \in \mathbb{N}\}.$$

For example, if time is measured in months, and benefits are paid annually, then $a = 12$ so that

$$k = 1, \quad t = 12 \text{ months} = 1 \text{ year},$$

$$k = 2, \quad t = 24 \text{ months} = 2 \text{ years}.$$

We also observe that $t_\tau \in T_a$, that is, the initiation time of the withdrawal phase, belongs to the set T_a .

We only refer to discrete annuities as the payments are made at some fixed and regularly spaced points in time. This type of contract is the most frequently encountered in real applications, see e.g. Bowie (2021).

2. The premiums are proportional to the fund value, i.e., $\forall t_i \in T_a, t_i < t_\tau$

$$I(t_i) = i(X(t_i)) \cdot H_{t_i}^-, \quad (2)$$

where $i : E \rightarrow [0, 1]$ is the % of the fund which is injected and $I(t_0) = h_0 \cdot i(j_0) > 0$. Implicitly, it is assumed that the policyholder has enough money to be able to pay the required premium at the time $t_i < t_\tau$. Equation (2) affirms that the premium paid at time t_i is proportional to the value of the fund one instant before through the factor $i(X(t))$, which depends on the health status of the policyholder.

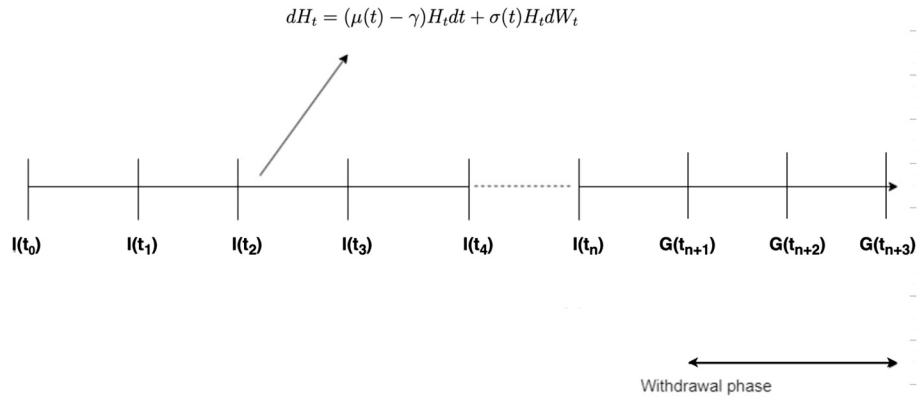


Fig. 1. Flow chart of the investment and Disinvestment.

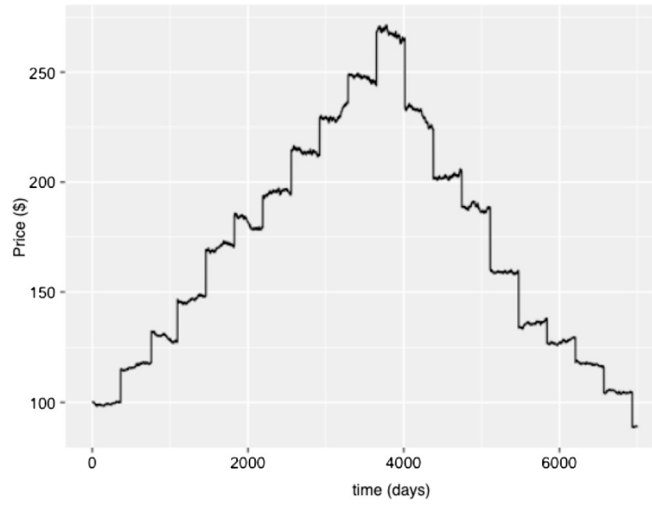


Fig. 2. Dynamics of fund.

3. The withdrawals, also known as the life benefits, are proportional to the fund value i.e. $\forall t_i \in T_a, t_i \geq t_\tau$

$$G(t_i) = g(X(t_i)) \cdot H_{t_i}^-, \quad (3)$$

where $g : E \rightarrow [0, 1]$ is the % of the fund that is withdrawn depending on the health status of the policyholder.

4. The time evolution of the fund is according to the following system of equations:

$$\begin{cases} dH_t = (\mu(t) - \gamma)H_t dt + \sigma(t)H_t dW_t & \forall t : ak < t < a(k+1), \\ H_t = (1 + i(X(t))) \cdot H_{t-} & \forall t \in T_a, t < t_\tau, \\ H_t = (1 - g(X(t))) \cdot H_{t-} & \forall t \in T_a, t \geq t_\tau. \end{cases} \quad (4)$$

Equation (4) represents the evolution of the underlying sub-account fund with time. The first part of the equation represents the evolution of the fund within two discrete-time epochs. The capital evolves as the risky asset $S(t)$ from which the fee rate γ is deduced instantaneously and with different initial values H_{t_i} given by subsequent equations. The second part describes the fund variation at the premium payment epochs. The premiums paid are proportional to the level of the fund H_{t-} through the function $i(X(t))$, which is dependent on the health-status process of the policyholder at that time. Hence, the fund level will be equal to $H_{t-}(1 + i(X(t)))$.

The final part describes the variation of the fund at the withdrawal epochs. The withdrawals are proportional to the level of the fund H_{t-} through the function $g(X(t))$, which is dependent on the health status process at that time. Hence, the fund level will be equal to $H_{t-}(1 - g(X(t)))$. The flow chart of the investment and disinvestment problem is shown in Fig. 1. One of the sample paths of the fund process H_t is shown in Fig. 2. In Fig. 2 the withdrawal initiation time is considered as day number 4000, before the initiation time the contract is in the accumulation phase and afterwards the contract is in the withdrawal phase. The fund value decreases and increases between two phases according to the Equation (4).

2.2. Expected wealth

In order to evaluate the contract, we write down the wealth balance equation for the policyholder, which includes the total outflow and inflow of the money. We obtain the conditional change of wealth of the policyholder from the inception of the contract to its maturity as follows:

$$\Delta W(t_\tau) = - \sum_{i=0}^{\tau-1} I(t_i) \psi(t_0, t_i) \mathbb{1}_{\{X(t_i) \neq D\}} + \sum_{i \geq \tau} G(t_i) \psi(t_0, t_i) \mathbb{1}_{\{X(t_i) \neq D\}} + \max\{0, \psi(t_0, c) H_c\}, \quad (5)$$

where $c \in \mathbb{R}$ is defined as $c = \inf\{t \in \mathbb{R} : X(t) = D\}$.

We observe that the change of wealth function is conditional on the switching time t_τ . Thus, we first compute the quantities assuming t_τ is known, and then we fix it according to an optimality criterion. The addenda are:

1. $\sum_{i=0}^{\tau-1} I(t_i) \psi(t_0, t_i) \mathbb{1}_{\{X(t_i) \neq D\}}$ = Discounted present value of premiums payment.
2. $\sum_{i \geq \tau} G(t_i) \psi(t_0, t_i) \mathbb{1}_{\{X(t_i) \neq D\}}$ = Discounted present value of life benefits.
3. $\max\{0, \psi(t_0, c) H_c\}$ = Discounted present value of death benefits.

Let $V_{j_0, h_0}(t_\tau, \theta)$ be the expected value of the wealth function conditioned to the initial state of the Markov chain $X(t_0) = j_0$ and the fund value h_0 , i.e., $V_{j_0, h_0}(t_\tau, \theta) = \mathbb{E}[\Delta W(t_\tau) | X(t_0) = j_0, H_{t_0}^- = h_0]$. The letter θ denotes the model parameters summarized in the vector

$\theta = \{[g(x)]_{x \in E}, [i(x)]_{x \in E}, r(t), \mu(t), \gamma, \sigma(t)\}$. A change in these parameters can be used for the sensitivity analysis. The objective of the policyholder is to maximize his/her wealth, that is, to choose t_τ such that $V_{j_0, h_0}(t_\tau, \theta)$ is maximized. Hence, we need to compute the following:

$$V_{j_0, h_0}(\theta) = \sup_{t_\tau \in T_a} V_{j_0, h_0}(t_\tau, \theta), \quad (6)$$

where, $V_{j_0, h_0}(t_\tau, \theta)$ is the expected wealth function explained above.

