

Valuation of equity-indexed annuities under correlated jump–diffusion processes

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ABSTRACT

This article studies the problem of valuation of equity-indexed annuities (EIA) when the stock index follows a Hawkes jump–diffusion model to account for the jump risk and clustering of index price jumps. Further, the interest rate is assumed to be driven by Vasicek type model, correlated to the dynamics of the stock index. In the proposed framework, an analytical expression is obtained for the price of annual reset and point-to-point designs of EIAs by employing the measure change technique. The effects of the model parameters governing the jump risk and the clustering of jumps on the EIAs pricing are illustrated through numerical experiments.

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1. Introduction

Equity indexed annuities (EIAs) gained phenomenal popularity since their introduction because they comprise of features of both the fixed and the variable annuities. More specifically, EIAs provide a minimum rate of return similar to fixed annuities and allow investors to participate in equity growth in the same way as in variable annuities. Further, they have bounded downside risk with a guaranteed minimum return, which makes them a more favourable alternative to other investment products, especially during the period of economic crisis.

Typically, the maturity of the EIA contract is fixed, ranging between 1 and 10 years. Based on the indexing method, the growth in EIA is modelled using various designs, e.g., point-to-point, annual reset design, Asian-end design and high-watermark design. In this article, we consider the pricing problem of the point-to-point design and annual reset design of EIAs. In the point-to-point design, the returns of the contract are calculated by measuring the index growth between two pre-fixed time points, usually the beginning and end dates of the EIA contract. On the other hand, in annual reset design, as the name suggests, the index growth is measured annually by comparing the index growth between the beginning and end of the year. The gain in each year is locked in, and any drop in the index is ignored.

Brennan and Schwartz [1] were the first to study the pricing of life insurance contracts, such as variable annuities and EIAs using the arbitrage principle in continuous time. After their seminal work, many research articles have studied the pricing of insurance contracts. One can observe that the valuation of EIAs mainly involves three types of risks, the investment risk, the mortality risk and the interest rate risk. The mortality risk is assumed to be diversifiable and hence ignored in most of the literature on EIAs. On the other hand, most of the literature has addressed the investment risk in the Black and Scholes framework (Piscopo [2]; Ballotta [3], Gerber and Shiu [4]). For example, Tiong [5] considered the valuation of EIAs (point-to-point, the cliquet, and the lookback) under the assumption that assets follow geometric Brownian motion (GBM) and the interest rate is constant.

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Due to the long maturity of an EIA contract, Lin and Tan [6] proposed a stochastic interest rate model for pricing EIAs. They assumed equity index to be driven by a GBM model. Similarly, Kijima et al. [7] also proposed a stochastic interest rate model for the pricing of ratchet EIAs with the interest rate following the extended Vasicek model. Qian et al. [8] derived closed-form solutions for pricing of EIAs assuming that the equity index is driven by a jump–diffusion (JD) process and interest rate is governed by a Vasicek model. They assumed mortality risk to be independent of the financial market, i.e., equity index and interest rate. Some of the recent literature on the valuation of EIAs includes work by Chiu et al. [9] and Qian et al. [10]. Qian et al. [10] valued EIAs using a regime-switching JD model. Chiu et al. [9] consider the valuation of a complex EIA design consisting of quanto feature. They further added the exchange rate model and foreign risk-free rate model in the valuation. However, the considered risky asset dynamics in their work is a GBM model.

Most of the literature on the pricing of EIAs have considered Lévy processes, nesting the Merton jumps or Poisson JD processes mentioned above, to model the growth of equity index. However, Lévy models have independent increments property, which implies that the occurrence of any two jumps is independent of each other. The clustering of jumps in equity markets is repeatedly observed and has been verified in literature too [11,12]. For instance, according to arguments of Ait-Sahalia et al. [13], it is not the initial jump which resulted in financial crisis; it is the amplification of these jumps which occurred over the time being one of the possible reasons for the crisis. This feature can be modelled using a Hawkes JD process. Hawkes processes are capable of reproducing jump clustering effect as under these process happening of a jump increases the probability of happening of future jumps. Therefore, Hawkes JD model captures empirical features of equities which are ignored by the classical Black–Scholes model as well as by JD models with Poisson jumps. Thus, these models have the edge over the standard Black–Scholes and JD models. As a typical example of their application in derivatives pricing, Ma et al. [14], and Liu and Zhu [15] adopted Hawkes JD processes to price vulnerable options and variance swaps, respectively. Pasricha and Goel [16] considered a Hawkes JD model to price power exchange options. Ma and Xu [17] adopted Hawkes JD processes to study a structural credit risk model. All these articles performed numerical experiments to demonstrate that jump clustering remarkably influences the final results.

Motivated by the increased popularity of the Hawkes process in quantitative finance, we extend the modelling framework of Quian et al. [8] and propose a Hawkes JD model for pricing EIAs. The jump model proposed by Quian et al. [8] has the independence of increment property and hence is unable to model the clustering of jumps feature. Therefore, we used a Hawkes JD process to capture features of Poisson JD models as well as to incorporate the clustering of jumps feature. Under these processes, the occurrence of a jump increases the intensity of occurrence of next jumps and hence the probability of arrival of next jumps increases. As a result, these processes are suitable to address the jump clustering in financial assets. The classical Black–Scholes model and the Poisson JD model are special cases of Hawkes JD model. In this paper, we model the risky asset dynamics using a Hawkes JD process. Further, considering the long maturity of EIA contracts, the interest rates are modelled through the Vasicek model. Additionally, we assumed the interest rate dynamics to be correlated with the dynamics of the risky asset. This article contributes to the existing literature in multiple ways: Firstly, jump clustering is incorporated in the model, and a closed-form formula for EIAs is presented. Secondly, the proposed model is general in the sense that the Black–Scholes framework, Merton JD framework can be obtained as special cases.

The rest of the paper is organized as follows. Section 2 presents the modelling of the underlying assets. In Section 3, the explicit pricing formulas of the point-to-point design and annual reset design of equity-indexed annuities are derived. Section 4 gives the sensitivity analysis concerning various parameters in the proposed model. Section 5 concludes the paper.

2. Model description

Consider a finite time horizon $T > 0$. Assume that the uncertainty in the economy is modelled by the filtered probability space $(\Omega, \mathcal{F}, Q, \mathcal{F}_{t \in [0, T]})$ and let the expectation with respect to the risk-neutral measure Q is denoted by E . Also, assume that the market comprises of only two traded assets: a risky asset with price process denoted as $S = \{S_t\}_{0 \leq t \leq T}$ and a risk-free asset with price process denoted as $S^{(0)} = \{S_t^{(0)}\}_{0 \leq t \leq T}$. The risky asset is generally referred to be a stock and the risk-free asset is like a savings account. Under the risk-neutral measure Q , suppose the asset price dynamics S_t follows Hawkes JD process given by the following equation:

$$\frac{dS_t}{S_{t-}} = r_t dt + \sigma_1 dB_{1,t} + d \left[\sum_{j=1}^{N_t} Z_j - m \int_0^t \lambda_s ds \right], \quad (1)$$

$$d\lambda_t = \delta(\lambda_\infty - \lambda_t)dt + \theta dN_t, \quad (2)$$

where S_0 is the initial price of the asset, σ_1 is the volatility (assumed to be a constant). The process $\{B_{1,t}, 0 \leq t \leq T\}$ is a standard Brownian motion and $\{r_t, 0 \leq t \leq T\}$ is the risk-free interest rate process. The process $\{N_t, t \geq 0\}$ is a Hawkes process with time varying intensity λ_t defined in Eq. (2). λ_t in Eq. (2) have initial value λ_0 and the parameters θ , δ and λ_∞ are constant. The i.i.d. random variables $\{Z_k, k = 1, 2, \dots\}$ with probability density function $f(x)$ with support $(-1, \infty)$ represent the magnitudes of jumps at the time of jumps of N_t . Assume that $\ln(1 + Z_k) \sim N(\tilde{m}, \sigma^2)$. And m is given by

$$m = E[Z_k] = \exp \left(\tilde{m} + \frac{1}{2} \sigma^2 \right) - 1. \quad (3)$$

The proposed model implies that with the previous jumps in the stock price, the intensity process λ_t of the Hawkes process immediately increases by θ and then this increment decays exponentially at rate δ . In other words, when a jump occurs in the asset price, it increases the probability of the occurrence of future jumps, thus resulting in jump clustering. Further, larger θ (smaller δ) implies more clustering of jumps. In Poisson jump models, due to independent increments, this self exciting behaviour is not observed, i.e., occurrence of a jump does not stimulate the future jumps. Thus, Hawkes processes (self exciting processes) are appropriate to model the jump clustering phenomena. Note that the process reduces to Poisson process with constant intensity λ_0 when the variable θ goes to zero and the parameter λ_∞ is same as the initial value λ_0 .

The proposed model can have GBM Model and JD models with Poisson jumps as special cases. Substituting $\theta = 0$ and $\lambda_\infty = \lambda_0$, the solution of Eq. (2) becomes $\lambda_t = \lambda_0$ and hence the stock price dynamics are reduced to those of Poisson jump models given by

$$\frac{dS_t}{S_{t-}} = r_t dt + \sigma_1 dB_{1,t} + d \left[\sum_{j=1}^{N_t} Z_j - m\lambda_0 t \right], \quad (4)$$

with $\{N_t, t \geq 0\}$ is a Poisson process with constant intensity λ_0 . Further assuming the intensity parameter to be 0, the above model reduces to a GBM model with asset dynamics given by

$$\frac{dS_t}{S_t} = r_t dt + \sigma_1 dB_{1,t}. \quad (5)$$

The dynamics of risk-free interest rate process $\{r_t, 0 \leq t \leq T\}$ are assumed to be

$$dr_t = a(b - r_t)dt + \sigma_2 dB_{2,t}, \quad (6)$$

where a, b are positive constants, σ_2 is the volatility and $\{B_{2,t}, 0 \leq t \leq T\}$ is a standard Brownian motion. The correlation coefficient between $\{B_{1,t}, 0 \leq t \leq T\}$ and $\{B_{2,t}, 0 \leq t \leq T\}$ is ρ and they are assumed to be independent of $\{Z_k, k = 1, 2, \dots\}$ and $\{N_t, t \geq 0\}$. Therefore, the price process of risk-free asset can be written as

$$S_t^{(0)} = S_0^{(0)} \exp \left\{ \int_0^t r_s ds \right\}. \quad (7)$$

Define τ_x as the random variable representing the policy holder's residual lifetime depending on the age x of the policy holder. Then the probability of survival till age $x + t$ for a person aged x is given by:

$${}_t p_x = P(\tau_x > t). \quad (8)$$

Assume τ_x and S_t to be independent. Also assume the mortality risk to be diversifiable, see for instance, Aase and Persson [18].