Now, there are two possible cases:

1. **Static:** Under this case, the policyholder fixes the withdrawal initiation time, i.e., t_τ at the beginning of the contract. This calculation is made using the available information at time zero, which is given by the initial state of the health status $X(t_0)$ and the initial value of the fund $H_{t_0}^- = h_0$.
2. **Dynamic:** The dynamic case can be treated as an investment decision problem, which depends on an option that gives the policyholder the flexibility to choose the optimal time t_τ , not necessarily at the beginning of the contract (i.e., $t = t_0$), but at any time $t_\tau \in T_a$ by deferring the decision from time t_i to the next decision time t_{i+1} .

3. Actuarial valuation formulas

This section shows how computing the actuarial functions involved in the solution of the optimization problem described before. The computations involve the expected present value of premium payments, life benefits, and death benefit. They are performed conditionally to the knowledge of the withdrawal regime's starting time, i.e., assuming t_τ as known and fixed. The computations we are going to present now generalize the contribution by D'Amico et al. (2023) and are later used to solve the optimal initiation time in both the static and dynamic cases where they represent parts of the implemented dynamic programming problem.

3.1. Computation of the expected discounted total premiums payment

In this subsection, we will compute the expected value of the total discounted premiums paid by the policyholder during the accumulation phase. Let us define $\forall j_0 \neq D$ and $h_0 \in \mathbb{R}_+$, $EP_{j_0, h_0}(t_0, t_\tau)$ be the expectation of the present value of the total premium paid by the policyholder conditioned to the initial health status and the initial fund value. Then we can write

$$\begin{aligned} EP_{j_0, h_0}(t_0, t_\tau) &= \mathbb{E}\left[\sum_{i=0}^{\tau-1} I(t_i) \psi(t_0, t_i) \mathbb{1}_{\{X(t_i) \neq D\}} | X(t_0) = j_0, H_{t_0}^- = h_0\right], \\ &= \mathbb{E}\left[\mathbb{E}\left[I(t_0) + \sum_{i=1}^{\tau-1} I(t_i) \psi(t_0, t_i) \mathbb{1}_{\{X(t_i) \neq D\}} | X(t_1), H_{t_1}^- \right] | X(t_0) = j_0, H_{t_0}^- = h_0\right], \\ &= h_0 \cdot i(j_0) + \mathbb{E}\left[\mathbb{E}\left[\sum_{i=1}^{\tau-1} I(t_i) \psi(t_0, t_i) \mathbb{1}_{\{X(t_i) \neq D\}} | X(t_1), H_{t_1}^- \right] | X(t_0) = j_0, H_{t_0}^- = h_0\right]. \end{aligned}$$

Now consider the random variable

$$\begin{aligned} \mathbb{E}\left[\sum_{i=1}^{\tau-1} I(t_i) \psi(t_0, t_i) \mathbb{1}_{\{X(t_i) \neq D\}} | X(t_1), H_{t_1}^- \right] &= \mathbb{E}\left[\sum_{i=1}^{\tau-1} I(t_i) \psi(t_1, t_i) \cdot \psi(t_0, t_1) \mathbb{1}_{\{X(t_i) \neq D\}} | X(t_1), H_{t_1}^- \right], \\ &= \psi(t_0, t_1) \cdot \mathbb{E}\left[\sum_{i=1}^{\tau-1} I(t_i) \psi(t_1, t_i) \mathbb{1}_{\{X(t_i) \neq D\}} | X(t_1), H_{t_1}^- \right], \\ &= \psi(t_0, t_1) \cdot EP_{X(t_1), H_{t_1}^-}(t_1, t_\tau), \end{aligned}$$

consequently we obtain that

$$EP_{j_0, h_0}(t_0, t_\tau) = h_0 \cdot i(j_0) + \mathbb{E}[\psi(t_0, t_1) \cdot EP_{X(t_1), H_{t_1}^-}(t_1, t_\tau) | X(t_0) = j_0, H_{t_0}^- = h_0]. \quad (7)$$

Thus, the evaluation of the formula in Equation (7) needs the joint distribution of $(X(t_1), H_{t_1}^-)$ conditional on $(X(t_0) = j_0, H_{t_0}^- = h_0)$. Let us denote by $t_{j_0, h_0}(j_1, b_1)$ this conditional density function, then for sufficiently small db_1 we have that

$$\begin{aligned} t_{j_0, h_0}(j_1, b_1)db_1 &\approx \mathbb{P}(X(t_1) = j_1, H_{t_1}^- \in (b_1, b_1 + db_1) | X(t_0) = j_0, H_{t_0}^- = h_0), \\ &= \mathbb{P}(X(t_1) = j_1 | H_{t_1}^- = b_1, X(t_0) = j_0, H_{t_0}^- = h_0) \cdot \mathbb{P}(H_{t_1}^- \in (b_1, b_1 + db_1) | X(t_0) = j_0, H_{t_0}^- = h_0). \end{aligned}$$

From the hypothesis of independence of $X(t)$ and W_t we obtain that also $X(t)$ and H_t are independent $\forall t \in (t_{i-1}, t_i)$. This implies that

$$\mathbb{P}(X(t_1) = j_1 | H_{t_1}^- = b_1, X(t_0) = j_0, H_{t_0}^- = h_0) = \mathbb{P}(X(t_1) = j_1 | X(t_0) = j_0) = (e^{Q(t_1-t_0)})_{j_0, j_1}.$$

Moreover, for db_1 small enough

$$\begin{aligned} \mathbb{P}(H_{t_1}^- \in (b_1, b_1 + db_1) | X(t_0) = j_0, H_{t_0}^- = h_0) \\ &= \mathbb{P}(H_{t_1}^- \in (b_1, b_1 + db_1) | X(t_0) = j_0, H_{t_0}^- = h_0, H_{t_0} = h_0(1 + i(j_0))), \\ &= \mathbb{P}(H_{t_1}^- \in (b_1, b_1 + db_1) | H_{t_0} = h_0(1 + i(j_0))), \\ &= f_{h_0(1+i(j_0))}(b_1; \mu_{[0, t_1]} - \gamma, \sigma_{[0, t_1]})db_1, \end{aligned}$$

where $f_{h_0(1+i(j_0))}(\cdot; \mu_{[0, t_1]} - \gamma, \sigma_{[0, t_1]})$ is the probability density function of a Lognormal distribution with:

$$\begin{aligned} \text{mean} &= h_0(1 + i(j_0)) \cdot e^{\int_0^{t_1} (\mu(s) - \gamma) ds}, \\ \text{variance} &= h_0^2(1 + i(j_0))^2 \cdot e^{\int_0^{t_1} 2(\mu(s) - \gamma) ds} \cdot (e^{\int_0^{t_1} \sigma^2(s) ds} - 1). \end{aligned}$$

Thus, we have

$$EP_{j_0, h_0}(t_0, t_\tau) = h_0 \cdot i(j_0) + \psi(t_0, t_1) \sum_{j_1 \neq D} \int_0^\infty (e^{Q(t_1)})_{j_0, j_1} \cdot f_{h_0(1+i(j_0))}(b_1; \mu_{[0, t_1]} - \gamma, \sigma_{[0, t_1]}) \cdot EP_{j_1, b_1}(t_1, t_\tau) db_1. \quad (8)$$

Equation (8) is of recurrent type and can be solved recursively observing that $\forall j_{\tau-1} \neq D, b_{\tau-1} \in \mathbb{R}_+$

$$\begin{aligned} EP_{j_{\tau-1}, b_{\tau-1}}(t_{\tau-1}, t_\tau) &= \mathbb{E}[I(t_{\tau-1}) \mathbb{1}_{\{X(t_{\tau-1}) \neq D\}} | X(t_{\tau-1}) = j_{\tau-1}, H_{t_{\tau-1}}^- = b_{\tau-1}], \\ &= b_{\tau-1} \cdot i(j_{\tau-1}). \end{aligned} \quad (9)$$

This gives the possibility to compute $EP_{j_0, h_0}(t_0, t_\tau)$ using Equation (8) and Equation (9). Indeed, once we dispose of Equation (9), we can use it to get $EP_{j_{\tau-2}, b_{\tau-2}}(t_{\tau-2}, t_\tau)$ as follows:

$$\begin{aligned} EP_{j_{\tau-2}, b_{\tau-2}}(t_{\tau-2}, t_\tau) &= b_{\tau-2} \cdot i(j_{\tau-2}) + \psi(t_{\tau-2}, t_{\tau-1}) \cdot \sum_{j_{\tau-1} \neq D} \int_0^\infty (e^{Q(t_{\tau-1}-t_{\tau-2})})_{j_{\tau-2}, j_{\tau-1}} \\ &\quad \cdot f_{b_{\tau-2}(1+i(j_{\tau-2}))}(b_{\tau-1}; (\mu_{[t_{\tau-2}, t_{\tau-1}]} - \gamma), \sigma_{[t_{\tau-2}, t_{\tau-1}]}) \cdot EP_{j_{\tau-1}, b_{\tau-1}}(t_{\tau-1}, t_\tau) db_{\tau-1}, \\ &= b_{\tau-2} \cdot i(j_{\tau-2}) + \psi(t_{\tau-2}, t_{\tau-1}) \cdot \sum_{j_{\tau-1} \neq D} \int_0^\infty (e^{Q(t_{\tau-1}-t_{\tau-2})})_{j_{\tau-2}, j_{\tau-1}} \\ &\quad \cdot f_{b_{\tau-2}(1+i(j_{\tau-2}))}(b_{\tau-1}; (\mu_{[t_{\tau-2}, t_{\tau-1}]} - \gamma), \sigma_{[t_{\tau-2}, t_{\tau-1}]}) \cdot b_{\tau-1} \cdot i(j_{\tau-1}) db_{\tau-1}. \end{aligned}$$

Working by iteration, we can get $EP_{j_0, h_0}(t_0, t_\tau)$.