3. Valuation of EIAs

In this section, for the valuation of point-to-point and annual reset designs of EIAs, we will apply the changing measure technique in the proposed framework.

3.1. Change of measure

We introduce $Q^{(r)}$, an equivalent measure to Q by the Radon–Nikodým derivative

$$\frac{dQ^{(r)}}{dQ} = \frac{e^{-\int_t^T r_s ds}}{E \left(e^{-\int_t^T r_s ds} \right)}. \quad (9)$$

Proposition 3.1. Define $M(u, T, a) = 1 - e^{-a(T-u)}$. Define the following processes,

$$\widehat{B}_{1,t} = B_{1,t} + \rho \int_0^t \frac{\sigma_2}{a} M(u, T, a) du, \quad (10)$$

$$\widehat{B}_{2,t} = B_{2,t} + \int_0^t \frac{\sigma_2}{a} M(u, T, a) du. \quad (11)$$

Under the probability measure $Q^{(r)}$ defined in Eq. (9), the process $(\widehat{B}_{1,t}, \widehat{B}_{2,t})$ is a two dimension Brownian motion with correlation coefficient ρ .

Proof. Using Fubini's theorem in Eq. (6), we have

$$-\int_t^T r_s ds = -b(T-t) - \frac{(r_t - b)}{a} (1 - e^{-a(T-t)}) - \int_t^T \frac{\sigma_2}{a} (1 - e^{-a(T-u)}) dB_{2,u}. \quad (12)$$

Denote $X(t, T) = E \left(e^{-\int_t^T r_s ds} \mid \mathcal{F}_t \right)$, then we have

$$X(t, T) = \exp \left\{ -b(T-t) - \frac{(r_t - b)}{a} M(t, T, a) + \frac{1}{2} \int_t^T \frac{\sigma_2^2}{a^2} M^2(u, T, a) du \right\}. \quad (13)$$

Hence, using Eqs. (12) and (13) in Eq. (9), we have

$$\frac{dQ^{(r)}}{dQ} = \exp \left\{ -\int_t^T \frac{\sigma_2}{a} M(u, T, a) dB_{2,u} - \frac{1}{2} \int_t^T \frac{\sigma_2^2}{a^2} M^2(u, T, a) du \right\}. \quad (14)$$

Therefore, by Girsanov's theorem, we have $\widehat{B}_{1,t}, \widehat{B}_{2,t}$ given in Eqs. (10) and (11) are standard Brownian motions under $Q^{(r)}$. Also, $\widehat{B}_{1,t}$ and $\widehat{B}_{2,t}$ have same correlation as that of $(B_{1,t}, B_{2,t})$. $\square$

3.2. Dynamics of S_t under new measure

In this subsection, we obtain the dynamics of S_t under the new measure $Q^{(r)}$. From Eqs. (1) and (6) and Proposition 3.1, we have the dynamics of S_t under the measure $Q^{(r)}$ given by

$$S_t = S_0 \exp \left\{ \Lambda_{0,t} + \sigma_1 \widehat{B}_{1,t} + \int_0^t \frac{\sigma_2}{a} M(u, t, a) d\widehat{B}_{2,u} + \int_0^t \ln(1 + Z_u) dN_u \right\}, \quad (15)$$

where

$$\begin{aligned} \Lambda_{0,t} = & bt + \frac{(r_0 - b)}{a} M(0, t, a) - \int_0^t \frac{\sigma_1 \sigma_2}{a} \rho M(u, T, a) du \\ & - \int_0^t \frac{\sigma_2^2}{a^2} M(u, t, a) M(u, T, a) du - \frac{1}{2} \sigma_1^2 t - m \int_0^t \lambda_s ds. \end{aligned} \quad (16)$$

We obtain a valuation formula by conditioning over the paths of the Hawkes process, i.e.,

$$\mathcal{G}_t = \{N_s, 0 \leq s \leq t\}, \quad 0 \leq t \leq T.$$

Let T_i denote the jump times of N_t till time t . Given $\mathcal{G}_t$, S_t becomes

$$S_t = S_0 \exp \left\{ \Lambda_{0,t} + \int_0^t \sigma_1 d\widehat{B}_{1,u} + \int_0^t \frac{\sigma_2}{a} M(u, t, a) d\widehat{B}_{2,u} + \sum_{k=1}^{N_t} \ln(1 + Z_{T_k}) \right\}.$$

Hence, given $\mathcal{G}_t$ and assuming $N_t = n$, we have that $\ln(S_t)$ follows normal distribution with mean M_t^r and variance V_t^r given by

$$M_t^r = E^r(\ln(S_t)) = \ln(S_0) + \Lambda_{0,t} + n\tilde{m}, \quad (17)$$

$$V_t^r = V^r(\ln(S_t)) = \sigma_1^2 t + \int_0^t \frac{\sigma_2^2}{a^2} M^2(u, t, a) du + \rho \int_0^t \frac{2\sigma_1 \sigma_2}{a} M(u, t, a) du + n\sigma^2. \quad (18)$$

3.3. The point-to-point design

Now, we consider the point-to-point design of EIA. This is one of the simplest classes of EIAs. The claim amount for this design at time t denoted by $C_{pp}(t)$ for a single unit of EIA can be given as

$$C_{pp}(t) = \max\{\min\{e^{\alpha Y_t}, e^{\gamma t}\}, e^{gt}\}, \quad (19)$$

where $Y_t = \ln(\frac{S_t}{S_0})$, α denotes the participation in equity growth rate, γ denotes the cap rate i.e., the maximum annualized rate that can be credited, and $g(< \gamma)$ is a continuously compounded guaranteed minimum return. In Eq. (19), the random term allows the investor to participate in market gains subject to a maximum cap rate γ . In case of adverse market situations, the downside moment of returns is bounded by a minimum guaranteed return g .

Assume that the policy can be terminated only in the year-end. Hence, the time of termination of the EIA contract is discrete, given by the set $\{1, 2, \dots, T\}$. The insured will get the amount $C_{pp}(T)$ at the maturity (T) if he/she survives till

the maturity. In case of demise of the insured in the time period $(t, t + 1]$, for $t = 0, 1, 2, \dots, T - 1$, the insurer will pay $C_{pp}(t + 1)$ at time $t + 1$ to the nominee. Hence, under the risk-neutral measure Q the value of this policy is given by

$$P_{pp} = E \left[e^{-\int_0^T r_s ds} C_{pp}(T) I_{\{\tau_x > T\}} + \sum_{t=0}^{T-1} e^{-\int_0^{t+1} r_s ds} C_{pp}(t + 1) I_{\{t < \tau_x \leq t+1\}} \right], \quad (20)$$

where I_A is the indicator function, i.e., $I_A(x) = 1$ if $x \in A$ and 0 otherwise.

Theorem 3.2. The present value of the point-to-point EIA design, P_{pp} is

$$P_{pp} = X(0, T)E^{(r)}[C_{pp}(T)]P(\tau_x > T) + \sum_{t=0}^{T-1} (X(0, t + 1)E^{(r)}[C_{pp}(t + 1)]P(t < \tau_x \leq t + 1)), \quad (21)$$

where $X(0, t)$ can be obtained from Eq. (13) and for $t = 0, 1, \dots, T - 1$, we have

$$\begin{aligned} E^{(r)}[C_{pp}(t + 1)] = & \tilde{E} \left[e^{g(t+1)} \Phi \left(\frac{\frac{g(t+1)}{\alpha} - M_{t+1}^r + \ln(S_0)}{\sqrt{V_{t+1}^r}} \right) \right. \\ & + e^{\alpha M_{t+1}^r + \frac{\alpha^2}{2} V_{t+1}^r} \left(\Phi \left(\frac{\frac{\gamma(t+1)}{\alpha} - M_{t+1}^r + \ln(S_0) - \alpha V_{t+1}^r}{\sqrt{V_{t+1}^r}} \right) \right. \\ & \left. \left. - \Phi \left(\frac{\frac{g(t+1)}{\alpha} - M_{t+1}^r + \ln(S_0) - \alpha V_{t+1}^r}{\sqrt{V_{t+1}^r}} \right) \right) \right. \\ & \left. + e^{\gamma(t+1)} \Phi \left(-\frac{\frac{\gamma(t+1)}{\alpha} - M_{t+1}^r + \ln(S_0)}{\sqrt{V_{t+1}^r}} \right) \right], \end{aligned} \quad (22)$$

where M_{t+1}^r and V_{t+1}^r can be obtained from Eqs. (17) and (18) respectively by replacing t by $t + 1$, and $\tilde{E}$ is the expectation over the paths of Hawkes process, i.e., over $\mathcal{G}_T$.

Proof. P_{pp} can be written as a sum of two terms I and II defined as follows:

$$\begin{aligned} I &= E \left[e^{-\int_0^T r_s ds} C_{pp}(T) I_{\{\tau_x > T\}} \right], \\ II &= E \left[\sum_{t=0}^{T-1} e^{-\int_0^{t+1} r_s ds} C_{pp}(t + 1) I_{\{t < \tau_x \leq t+1\}} \right]. \end{aligned}$$

Now, calculate the value of term I . Since the mortality risk and financial market are considered to be independent, therefore, the term I can be written as

$$I = E \left[e^{-\int_0^T r_s ds} C_{pp}(T) \right] E [I_{\{\tau_x > T\}}]. \quad (23)$$

Using Proposition 3.1, we can rewrite Eq. (23) as follows

$$I = X(0, T)E^{(r)}[C_{pp}(T)]P(\tau_x > T),$$

where $E^{(r)}$ represent the expectation under the measure $Q^{(r)}$.

Then, we can write it as

$$E^{(r)}[C_{pp}(T)] = E^{(r)} \left[e^{gT} I_{\{Y_T \leq \frac{gT}{\alpha}\}} + e^{\alpha Y_T} I_{\{\frac{gT}{\alpha} < Y_T \leq \frac{\gamma T}{\alpha}\}} + e^{\gamma T} I_{\{Y_T \geq \frac{\gamma T}{\alpha}\}} \right].$$