3.2. Computation of the expected discounted total withdrawals

The next step is the computation of the expected discounted total withdrawals. To this end, we now define $\forall j_0 \neq D, \forall h_0 \in \mathbb{R}_+$, and $\forall t_\tau \in T_a$ the expected discounted total withdrawals as follows:

$$ELB_{j_0, h_0}(t_0, t_\tau) = \mathbb{E}[\sum_{i \geq \tau} G(t_i) \psi(t_0, t_i) \mathbb{1}_{\{X(t_i) \neq D\}} | X(t_0) = j_0, H_{t_0}^- = h_0].$$

The application of the tower property of the conditional expectation operator gives

$$ELB_{j_0, h_0}(t_0, t_\tau) = \mathbb{E}[\mathbb{E}[\sum_{i \geq \tau} G(t_i) \psi(t_0, t_i) \mathbb{1}_{\{X(t_i) \neq D\}} | X(t_\tau), H_{t_\tau}^-] | X(t_0) = j_0, H_{t_0}^- = h_0], \quad (10)$$

$$= \mathbb{E}[\psi(t_0, t_\tau) \mathbb{E}[\sum_{i \geq \tau} G(t_i) \psi(t_\tau, t_i) \mathbb{1}_{\{X(t_i) \neq D\}} | X(t_\tau), H_{t_\tau}^-] | X(t_0) = j_0, H_{t_0}^- = h_0], \quad (11)$$

$$= \psi(t_0, t_\tau) \mathbb{E}[ELB_{X(t_\tau), H_{t_\tau}^-}(t_\tau, t_\tau) | X(t_0) = j_0, H_{t_0}^- = h_0]. \quad (12)$$

The evaluation of the formula in Equation (12) needs the use of the joint law of $(X(t_\tau), H_{t_\tau}^-)$ given $(X(t_0) = j_0, H_{t_0}^- = h_0)$. Now, for db_τ sufficiently small and $j_\tau \neq D$, the conditional joint distribution can be obtained as:

$$t_{j_0, h_0}(j_\tau, b_\tau)db_\tau = \mathbb{P}(X(t_\tau) = j_\tau, H_{t_\tau}^- \in (b_\tau, b_\tau + db_\tau) | X(t_0) = j_0, H_{t_0}^- = h_0), \quad (13)$$

$$\begin{aligned}
&= \sum_{j_1 \neq D} \int_0^\infty \mathbb{P}(X(t_\tau) = j_\tau, H_{t_\tau}^- \in (b_\tau, b_\tau + db_\tau), X(t_1) = j_1, H_{t_1}^- \in (b_1, b_1 + db_1) | X(t_0) = j_0, H_{t_0}^- = h_0), \\
&= \sum_{j_1 \neq D} \int_0^\infty \mathbb{P}(X(t_\tau) = j_\tau, H_{t_\tau}^- \in (b_\tau, b_\tau + db_\tau) | X(t_1) = j_1, H_{t_1}^- = b_1) \cdot \mathbb{P}(X(t_1) = j_1, H_{t_1}^- \in (b_1, b_1 + db_1) | X(t_0) = j_0, H_{t_0}^- = h_0), \\
&= \sum_{j_1 \neq D} \int_0^\infty db_\tau t_{j_1, b_1}(j_\tau, b_\tau) \cdot t_{j_0, h_0}(j_1, b_1) db_1,
\end{aligned}$$

using the outcomes for the development of the expected discounted total premiums payment function for $t_{j_0, h_0}(j_1, b_1)$ that we previously obtained, we get

$$t_{j_0, h_0}(j_\tau, b_\tau) db_\tau = \sum_{j_1 \neq D} \int_0^\infty db_\tau t_{j_1, b_1}(j_\tau, b_\tau) \cdot (e^{Q_{t_1}})_{j_0, j_1} \cdot f_{h_0(1+i(j_0))}(b_1; \mu_{[0, t_1]} - \gamma, \sigma_{[0, t_1]}) db_1. \quad (14)$$

Using Equations (12) and (14) we get

$$ELB_{j_0, h_0}(0, t_\tau) = \psi(t_0, t_\tau) \sum_{j_\tau \neq D} \int_0^\infty ELB_{j_\tau, b_\tau}(t_\tau, t_\tau) \cdot t_{j_0, h_0}(j_\tau, b_\tau) db_\tau.$$

From this last relation, we see now that a representation formula for the integrand $ELB_{j_\tau, b_\tau}(t_\tau, t_\tau)$ is required. Using both the properties of the conditional expectation and the joint distribution of $(X(t_{\tau+1}), H_{t_{\tau+1}}^- | X(t_\tau) = j_\tau, H_{t_\tau}^- = b_\tau)$ given by

$$t_{j_\tau, b_\tau}(j_{\tau+1}, b_{\tau+1}) = (e^{Q_{t_{\tau+1}-t_\tau}})_{j_\tau, j_{\tau+1}} \cdot f_{b_\tau(1-g(j_\tau))}(b_{\tau+1}; \mu_{[t_\tau, t_{\tau+1}]} - \gamma, \sigma_{[t_\tau, t_{\tau+1}]}),$$

we have

$$\begin{aligned}
ELB_{j_\tau, b_\tau}(t_\tau, t_\tau) &= \mathbb{E}[\sum_{i \geq \tau} G(t_i) \psi(t_\tau, t_i) \mathbb{1}_{\{X(t_i) \neq D\}} | X(t_\tau) = j_\tau, H_{t_\tau}^- = b_\tau], \\
&= g(j_\tau) b_\tau + \mathbb{E}[\sum_{i \geq \tau+1} G(t_i) \psi(t_{\tau+1}, t_i) \cdot \psi(t_\tau, t_{\tau+1}) \mathbb{1}_{\{X(t_i) \neq D\}} | X(t_{\tau+1}), H_{t_{\tau+1}}^-] | X(t_\tau) = j_\tau, H_{t_\tau}^- = b_\tau], \\
&= g(j_\tau) b_\tau + \psi(t_\tau, t_{\tau+1}) \mathbb{E}[\sum_{i \geq \tau+1} G(t_i) \psi(t_{\tau+1}, t_i) \mathbb{1}_{\{X(t_i) \neq D\}} | X(t_{\tau+1}), H_{t_{\tau+1}}^-] | X(t_\tau) = j_\tau, H_{t_\tau}^- = b_\tau], \\
&= g(j_\tau) b_\tau + \psi(t_\tau, t_{\tau+1}) \mathbb{E}[ELB_{X(t_{\tau+1}), H_{t_{\tau+1}}^-}(t_{\tau+1}, t_{\tau+1}) | X(t_\tau) = j_\tau, H_{t_\tau}^- = b_\tau],
\end{aligned}$$

which finally gives $ELB_{j_\tau, b_\tau}(t_\tau, t_\tau)$ as

$$\begin{aligned}
ELB_{j_\tau, b_\tau}(t_\tau, t_\tau) &= g(j_\tau) b_\tau + \psi(t_\tau, t_{\tau+1}) \sum_{j_{\tau+1} \neq D} \int_0^\infty (e^{Q_{t_{\tau+1}-t_\tau}})_{j_\tau, j_{\tau+1}} \cdot f_{b_\tau(1-g(j_\tau))}(b_{\tau+1}; \mu_{[t_\tau, t_{\tau+1}]} - \gamma, \sigma_{[t_\tau, t_{\tau+1}]}), \\
&ELB_{j_{\tau+1}, b_{\tau+1}}(t_{\tau+1}, t_{\tau+1}) db_{\tau+1}. \quad (15)
\end{aligned}$$