Given the information $\mathcal{G}_T$, the conditional value of $E^{(r)}[C_{pp}(T)]$ can be calculated as

$$\begin{aligned} & E^{(r)} \left[e^{gT} I_{\{Y_T \leq \frac{gT}{\alpha}\}} + e^{\alpha Y_T} I_{\{\frac{gT}{\alpha} < Y_T \leq \frac{\gamma T}{\alpha}\}} + e^{\gamma T} I_{\{Y_T \geq \frac{\gamma T}{\alpha}\}} \mid \mathcal{G}_T \right] \\ &= e^{gT} \Phi \left(\frac{\frac{gT}{\alpha} - M_T^r + \ln(S_0)}{\sqrt{V_T^r}} \right) + e^{\alpha(M_T^r - \ln(S_0)) + \frac{\alpha^2}{2} V_T^r} \left[\Phi \left(\frac{\frac{\gamma T}{\alpha} - M_T^r + \ln(S_0) - \alpha V_T^r}{\sqrt{V_T^r}} \right) - \right. \\ & \quad \left. \Phi \left(\frac{\frac{gT}{\alpha} - M_T^r + \ln(S_0) - \alpha V_T^r}{\sqrt{V_T^r}} \right) \right] + e^{\gamma T} \Phi \left(-\frac{\frac{\gamma T}{\alpha} - M_T^r + \ln(S_0)}{\sqrt{V_T^r}} \right), \end{aligned}$$

where $\Phi(\cdot)$ denotes the cumulative distribution function of a standard normal random variable. Therefore, the unconditional value of $E^{(r)}[C_{pp}(T)]$ can be obtained by taking expectation over all possible paths of jump processes, i.e., over $\mathcal{G}_T$

and is given by

$$E^{(r)}[C_{pp}(T)] = \tilde{E} \left(e^{\frac{gT}{\alpha}} \Phi \left(\frac{\frac{gT}{\alpha} - M_T^r + \ln(S_0)}{\sqrt{V_T^r}} \right) + e^{\alpha(M_T^r - \ln(S_0)) + \frac{\alpha^2}{2} V_T^r} \left[\Phi \left(\frac{\frac{\gamma T}{\alpha} - M_T^r + \ln(S_0) - \alpha V_T^r}{\sqrt{V_T^r}} \right) - \Phi \left(\frac{\frac{gT}{\alpha} - M_T^r + \ln(S_0) - \alpha V_T^r}{\sqrt{V_T^r}} \right) \right] + e^{\gamma T} \Phi \left(-\frac{\frac{\gamma T}{\alpha} - M_T^r + \ln(S_0)}{\sqrt{V_T^r}} \right) \right), \quad (24)$$

where $\tilde{E}$ is expectation over the paths of Hawkes process (i.e., $\mathcal{G}_T$). Similarly, we can write the value of II , which is given by

$$\begin{aligned} II &= \sum_{t=0}^{T-1} \left(E[e^{-\int_0^{t+1} r_s ds}] E^{(r)}[C_{pp}(t+1)] P(t < \tau_x \leq t+1) \right) \\ &= \sum_{t=0}^{T-1} (X(0, t+1) E^{(r)}[C_{pp}(t+1)] P(t < \tau_x \leq t+1)), \end{aligned}$$

where $X(0, t+1)$, $t = 0, 1, \dots, T-1$ can be obtained using $X(t, T)$ given in Eq. (13) by substituting $t = 0$ and $T = t+1$. Further, $E^{(r)}[C_{pp}(t+1)]$ can be written similar to Eq. (24) and is given by

$$\begin{aligned} E^{(r)}[C_{pp}(t+1)] &= \tilde{E} \left[e^{\frac{g(t+1)}{\alpha}} \Phi \left(\frac{\frac{g(t+1)}{\alpha} - M_{t+1}^r + \ln(S_0)}{\sqrt{V_{t+1}^r}} \right) + e^{\alpha(M_{t+1}^r - \ln(S_0)) + \frac{\alpha^2}{2} V_{t+1}^r} \left[\Phi \left(\frac{\frac{\gamma(t+1)}{\alpha} - M_{t+1}^r + \ln(S_0) - \alpha V_{t+1}^r}{\sqrt{V_{t+1}^r}} \right) - \Phi \left(\frac{\frac{g(t+1)}{\alpha} - M_{t+1}^r + \ln(S_0) - \alpha V_{t+1}^r}{\sqrt{V_{t+1}^r}} \right) \right] + e^{\gamma(t+1)} \Phi \left(-\frac{\frac{\gamma(t+1)}{\alpha} - M_{t+1}^r + \ln(S_0)}{\sqrt{V_{t+1}^r}} \right) \right]. \quad \square \end{aligned} \quad (25)$$

Remark 3.1.

The other possible approach to obtain present value of the point-to-point EIA design, P_{pp} , is to obtain a partial integro-differential equation (PIDE) governing its price. In particular, a PIDE corresponding to $E[e^{-\int_0^t r_s ds} C_{pp}(t)]$ in Eq. (20). For $t = 1, 2, \dots, T$, define

$$V_t = E \left[e^{-\int_0^t r_s ds} C_{pp}(t) \right].$$

Then, the present value of the EIA P_{pp} (from Eq. (20)) becomes

$$P_{pp} = V_T P(\tau_x > T) + \sum_{t=0}^{T-1} (V_{t+1} P(t < \tau_x \leq t+1)). \quad (26)$$

Note that we have not changed measure while writing the above equation as we defined V_t as the discounted value of $C_{pp}(t)$. This means we are still under the dynamics defined in Section 2.

Let $Y_t = \ln(\frac{S_t}{S_0})$ be the log-price. Now, in order to solve for V_t , $t = 1, 2, \dots, T$, we utilize the Feynman–Kac theorem to obtain the PIDE satisfied by $v(\tau, t, y, r, \lambda) = E[e^{-\int_t^\tau r_s ds} C_{pp}(t)]$ which, in turn, gives $V_t = v(0, t, y, r, \lambda)$. The PDE satisfied by $v(\tau, t, y, r, \lambda)$ given by

$$\begin{aligned} &\frac{\partial v}{\partial \tau} + (r - \frac{1}{2}\sigma_1^2 - m\lambda) \frac{\partial v}{\partial y} + a(b-r) \frac{\partial v}{\partial r} + \frac{1}{2}\sigma_1^2 \frac{\partial^2 v}{\partial y^2} + \frac{1}{2}\sigma_2^2 \frac{\partial^2 v}{\partial r^2} + \rho\sigma_1\sigma_2 \frac{\partial^2 v}{\partial y \partial r} \\ &+ \delta(\lambda_\infty - \lambda) \frac{\partial v}{\partial \lambda} \\ &+ \lambda \left[\int_{-\infty}^{\infty} (v(\tau, t, y + x_1, r, \lambda_t + \theta) - v(\tau, t, y, r, \lambda)) f(x_1) dx_1 \right] - rv = 0, \quad \tau < t \end{aligned}$$

with the terminal condition $v|_{\tau=t} = \max\{\min\{e^{\alpha Y_t}, e^{\gamma t}\}, e^{g_t}\}$ and $f(x)$ denoting the pdf of a normal random variable with mean $\tilde{m}$ and variance σ^2 . We are interested in the value of v at $\tau = 0$. One can use any numerical scheme to solve this PIDE to get the value of $E\left[e^{-\int_{\tau}^t r_s ds} C_{pp}(t)\right]$ which when substituted into Eq. (26) gives the present value of the EIA P_{pp} .

3.4. The annual reset design

Annual reset EIA or ratchet EIA is one of the popular EIA designs. Under this policy the payoffs are reset every year till end of the contract. Further, the gains for a year are credited only if they are positive otherwise they are ignored. And the returns once credited get locked in and never decreases. The payoff for one unit of annual reset EIA in year t is given by

$$C_{ar}(t) = \prod_{i=1}^t \max\{\min\{e^{\alpha \tilde{S}_i}, e^{\gamma}\}, e^g\}, \quad (27)$$

where $\tilde{S}_i = \ln(S_i) - \ln(S_{i-1})$. By replacing the term $C_{pp}(t)$ in pricing of point-to-point EIA by $C_{ar}(t)$, we will obtain the equation for pricing of annual reset EIA. Thus, the value of this EIA policy is

$$P_{ar} = E\left[e^{-\int_0^T r_s ds} C_{ar}(T) I_{\{\tau_X > T\}} + \sum_{t=1}^T e^{-\int_0^{t+1} r_s ds} C_{ar}(t) I_{\{t-1 < \tau_X \leq t\}}\right]. \quad (28)$$

Hence, the difference between pricing of the two designs lies in the valuation of $C_{pp}(t)$ and $C_{ar}(t)$ terms.

Proposition 3.3. For any $0 < t \leq T$, under the measure $Q^{(r)}$, and given information $\mathcal{G}_T$, $(\tilde{S}_1, \tilde{S}_2, \dots, \tilde{S}_t | \mathcal{G}_T)$ is a multivariate normal vector with mean $(m_1, m_2, \dots, m_t)$ and variance-covariance matrix $\Sigma_t = (\sigma_{ij})_{1 \leq i, j \leq t}$, where

$$\begin{aligned} \tilde{A}_{i-1,i} &= b + \frac{(r_0 - b)}{a} M(0, 1, a) e^{-a(i-1)} - \left(\frac{1}{2} \sigma_1^2 + k_s\right) \\ &+ \int_0^{i-1} \frac{\sigma_1 \sigma_2}{a} \rho_{12} e^{-a(i-1-u)} M(0, 1, a) du + \int_{i-1}^i \frac{\sigma_1 \sigma_2}{a} \rho_{12} M(u, i, a) du \\ &+ \int_0^{i-1} \frac{\sigma_2^2}{a^2} [e^{-2a(i-1-u)} - 2e^{a(i-1-u)}] M(0, 1, a) du + \int_{i-1}^i \frac{\sigma_2^2}{a^2} M^2(u, i, a) du \end{aligned}$$

$$m_i = E^{(r)}(\tilde{S}_i) = \tilde{A}_{i-1,i} + \sum_{k=N_{(i-1)}+1}^{N_i} \tilde{m}$$

$$\begin{aligned} \sigma_{ii} &= V^{(r)}(\tilde{S}_i) \\ &= \sigma_1^2 + \int_0^{i-1} \frac{\sigma_2^2}{a^2} e^{-2a(i-1-u)} M^2(0, 1, a) du \\ &+ \int_{i-1}^i \frac{\sigma_2^2}{a^2} M^2(u, i, a) du + 2\rho_{12} \int_{i-1}^i \frac{\sigma_1 \sigma_2}{a} M(u, i, a) du + \sum_{k=N_{(i-1)}+1}^{N_i} \sigma^2 \end{aligned}$$

for $i = 1, 2, \dots, t$ and for $1 \leq i < j \leq t$, we have

$$\begin{aligned} \sigma_{ij} &= \text{Cov}^{(r)}(\tilde{S}_i, \tilde{S}_j) \\ &= \int_0^{i-1} \frac{\sigma_2^2}{a^2} e^{-a(i-1-u)} e^{-a(j-1-u)} M^2(0, 1, a) du \\ &+ \int_{i-1}^i \frac{\sigma_2^2}{a^2} M(0, 1, a) M(u, i, a) e^{-a(j-1-u)} du + \int_{i-1}^i \frac{\rho_{12} \sigma_1 \sigma_2}{a} M(0, 1, a) e^{-a(j-1-u)} du \end{aligned}$$