To compute Equation (15), we consider the recursive structure of the equation again but introduce a killing time for the policyholder. In other words, we suppose that policyholder's lifetime cannot exceed a time $t_D \in T_a$. Accordingly, the health state is forced to be in health state D at time t_D , i.e., $X(t_D) = D$. This is not such a big issue from a practical point of view and can be formalized according to the switching operator introduced in D'Amico (2009) that checks the state of the process at time t_D and switches it instantaneously to the death state according to the rule

$$X(t_D) \circ S(t_D) = D.$$

The operator $\circ S(t_D)$ acts by the mean of a stochastic matrix $\beta = (\beta_{i,j})_{i,j \in E}$ such that

$$\mathbb{P}(X(t_D) \circ S(t_D) = j | X(t_D) = i) = \beta_{i,j} = \begin{cases} 1 & \forall i \in E, \quad j = D, \\ 0 & \forall i \in E, \quad j \neq D. \end{cases}$$

We can now solve Equation (15) recursively. First we set $t_\tau = t_D$ and note that $j_{t_D} = D$ and $\forall b_{t_D} \in \mathbb{R}_+$ we have that $ELB_{D, b_{t_D}}(t_D, t_D) = 0$. Then, set $t_\tau = t_D - a \in T_a$ and $j_{t_D-a} \in E_D, \forall b_{t_D-a} \in \mathbb{R}_+$ to have

$$\begin{aligned}
ELB_{j_{t_D-a}, b_{t_D-a}}(t_D - a, t_D - a) &= g(j_{t_D-a}) b_{t_D-a} + \psi(t_D - a, t_D) \sum_{j_{t_D} \in E_D} \int_0^\infty (e^{Q_a})_{j_{t_D-a}, j_{t_D}} \\
&\cdot f_{b_{t_D-a}(1-g(j_{t_D-a}))}(b_{t_D}; \mu_{[t_D-a, t_D]} - \gamma, \sigma_{[t_D-a, t_D]}) \cdot ELB_{j_{t_D}, b_{t_D}}(t_D, t_D) db_{t_D}.
\end{aligned}$$

Here, we observe that $ELB_{j_{t_D}, b_{t_D}}(t_D, t_D) = 0$ because $(e^{Q_a})_{j_{t_D-a}, j_{t_D}} = \mathbb{P}(X(t_0) = j_{t_D} \in E_D | X(t_D - a) = j_{t_D-a} \in E_D) = 1_{\{j_{t_D}=D\}}$ as the system is instantaneously moved into D at time t_D according to the random matrix β so that

$$ELB_{j_{t_D-a}, b_{t_D-a}}(t_D - a, t_D - a) = g(j_{t_D-a}) \cdot b_{t_D-a}.$$

Set $t_\tau = t_D - 2a$, then $j_{t_D-2a} \in E_D, \forall b_{t_D-2a} \in \mathbb{R}_+$ we have

$$ELB_{j_{t_D-2a}, b_{t_D-2a}}(t_D - 2a, t_D - 2a) = g(j_{t_D-2a})b_{t_D-2a} + \psi(t_D - 2a, t_D - a) \sum_{j_{t_D-a} \in E_D} \int_0^\infty (e^{Qa})_{j_{t_D-2a}, j_{t_D-a}} \\ \cdot f_{b_{t_D-2a}(1-g(j_{t_D-2a}))}(b_{t_D-a}; \mu_{[t_D-2a, t_D-a]} - \gamma, \sigma_{[t_D-2a, t_D-a]}) \cdot ELB_{j_{t_D-a}, b_{t_D-a}}(t_D - a, t_D - a) db_{t_D-a}.$$

The above equation contains all the parameters that are well-known from the previous steps. This reverse recursion produces $ELB_{j_\tau, b_\tau}(t_\tau, t_\tau)$ with each iteration.

3.3. Computation of the expected death benefits

The final quantity we need to evaluate is the expected value of death benefits. The death benefit random variable is defined as $DB(t_0, c) = \max\{0, \psi(t_0, c)H_c\}$, where $c = \inf\{t \in \mathbb{R}_+ : X(t) = D\}$. Since the death benefit depends on the remaining value of the fund at the time of death of the insured, in turn it depends on the time t_τ .

Consider the conditional expectation of the discounted death benefits given initial conditions about the health status $X(t_0) = j_0$, the initial premium payment $H_{t_0}^- = h_0$, the knowledge of the switching time t_τ and of the time of death c . Formally, we denote it by

$$EDB_{j_0, h_0}(t_\tau; c) = \mathbb{E}[DB(t_\tau, c) | X(t_0) = j_0, H_{t_0}^- = h_0]. \quad (16)$$

This can be further represented according to:

$$EDB_{j_0, h_0}(t_\tau; c) = \mathbb{E}[\max\{0, \psi(t_0, c)H_c\} \mathbb{1}_{\{c < t_\tau\}}] + \mathbb{E}[\max\{0, \psi(t_0, c)H_c\} \mathbb{1}_{\{c \geq t_\tau\}}], \\ =: EDB_{j_0, h_0}^{<}(t_\tau; c) + EDB_{j_0, h_0}^{\geq}(t_\tau; c).$$

Let's start computing $EDB_{j_0, h_0}^{<}(t_\tau; c)$, we know that:

$$DB^{<}(\tau; c) = \psi(t_0, c) \cdot \max\{0, h_0 \xi(t_0, c) + \sum_{s=0}^{\lfloor c \rfloor} H_{t_s}^- \cdot i(X(t_s)) \xi(t_s, c)\},$$

where, in general $\xi(s, t) := e^{\int_s^t (\mu(v) - \gamma - \frac{\sigma^2(v)}{2}) dv + \int_s^t \sigma(v) dW_v}$ and $\lfloor c \rfloor = \max\{k \in \mathbb{N} : k \cdot a \leq c\}$. Observe now that in the case when $c < t_\tau$ we are injecting money in the fund by paying premiums without entering the withdrawal regime, thus

$$\max\{0, h_0 \xi(t_0, c) + \sum_{s=0}^{\lfloor c \rfloor} H_{t_s}^- \cdot i(X(t_s)) \xi(t_s, c)\} = h_0 \xi(t_0, c) + \sum_{s=0}^{\lfloor c \rfloor} H_{t_s}^- \cdot i(X(t_s)) \xi(t_s, c).$$

Then

$$\mathbb{E}[DB^{<}(\tau; c) | X(t_0) = j_0, H_{t_0}^- = h_0] = \psi(t_0, c) h_0 \mathbb{E}[\xi(t_0, c) | X(t_0) = j_0, H_{t_0}^- = h_0] + \psi(t_0, c) \sum_{s=0}^{\lfloor c \rfloor} \mathbb{E}[\xi(t_s, c) H_{t_s}^- \cdot i(X(t_s)) | X(t_0) = j_0, H_{t_0}^- = h_0]. \quad (17)$$

Now since $\int_{t_0}^c \sigma(v) dW_v$ follows normal distribution with zero mean and variance equal to $\int_{t_0}^c \sigma^2(v) dv$, then its exponential transform is log-normally distributed with mean $\mathbb{E}[e^{\int_{t_0}^c \sigma(v) dW_v}] = e^{\int_{t_0}^c \frac{\sigma^2(v)}{2} dv}$. Therefore

$$\psi(t_0, c) \mathbb{E}[\xi(t_0, c) \cdot h_0 | X(t_0) = j_0, H_{t_0}^- = h_0] = h_0 e^{\int_{t_0}^c (\mu(v) - \gamma - r(v)) dv}.$$

We now derive the second expectation in (17). Note that

$$\psi(t_0, c) \mathbb{E}[\xi(t_s, c) H_{t_s}^- \cdot i(X(t_s)) | X(t_0) = j_0, H_{t_0}^- = h_0], \\ = \psi(t_0, c) \mathbb{E}[\mathbb{E}[\xi(t_s, c) H_{t_s}^- \cdot i(X(t_s)) | X(t_s), H_{t_s}^-] | X(t_0) = j_0, H_{t_0}^- = h_0], \\ = \psi(t_0, c) \mathbb{E}[H_{t_s}^- \cdot i(X(t_s)) \mathbb{E}[\xi(t_s, c) | X(t_s), H_{t_s}^-] | X(t_0) = j_0, H_{t_0}^- = h_0].$$

The process $\xi(t_s, c)$ is independent of the joint process $(X(t_s), H_{t_s}^-)$ because of the independence assumption A3 and the fact that the Wiener process has independent increments, hence

$$\mathbb{E}[\xi(t_s, c) | X(t_s), H_{t_s}^-] = e^{\int_{t_s}^c (\mu(v) - \gamma) dv}.$$

By substitution, it gives

$$\mathbb{E}[\xi(t_s, c) \cdot H_{t_s}^- \cdot i(X(t_s)) | X(t_0) = j_0, H_{t_0}^- = h_0] = e^{\int_{t_s}^c (\mu(v) - \gamma) dv} \mathbb{E}[H_{t_s}^- \cdot i(X(t_s)) | X(t_0) = j_0, H_{t_0}^- = h_0].$$

Now, according to the conditional joint law of $(H_{t_s}^-, X(t_s))$ we get

$$\mathbb{E}[H_{t_s}^- \cdot i(X(t_s)) | X(t_0) = j_0, H_{t_0}^- = h_0] = \sum_{j_s \neq D} \int_0^\infty t_{j_0, h_0}(j_s, b) \cdot b \cdot i(j_s) db$$

$$= \sum_{j_s \neq D} \int_0^\infty (e^{\mathbf{Q}(t_s - t_0)})_{j_0, j_s} \cdot f_{h_0(1+i(j_0))}(b; \mu_{[t_0, t_1]} - \gamma, \sigma_{[t_0, t_1]}) \cdot b \cdot i(j_s) db.$$

From which we get the final result:

$$\begin{aligned} EDB_{j_0, h_0}^<(t_0; c) &= h_0 e^{\int_{t_0}^c (\mu(v) - \gamma - r(v)) dv} \\ &+ \sum_{s=0}^{\lfloor c \rfloor} e^{-\int_{t_0}^{t_s} r(v) dv} e^{\int_{t_s}^c (\mu(v) - \gamma - r(v)) dv} \sum_{j_s \neq D} \int_0^\infty (e^{\mathbf{Q}(t_s - t_0)})_{j_0, j_s} \cdot f_{h_0(1+i(j_0))}(b; \mu_{[t_0, t_1]} - \gamma, \sigma_{[t_0, t_1]}) \cdot b \cdot i(j_s) db. \end{aligned} \quad (18)$$

Our next task is to compute

$$\begin{aligned} EDB_{j_0, h_0}^>(t_\tau; c) &= \mathbb{E}[\max\{0, \psi(t_0, c) H_c\} \cdot \mathbb{1}_{\{c \geq t_\tau\}} | X(t_0) = j_0, H_{t_0}^- = h_0] \\ &= \mathbb{E}[\mathbb{E}[\max\{0, \psi(t_0, c) H_c\} \cdot \mathbb{1}_{\{c \geq t_\tau\}} | H_{t_\tau}^-, X(t_\tau)] | X(t_0) = j_0, H_{t_0}^- = h_0]. \end{aligned}$$

Consider the inner expectation and denote its conditional version by

$$EDB_{j_\tau, b_\tau}^>(t_\tau; c) = \mathbb{E}[\max\{0, \psi(t_0, c) H_c\} \cdot \mathbb{1}_{\{c \geq t_\tau\}} | H_{t_\tau}^- = b_\tau, X(t_\tau) = j_\tau]. \quad (19)$$

In line with the analysis developed in Milevsky and Salisbury (2006) and D'Amico et al. (2023), we have that:

$$EDB_{j_\tau, b_\tau}^>(t_\tau; c) = \psi(t_0, c) \mathbb{E}[\xi(t_\tau, c) \cdot \max\{0, H_{t_\tau}^-(1 - g(X(t_\tau))) - \sum_{s=\tau+1}^{\lfloor c \rfloor} H_{t_s}^- \cdot g(X(t_s))\} \cdot \xi^{-1}(t_\tau, t_s) | X(t_\tau) = j_\tau, H_{t_\tau}^- = b_\tau]. \quad (20)$$

To evaluate the expectation in Equation (20), we first observe that:

$$\begin{aligned} H_{t_s}^- &= H_{t_{s-1}}^- \cdot \xi(t_{s-1}, t_s) = H_{t_{s-1}}^- \cdot (1 - g(X(t_{s-1}))) \cdot \xi(t_{s-1}, t_s) \\ &= H_{t_{s-2}}^- \cdot \xi(t_{s-2}, t_{s-1}) \xi(t_{s-1}, t_s) \cdot (1 - g(X(t_{s-1}))) \\ &= H_{t_{s-2}}^- \cdot \xi(t_{s-2}, t_s) \cdot (1 - g(X(t_{s-1}))) (1 - g(X(t_{s-2}))), \end{aligned}$$

a repetition of the previous argument leads us to

$$H_{t_s}^- = H_{t_\tau}^- \cdot \xi(t_\tau, t_s) \prod_{k=1}^{s-\tau} (1 - g(X(t_{s-k}))),$$

and by substituting in Equation (20) we have

$$\begin{aligned} EDB_{j_\tau, b_\tau}^>(t_\tau; c) &= \psi(t_0, c) \mathbb{E}[\max(0, b_\tau(1 - g(j_\tau)) \cdot \xi(t_\tau, c) - \xi(t_\tau, c) \sum_{s=\tau+1}^{\lfloor c \rfloor} H_{t_s}^- \cdot \prod_{k=1}^{s-\tau} (1 - g(X(t_{s-k}))) \cdot \\ &\quad \xi(t_\tau, t_s) \cdot g(X(t_s)) \cdot \xi^{-1}(t_\tau, t_s) | X(t_\tau) = j_\tau, H_{t_\tau}^- = b_\tau]. \end{aligned}$$

We note that $H_{t_\tau}^- = b_\tau$ and hence

$$\begin{aligned} EDB_{j_\tau, b_\tau}^>(t_\tau; c) &= \psi(t_0, c) \mathbb{E}[\max(0, b_\tau(1 - g(j_\tau)) \cdot \xi(t_\tau, c) - \sum_{s=\tau+1}^{\lfloor c \rfloor} b_\tau \\ &\quad \cdot \prod_{k=1}^{s-\tau} (1 - g(X(t_{s-k}))) \cdot g(X(t_s)) \cdot \xi(t_\tau, c) | X(t_\tau) = j_\tau, H_{t_\tau}^- = b_\tau] \\ &= \psi(t_0, c) \mathbb{E}[\max(0, b_\tau \cdot \xi(t_\tau, c) \cdot \{(1 - g(j_\tau)) - \sum_{s=\tau+1}^{\lfloor c \rfloor} \prod_{k=1}^{s-\tau} (1 - g(X(t_{s-k}))) \cdot g(X(t_s))\} | X(t_\tau) = j_\tau, H_{t_\tau}^- = b_\tau]. \end{aligned}$$

We further observe that due to assumption A3, the two random variables $\xi(t_\tau, c)$ and

$$A(\tau, \lfloor c \rfloor) := \sum_{s=\tau+1}^{\lfloor c \rfloor} \prod_{k=1}^{s-\tau} (1 - g(X(t_{s-k}))) g(X(t_s)),$$

are independent. Therefore, we have:

$$\begin{aligned} EDB_{j_\tau, b_\tau}^>(t_\tau; c) &= \psi(t_0, c) \mathbb{E}[\max(0, b_\tau \cdot \xi(t_\tau, c) \cdot \{(1 - g(j_\tau)) - A(\tau, \lfloor c \rfloor)\} | X(t_\tau) = j_\tau, H_{t_\tau}^- = b_\tau] \\ &= \psi(t_0, c) \int_0^\infty \mathbb{P}(b_\tau \cdot \xi(t_\tau, c) ((1 - g(j_\tau)) - A(\tau, \lfloor c \rfloor)) > l | X(t_\tau) = j_\tau, H_{t_\tau}^- = b_\tau) dl \\ &= \psi(t_0, c) \int_0^\infty \mathbb{P}((1 - A(\tau, \lfloor c \rfloor)) > \frac{l}{b_\tau \cdot \xi(t_\tau, c)} + g(j_\tau) | X(t_\tau) = j_\tau, H_{t_\tau}^- = b_\tau) dl \end{aligned}$$

$$= \psi(t_0, c) \int_0^\infty \int_0^\infty \mathbb{P}((1 - A(\tau, \lfloor c \rfloor)) > \frac{l + g(j_\tau) b_\tau x}{b_\tau \cdot x} | \xi(t_\tau, c) = x, X(t_\tau) = j_\tau, H_{t_\tau}^- = b_\tau) \\ \cdot \mathbb{P}(\xi(t_\tau, c) \in (x, x + dx) | X(t_\tau) = j_\tau, H_{t_\tau}^- = b_\tau) dl.$$