Proof. From Eq. (15), we have

$$\begin{aligned} \tilde{S}_i &= \ln\left(\frac{S_i}{S_{i-1}}\right) \\ &= \tilde{A}_{i-1,i} + \int_{i-1}^i \sigma_1 d\hat{B}_{1u} + \int_0^{i-1} \frac{\sigma_2}{a} e^{-a(i-1-u)} M(0, 1, a) d\hat{B}_{2u} \\ &+ \int_{i-1}^i \frac{\sigma_2}{a} M(u, i, a) d\hat{B}_{2u} + \sum_{k=N_{(i-1)}+1}^{N_i} \ln(1 + Z_{T_k}). \end{aligned}$$

Then the assertion follows immediately by the direct calculation. $\square$

Theorem 3.4. The present value of the EIA P_{ar} is given by

$$P_{ar} = X(0, T)E^{(r)}[C_{ar}(T)]P(\tau_x > T) + \sum_{t=0}^{T-1} (X(0, t+1)E^{(r)}[C_{ar}(t+1)]P(t < \tau_x \leq t+1)) \quad (29)$$

where for $t = 0, 1, \dots, T-1$

$$E^{(r)}[C_{ar}(t+1)] = \tilde{E} \left(\int_{\mathbb{R}} \dots \int_{\mathbb{R}} \prod_{i=1}^t \max\{\min\{e^{\alpha \tilde{S}_i}, e^\gamma\}, e^g\} g(\tilde{S}_1, \dots, \tilde{S}_t) d\tilde{S}_1 \dots d\tilde{S}_t \right) \quad (30)$$

where $g(\tilde{S}_1, \dots, \tilde{S}_t)$ is density of t dimensional normal variable with mean $(m_1, m_2, \dots, m_t)$ and variance-covariance matrix Σ_t given in Proposition 3.3. $\tilde{E}$ is expectation over paths of Hawkes processes and $X(0, t)$ is given in Theorem 3.2.

Proof. The assertion follows immediately by taking expectation of Eq. (27) and using Proposition 3.3. $\square$

Remark 3.2. One can also derive a PIDE governing

$$E(C_{ar}(t)) = E \left(e^{-\int_0^{t+1} r_s ds} \prod_{i=1}^t \max\{\min\{e^{\alpha \tilde{S}_i}, e^\gamma\}, e^g\} \right),$$

where $\tilde{S}_i = \ln(S_i) - \ln(S_{i-1})$. The idea is to obtain the stochastic differential equation of the multi-dimension process $(\tilde{S}_i)_{i=1,2,\dots,t}$ and then use Feynman-Kac theorem to obtain the multi-dimensional PIDE governing the $E(C_{ar}(t))$. We believe that it is not computationally efficient to solve the multi-dimensional PIDE (i.e., dimension t) when t becomes large. Therefore, we adopted the procedure in Theorem 3.4 to derive the price of annual reset EIA. The procedure to solve the multiple dimensional integral in (30) is discussed in the next remark.

Remark 3.3. It is possible to derive an analytical expression for the multiple integral in Eq. (30), i.e., $E^{(r)}[C_{ar}(t+1)]$, for $t = 0, 1, \dots, T-1$. We follow the steps in Kijima and Wong (2007) [7], where a similar analytical expression is derived for a compound Ratchet EIA.

We shall fix a $t = N$ here. From Proposition 3.3, we know that $(\tilde{S}_1, \dots, \tilde{S}_N)$ follows a multivariate normal distribution with mean m and covariance matrix Λ (say). Now, define the following sets in $\mathbb{R}$ for $i = 1, 2, \dots, N$,

$$E(i, 0) \equiv \left(-\infty, \frac{g}{\alpha}\right], \quad E(i, 1) \equiv \left[\frac{\gamma}{\alpha}, \infty\right), \quad \text{and} \quad E(i, 2) \equiv \left(\frac{g}{\alpha}, \frac{\gamma}{\alpha}\right), \quad (31)$$

and the following sets in $\mathbb{R}^N$:

$$H_N(e) \equiv E(1, e_1) \times E(2, e_2) \times \dots \times E(N, e_N), \quad (32)$$

for each $e = (e_1, e_2, \dots, e_N) \in \{0, 1, 2\}^N$. Clearly, $\{H_N(e) \mid e \in \{0, 1, 2\}^N\}$ forms a partition of $\mathbb{R}^N$. Moreover,

$$\begin{aligned} & \int_{\mathbb{R}} \dots \int_{\mathbb{R}} \prod_{i=1}^N \max\{\min\{e^{\alpha \tilde{S}_i}, e^\gamma\}, e^g\} g(\tilde{S}_1, \dots, \tilde{S}_N) d\tilde{S}_1 \dots d\tilde{S}_N \\ &= \sum_{e \in \{0, 1, 2\}^N} \int_{H_N(e)} e^{\delta_1 g} e^{\delta_2 \gamma} \exp(\alpha \mathbf{b} \cdot \tilde{\mathbf{S}}^T) g(\tilde{S}_1, \dots, \tilde{S}_N) d\tilde{S}_1 \dots d\tilde{S}_N, \end{aligned} \quad (33)$$

where $\mathbf{b} = (b_1, b_2, \dots, b_N)$, a row vector with $b_j = \max\{e_j - 1, 0\}$ and $\tilde{\mathbf{S}} = (\tilde{S}_1, \tilde{S}_2, \dots, \tilde{S}_N)$, a row vector. The variables δ_1 and δ_2 count the number of 0s and 1s. For example, one can use the following

$$\delta_1 = \sum_{i=1}^N \max\{e_i - 1, 0\}, \quad \text{and} \quad \delta_2 = \sum_{i=1}^N e_i \pmod{2}.$$