It now becomes of interest to determine the probability distribution of $\xi(t_\tau, c)$. In particular,

$$\begin{aligned} \mathbb{P}(\xi(t_\tau, c) \in (x, x + dx) | X(t_\tau) = j_\tau, H_{t_\tau}^- = b_\tau) \\ = \mathbb{P}(e^{\int_{t_\tau}^c (\mu(v) - \gamma - \frac{\sigma^2(v)}{2}) dv} e^{\int_{t_\tau}^c \sigma(v) dW_v} \in (x, x + dx) | X(t_\tau) = j_\tau, H_{t_\tau}^- = b_\tau) \\ = \mathbb{P}(e^{\int_{t_\tau}^c \sigma(v) dW_v} \in (\frac{x}{e^{\int_{t_\tau}^c (\mu(v) - \gamma - \frac{\sigma^2(v)}{2}) dv}}, \frac{x + dx}{e^{\int_{t_\tau}^c (\mu(v) - \gamma - \frac{\sigma^2(v)}{2}) dv}}) | X(t_\tau) = j_\tau, H_{t_\tau}^- = b_\tau). \end{aligned}$$

Now, by setting $z = x \cdot e^{-\int_{t_\tau}^c (\mu(v) - \gamma - \frac{\sigma^2(v)}{2}) dv}$, we obtain

$$\mathbb{P}(\xi(t_\tau, c) \in (x, x + dx) | X(t_\tau) = j_\tau, H_{t_\tau}^- = b_\tau) = \mathbb{P}(e^{\int_{t_\tau}^c \sigma(v) dW_v} \in (z, z + dz) | X(t_\tau) = j_\tau, H_{t_\tau}^- = b_\tau). \quad (21)$$

As previously observed, we know that $e^{\int_{t_\tau}^c \sigma(v) dW_v}$ follows a Lognormal distribution with mean value $\int_{t_\tau}^c \frac{\sigma^2(v)}{2} dv$ and variance $[e^{\int_{t_\tau}^c \sigma^2(v) dv} - 1]$. Its probability density function will be denoted simply by $\phi(\cdot)$. Therefore, it is clear that for dz sufficiently small, equation (21) is equal to $\phi(z)dz$.

We are now in a position to provide an integral representation of the function $EDB_{j_\tau, b_\tau}^\geq(t_\tau; c)$. Indeed, we have that

$$\begin{aligned} EDB_{j_\tau, b_\tau}^\geq(t_\tau; c) &= \psi(t_0, c) \cdot \int_0^\infty \int_0^\infty \mathbb{P}((1 - A(\tau, \lfloor c \rfloor)) > \frac{l + g(j_\tau) \cdot z \cdot e^{\int_{t_\tau}^c (\mu(v) - \gamma - \frac{\sigma^2(v)}{2}) dv} \cdot b_\tau}{b_\tau \cdot z \cdot e^{\int_{t_\tau}^c (\mu(v) - \gamma - \frac{\sigma^2(v)}{2}) dv}} | \\ \xi(t_\tau, c) &= z \cdot e^{\int_{t_\tau}^c (\mu(v) - \gamma - \frac{\sigma^2(v)}{2}) dv}, X(t_\tau) = j_\tau, H_{t_\tau}^- = b_\tau) \times \phi(z) dl dz \\ &= \psi(t_0, c) \cdot \int_0^\infty \int_0^\infty F_{j_\tau}(\tau, \lfloor c \rfloor; 1 - \frac{l + g(j_\tau) \cdot z \cdot e^{\int_{t_\tau}^c (\mu(v) - \gamma - \frac{\sigma^2(v)}{2}) dv} \cdot b_\tau}{b_\tau \cdot z \cdot e^{\int_{t_\tau}^c (\mu(v) - \gamma - \frac{\sigma^2(v)}{2}) dv}}) \cdot \phi(z) dl dz, \end{aligned}$$

where

$$\begin{aligned} F_{j_\tau}(\tau, \lfloor c \rfloor; \theta) &= \mathbb{P}(A(\tau, \lfloor c \rfloor) \leq \theta | X(t_\tau) = j_\tau) = \sum_{j_{\tau+1} \in E} \mathbb{P}(A(\tau, \lfloor c \rfloor) \leq \theta, X(t_{\tau+1}) = j_{\tau+1} | X(t_\tau) = j_\tau) \\ &= \sum_{j_{\tau+1} \in E} (e^{Q(t_{\tau+1} - t_\tau)})_{j_\tau, j_{\tau+1}} \cdot F_{j_{\tau+1}}(\tau + 1, \lfloor c \rfloor; \frac{\theta}{j_{\tau+1}}). \end{aligned}$$

Summarizing,

$$\begin{aligned} EDB^\geq(t_\tau; c) &= \sum_{j_\tau \neq D} \int_0^\infty t_{j_0, h_0}(j_\tau, b_\tau) \cdot EDB_{j_\tau, b_\tau}^\geq(t_\tau; c) db_\tau \\ &= \sum_{j_\tau \neq D} \int_0^\infty (e^{Q(t_\tau - t_0)})_{j_0, j_\tau} \cdot f_{h_0(1 + l(j_0))}(b_\tau; \mu_{[t_0, t_\tau]} - \gamma, \sigma_{[t_0, t_\tau]}) \cdot \psi(t_0, c) \left(\int_0^\infty \int_0^\infty F_{j_\tau}(\tau, \lfloor c \rfloor; 1 - \right. \\ &\quad \left. \frac{l + g(j_\tau) \cdot z \cdot e^{\int_{t_\tau}^c (\mu(v) - \gamma - \frac{\sigma^2(v)}{2}) dv} \cdot b_\tau}{b_\tau \cdot z \cdot e^{\int_{t_\tau}^c (\mu(v) - \gamma - \frac{\sigma^2(v)}{2}) dv}}) \cdot \phi(z) dl dz \right) db_\tau. \end{aligned}$$

Finally, we can conclude the computation of the EDB by considering the probability of the death event exactly at time c . According to D'Amico et al. (2023) we have:

$$EDB = \int_0^\infty EDB(t_0; c) \cdot \lambda(c) dc \quad (22)$$

$$= \int_0^{t_\tau} EDB^\leq(t_\tau; c) \cdot \lambda(c) dc + \int_{t_\tau}^\infty EDB^\geq(t_\tau; c) \cdot \lambda(c) dc, \quad (23)$$

where

Table 1
Drift and volatility parameters
for the partitioned data.

S&P 500	μ	σ
Data 1	-0.0046	0.1350
Data 2	0.0257	0.0710
Data 3	0.0512	0.2400
Data 4	0.1248	0.4347
Data 5	0.2201	0.7545
Data 6	0.3186	2.5322
Data 7	0.2467	1.4116
Data 8	-0.0147	2.0111
Data 9	0.7386	2.9948

$$\lambda(c) = \lim_{h \rightarrow 0} \frac{\mathbb{P}(c < \tau \leq c + h \mid \tau > c)}{h},$$

is the so-called failure rate function. For an n -state continuous-time Markov chain, it can be expressed according to Sadek and Limnios (2005)

$$\lambda(c) = \begin{cases} \frac{-1}{R(c)} \cdot \frac{dR(c)}{dc} = -\frac{\alpha_1 e^{cQ_{(n-1,n-1)}} \cdot Q_{(n-1,n-1)} \mathbb{1}_{(n-1)}}{\alpha_1 e^{cQ_{(n-1,n-1)}} \mathbb{1}_{(n-1)}} & \text{if } R(c) \neq 0, \\ 0 & \text{if } R(c) = 0, \end{cases} \quad (24)$$

where Q is the generator matrix of the Markov chain, α is the initial probability distribution, $R(c)$ the reliability function and $\mathbb{1}_{(n-1)}$ is the unitary vector of the dimension $n - 1$.

3.4. Solving the optimal decision problems

This section discusses the solutions to the optimal decision problems raised in the static and dynamic cases. Once the actuarial functions have been calculated according to the formulas developed in the previous subsections, we can use them to solve static and dynamic optimization problems. In relation to the static problem First, we observe that t_τ is given, so the value function $V_{x_0, h_0}(t_\tau, \theta)$ can be computed simply by taking the maximum values over all possible values of t_τ . Successively, we can solve the dynamic optimization problem by determining and treating it as an investment. The conditional value of the dynamic opportunity can be computed using the real option approach and solved using dynamic programming, e.g., Dixit et al. (1994). Indeed, denoted by $F_{j_h, b_h}(t_h)$ the value of the deferring strategy conditional on $X(t_h) = j_h$, $H_{t_h}^- = b_h$ and $t_\tau \geq t_h$, it satisfies the equations

$$F_{j_h, b_h}(t_h) = V_{j_h, b_h}(t_h; \theta),$$

where $t_h = T$.

$$F_{j_h, b_h}(t_h) = \max\{V_{j_h, b_h}(\theta), -b_h \cdot i(j_h) + \psi(t_h, t_{h+1}) \cdot \mathbb{E}[V_{X(t_{h+1}), H_{t_{h+1}}^-}(\theta) \mid X(t_h) = j_h, H_{t_h}^- = b_h]\}.$$

It should be noticed that $V_{j_h, b_h}(t_h; \theta)$ can be directly computed according to the computation illustrated in the subsections 3.1, 3.2, and 3.3. Successively to the computation of $V_{j_h, b_h}(t_h; \theta)$, the quantity $F_{j_h, b_h}(t_h)$ can be obtained using backward dynamic programming, which in turn gives the optimal switching times t_τ . The difference $F_{j_0, h_0}(t_0) - V_{j_0, h_0}(\theta)$ denotes the value of the deferral option. This set of points at which the investor exhibits indifference toward exercising and keeping the option alive is known as the exercise frontier. If the investment's discounted cash flows at time t , $V_{j_t, b_t}(t, \theta)$ exceeds V_t^* , exercising the option at time t is the optimal strategy, and waiting otherwise.

4. Numerical results and sensitivity analysis

In this section, we describe the data used for the numerical calculations and the numerical results obtained for the given model.

4.1. Data description

This research paper focuses on conducting a numerical analysis using the US market's S&P 500 composite index prices. This index was selected based on its popularity and data availability in the country. The time frame for the analysis spans from January 1, 1970, to October 31, 2019. To obtain the S&P 500 index data, we accessed Thomson and Reuters Datastream.

The parameters $\mu(t)$ and $\sigma(t)$ are derived from the S&P 500 dataset, which is partitioned into nine equally sized segments. These parameters are then approximated through the utilization of a step function, serving as a basis for subsequent numerical calculations. The parameters that result are detailed in Table 1.

Additionally, historical annual interest rate data for the US market spanning the previous 50 years was obtained from a reliable source (Fred Economic Data, 2021).

In evaluating an insurance health product, the health condition of policyholders and the corresponding transition probabilities are of utmost importance. The classification of an individual's health status aligns with the approach proposed by D'Amico et al. (2023). This study utilizes two-year transition probabilities between disability states, calculated based on data from the 1982 and 1984 NLTC (National Long-Term Care) Survey (Table 2). The transition probabilities, expressed as percentages, are specific to males and females, as mentioned in Pritchard's (2006) work. Table 3

Table 2

Two-year transition probabilities between disability states calculated from the 1982 and 1984 NLTCS, as a percentage, for males and females according to Pritchard (2006).

1984 status								
Age	1982 status	Healthy	IADL only	1-2 ADLs	3-4 ADLs	5-6 ADLs	Inst'd	Dead
65	Healthy	90.01	2.41	1.92	0.59	0.56	0.47	4.04
71		83.73	3.60	2.41	0.95	0.93	1.11	7.28
77		76.74	5.40	4.08	1.30	1.24	1.94	9.30
65	IADL Only	30.69	31.53	15.70	4.55	4.28	2.19	11.06
71		25.27	32.53	17.48	3.64	2.77	4.93	13.38
77		15.39	30.65	18.73	5.01	5.31	7.53	17.38
65	1-2 ADLs	16.25	13.44	35.96	10.50	6.05	2.94	14.87
71		10.49	13.22	34.40	12.35	5.72	6.92	16.89
77		9.20	12.36	34.83	10.34	6.71	6.08	20.48
65	3-4 ADLs	7.19	4.69	28.22	23.10	13.19	4.74	18.85
71		8.40	3.31	19.30	28.75	18.52	4.00	17.72
77		3.76	3.51	18.31	20.65	16.51	11.01	26.25
65	5-6 ADLs	7.02	5.77	8.75	9.05	32.98	5.86	30.57
71		6.13	5.07	8.92	10.24	28.49	8.02	33.12
77		3.97	3.64	4.98	7.71	34.73	11.56	33.42
65	Inst'd	7.54	0.97	1.63	2.30	0.12	72.22	15.22
71		2.97	0.75	1.81	1.25	2.36	52.39	38.48
77		1.74	1.48	0.88	0.74	1.48	56.87	36.81
65	Dead	0	0	0	0	0	0	1
71		0	0	0	0	0	0	1
77		0	0	0	0	0	0	1

Table 3

Optimal time and option value corresponding to different product fee and age for dynamic case.

Fee (basis points)	Age = 65		Age = 71		Age = 77	
	τ (years)	value	τ (years)	value	τ (years)	value
$\gamma = 1000$	14	15606.50	11	13903.45	10	12578.00
$\gamma = 1100$	13	14563.89	10	13963.32	9	12630.78
$\gamma = 1200$	11	12630.78	8	12506.77	8	12487.70

provides an overview of these two-year transition probabilities, where Healthy; IADL only; 1–2 ADLs; 3–4 ADLs; 5–6 ADLs; Institutionalized; and Dead represent different health states for a detailed description (see Pritchard's 2006).

The numerical calculations were performed using RStudio version 2021.09.0 Build 351 on a system with the following specifications: AMD Ryzen 5 3500U with Radeon Vega Mobile Gfx 2.10 GHz, a 64-bit operating system, an x64-based processor, and 8 GB of RAM.

4.2. Numerical calculation

For the numerical calculation, the following conditions are considered:

- Initially, the state of the policyholder begins with state 1 (healthy), i.e., $\alpha = (1, 0, 0, 0, 0, 0)$.
- To remove the complexity of numerical calculation, the upper limit to initiate the withdrawal phase is set to 15 years.
- The investment amount in this scenario is determined on the basis of the health status of the patient. Specifically, for health states 1 and 2, the investment amount is assumed to be 5% of the fund value. For states 3 and 4, it is considered 4% of the fund value. Lastly, for the 5th and 6th states, the investment amount is supposed to be 3% of the fund value. That is

$$\mathbf{i} = [5\%, 5\%, 4\%, 4\%, 3\%, 3\%]^T.$$

- The withdrawal amount in this context is determined on the basis of the health status of the patient. Specifically, for health states 1 and 2, the withdrawal amount is assumed to be 2% of the fund value. For states 3 and 4, it is supposed to be 3% of the fund value. Lastly, for states 5 and 6, the withdrawal amount is considered to be 4% of the fund value. That is

$$\mathbf{g} = [2\%, 2\%, 3\%, 3\%, 4\%, 4\%]^T.$$

To proceed with the application of the model, it is necessary to estimate the Markov chain's generator matrix $\mathbf{Q}$. Due to the lack of precise data, we utilize the 2-year transition probability matrices presented in Table 3. In some instances, it is possible to derive the generator matrix from these matrices.

The maximum likelihood estimator $\hat{\mathbf{Q}}$ of the generator matrix $\mathbf{Q}$ satisfies the relationship $\hat{\mathbf{P}} = \exp(a\hat{\mathbf{Q}})$, and it can be obtained by taking the logarithm of the matrix as follows:

$$\hat{\mathbf{Q}} = \frac{\log(\hat{\mathbf{P}})}{a}, \quad (25)$$

where $\hat{\mathbf{P}}$ is the estimated transition probability matrix observed periodically at integer multiples of the real number a . The matrix $\hat{\mathbf{P}} = (\hat{p}_{ij})_{i,j \in E}$ can be estimated using the formula:

Table 4

Optimal time and option value corresponding to different product fee and age for static case.

Fee (basis points)	Age = 65		Age = 71		Age = 77	
	τ (years)	value	τ (years)	value	τ (years)	value
$\gamma = 1000$	11	10473.90	9	7934.78	8	8076.30
$\gamma = 1100$	9	9355.34	6	7700.00	4	7589.83
$\gamma = 1200$	8	8543.43	5	7598.09	6	7499.80

$$\hat{p}_{ij} = \frac{K_{ij}}{K_i},$$

where K_{ij} is the number of transitions from state i to state j , and $K_i = \sum_{j=1}^D K_{ij}$ is the total number of times the policyholder has health state i . In our application, the observation period is $a = 2$ years. It is important to note that the existence and uniqueness of the maximum likelihood estimator $\mathbf{Q}$ under this observational scheme are not guaranteed (see, for example, Bladt and Sørensen, 2005; Regnault, 2012). However, according to the methodology developed in D'Amico and Regnault (2018), the estimator $\hat{\mathbf{Q}}$ exists and is unique in our case. Table 3 provides the estimated transition probability matrices. Therefore, the estimated generator matrices for ages 65, 71, and 76 are denoted as $\hat{\mathbf{Q}}_{65}$, $\hat{\mathbf{Q}}_{71}$ and $\hat{\mathbf{Q}}_{76}$:

$$\hat{\mathbf{Q}}_{65} = \begin{pmatrix} -0.1157 & 0.0396 & 0.0237 & 0.0054 & 0.0057 & 0.0044 & 0.0367 \\ 0.5195 & -1.2726 & 0.4570 & 0.0689 & 0.0876 & 0.0275 & 0.1122 \\ 0.1952 & 0.3982 & -1.2717 & 0.3801 & 0.0983 & 0.0356 & 0.1638 \\ 0.0174 & 0.00008 & 1.0290 & -1.7539 & 0.4354 & 0.0749 & 0.1976 \\ 0.0608 & 0.1349 & 0.1039 & 0.3188 & -1.1832 & 0.1036 & 0.4548 \\ 0.0883 & 0.0135 & 0.01204 & 0.0453 & 0.000004 & -0.3273 & 0.1702 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\hat{\mathbf{Q}}_{71} = \begin{pmatrix} -0.1913 & 0.0629 & 0.0265 & 0.0110 & 0.0117 & 0.0137 & 0.0676 \\ 0.4540 & -1.2719 & 0.5823 & 0.0258 & 0.0585 & 0.0754 & 0.1276 \\ 0.0940 & 0.4279 & -1.3158 & 0.4342 & 0.0408 & 0.1412 & 0.1764 \\ 0.1069 & 0.0002 & 0.5720 & -1.4850 & 0.6984 & 0.0047 & 0.1145 \\ 0.0707 & 0.0850 & 0.1921 & 0.3470 & -1.3976 & 0.1870 & 0.5019 \\ 0.0382 & 0.0074 & 0.0305 & 0.0176 & 0.0538 & -0.6540 & 0.5065 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\hat{\mathbf{Q}}_{76} = \begin{pmatrix} -0.2799 & 0.1012 & 0.0507 & 0.0144 & 0.0110 & 0.0183 & 0.0865 \\ 0.2782 & -1.3390 & 0.6011 & 0.0628 & 0.1035 & 0.1340 & 0.0872 \\ 0.1279 & 0.4041 & -1.3039 & 0.4069 & 0.0797 & 0.0643 & 0.2218 \\ 0.0185 & 0.0147 & 0.7587 & -1.8751 & 0.6317 & 0.2237 & 0.2774 \\ 0.0540 & 0.0900 & 0.0340 & 0.2974 & -1.1518 & 0.2267 & 0.4483 \\ 0.0212 & 0.0321 & 0.0054 & 0.0140 & 0.0275 & -0.5715 & 0.4718 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

4.2.1. Optimal time

According to Equation (6), the optimal time to initiate the withdrawal phase is the value of t_r for which Equation (6) is satisfied. Corresponding to the fees charged by the insurance company for ages 65, 71, and 77, the optimal time to initiate the withdrawal phase is shown in Table 3 for the dynamic case and Table 4 for the static case. In the table's first column, we consider the fee γ charged by the insurance company. Results from Table 3 and Table 4 illustrate that with the increase in γ , the optimal initiation time period t_r associated with the contract decreases for both static and dynamic cases. Furthermore, the policyholder's age increment also affects the optimal time. For the same fee, senior policyholders should wait less for the withdrawal phase to initiate. For young policyholders, it's better not to initiate the withdrawal phase at the early stages.

Moreover, it is observed that for the static case, the value of the expected wealth is less compared to the dynamic case.

4.3. Sensitivity analysis

In this section, we will study the change in the optimal time to initiate the withdrawal phase of the contract with respect to the different parameters.

- Insurance fee:** In Fig. 3, the plot of the fee charged by the insurance company vs. Expected wealth is drawn to show the change in the behavior of the Expected wealth value with the change in the fee parameter γ . An increment in the fee leads to a decrement in the sub-account fund value given by Equation (4), decreasing the overall benefit amounts, because the value of H_t (sub-account fund value) decides the value of expected death benefit as well as the expected discounted total withdrawals. Hence, an increment in fees decreases the value of expected wealth.
- Age:** According to Fig. 4, it is evident that the initial insured age has an impact on the expected value of wealth. Specifically, as the initial age increases, the wealth value decreases. We investigate how a change in the age at inception, denoted as y , influences the death and living benefits of the Guaranteed Lifetime Withdrawal Benefit (GLWB) value.

When the initial insured age increases, the contract length decreases, leading to higher health status. Consequently, there are fewer withdrawals with an increased withdrawal amount, resulting in a lower living benefit amount. Moreover, a increase in the initial insured age reduces the time required to discount the death benefit value, leading to a more substantial death benefit. Conversely, as the initial insured age increases, the investment amount also increases.

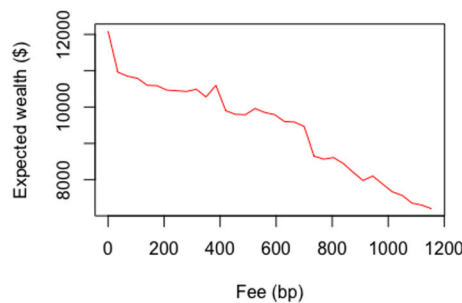


Fig. 3. Plot of Fee vs. Expected wealth.

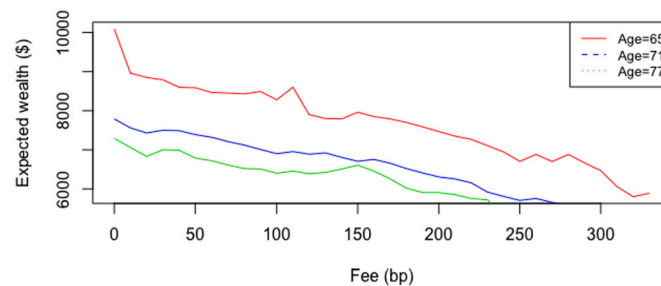


Fig. 4. Plot of optimal Expected wealth vs. fee for different ages.

Overall, an increase in the initial insured age leads to decreased GLWB value. It is primarily because the impact of investment becomes more significant than the combined living and death benefit value.

5. Conclusion and future work

In this article, real option pricing is applied to an investment-disinvestment problem in a variable annuity that includes the guaranteed lifetime withdrawal benefit as well as the health status benefit. A mixed continuous-discrete time model with time-varying volatility is used to assist the policyholder in making investment and disinvestment decisions based on his/her health. Withdrawal initiation time is calculated for both dynamic and static cases, which maximizes the expected wealth function. Our calculations reveal considerable differences between the static and dynamic cases in the context of withdrawal initiation time and the expected value of wealth. The results point towards the relation between the fee charged by the insurer versus the optimal initiation time. The fee charged by the insurance company forms the basis of the disinvestment decision.

People close to retiring need insurance against outliving their assets and protection from aging issues. This product might offer a more complete answer to all of these requirements. These products can be made more attractive if the health benefits can be modeled better to increase medical and hospitalization costs to the policyholder. As discussed in the sensitivity analysis section, changes in insurance premiums and age significantly impact the ideal initiation time.

Future works include applying more sophisticated models of health evolution like semi-Markov processes and considering path-dependent health-related rewards (see, e.g., D'Amico, 2011).

Generalizations of the investment variable using a suitable process, the removal of various financial presumptions, the subsequent application of stochastic volatility, and the interest rate can also be included in future work. An interesting challenge will be to apply machine-learning techniques to calculate the exercise frontiers.

CRediT authorship contribution statement

Guglielmo D'Amico: Writing – review & editing, Supervision, Methodology, Formal analysis, Conceptualization. **Shakti Singh:** Writing – review & editing, Writing – original draft, Visualization, Validation, Software, Methodology, Data curation. **Dharmaraja Selvamuthu:** Writing – review & editing, Supervision, Resources, Project administration, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

The authors do not have permission to share data.

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