It remains to compute the integral $I = \int_{H_N(e)} \exp(\alpha \mathbf{b} \cdot \tilde{\mathbf{S}}^T) g(\tilde{S}_1, \dots, \tilde{S}_N) d\tilde{S}_1 \dots d\tilde{S}_N$ for each $e = \{0, 1, 2\}^N$. Define $u(\mathbf{b}) = \mathbf{b} \Lambda$ and consider the integral I , i.e.,

$$\begin{aligned} I &= \int_{H_N(e)} \exp(\alpha \mathbf{b} \cdot \tilde{\mathbf{S}}^T) g(\tilde{S}_1, \dots, \tilde{S}_N) d\tilde{S}_1 \dots d\tilde{S}_N \\ &= \int_{H_N(e)} \exp(\alpha \mathbf{b} \cdot \tilde{\mathbf{S}}^T) \frac{1}{\sqrt{(2\pi)^N \det(\Lambda)}} e^{-\frac{1}{2}(\tilde{\mathbf{S}}-m)\Lambda^{-1}(\tilde{\mathbf{S}}-m)^T} d\tilde{S}_1 \dots d\tilde{S}_N \end{aligned}$$

$$\begin{aligned}
&= e^{-\frac{1}{2}u(\mathbf{b})^T \Lambda^{-1}u(\mathbf{b})} \int_{H_N(e)} \frac{1}{\sqrt{(2\pi)^N \det(\Lambda)}} e^{-\frac{1}{2}(\tilde{\mathbf{s}}-m-u(\mathbf{b}))\Lambda^{-1}(\tilde{\mathbf{s}}-m-u(\mathbf{b}))^T} d\tilde{s}_1 \dots d\tilde{s}_N \\
&= e^{-\frac{1}{2}u(\mathbf{b})^T \Lambda^{-1}u(\mathbf{b})} q(e)
\end{aligned}$$

where

$$q(e) = \int_{H_N(e)} f(\tilde{\mathbf{s}}, m + u(\mathbf{b})), \Lambda d\tilde{s}_1 \dots d\tilde{s}_N,$$

with $f(\tilde{\mathbf{s}}, m + u(\mathbf{b}))$, Λ denoting pdf of a multivariate normal random variable with mean $m + u(\mathbf{b})$ and covariance matrix Λ . Note that $q(e)$ denotes the CDF of a multivariate normal random variable which can be evaluated using many computing packages that exist and provides numerical implementation for evaluating multivariate normal CDFs.

From [Theorems 3.2](#) and [3.4](#), we observe that the analytical solutions to completely determine the present value of point-to-point design and annual reset design depend on the paths of the Hawkes processes. Thus in order to estimate these values, we have to rely on the simulations. The paths of Hawkes process are simulating using the thinning algorithm (Ogata [19]). Then, the prices are estimated by taking sample means of the prices over the simulated paths. Thinning algorithm for N_t is given by Algorithm 1.

Algorithm 1 Simulation of a Hawkes Process with parameters $(\lambda_0, \theta, \delta)$ on $[0, T]$

Require: Jump times of N_T

- 1: Set $\lambda \leftarrow \lambda_0, n \leftarrow 1$
 - 2: Generate a random number U in $(0, 1)$, such that, $U \sim U(0, 1)$. Let $s \leftarrow -\ln \frac{U}{\lambda}$. If $s \leq T$, then $t_1 \leftarrow s$; else go to Step 6.
 - 3: Set $n \leftarrow n + 1$, $\lambda \leftarrow \lambda_{t_{n-1}} + \theta$. If $s \leq T$, then go to Step 4; else go to Step 6.
 - 4: Generate two random numbers $U, V \sim U(0, 1)$, and let $s \leftarrow s - \ln \frac{U}{\lambda}$.
 - 5: If $V \leq \frac{\lambda s}{\lambda}$, a new event occurs at time s , i.e., $t_n \leftarrow s$, and then return to Step 3; else return to Step 4.
 - 6: Stop. The times of occurrence of the jump events are $t_1, t_2, \dots, t_n$.
-

4. Numerical experiments and sensitivity analysis

In this section, we analyse the proposed model numerically by considering the valuation of point-to-point design of EIA. The pricing analysis for EIA focuses on the break-even participation rate. The break-even participation rate is the rate at which the initial premium equals its notional principal.

The present value of a point-to-point EIA design (P_{pp}) is obtained as a function of α using [Theorem 3.2](#) by keeping the value of other parameters fixed (as given in [Table 1](#) for the numerical experiment). Considering the initial amount of investment by the policyholder to be 1, i.e., $P_p p = 1$, the break-even participation rate α is a solution of following equation:

$$X(0, T)E^{(r)}[C_{pp}(T)]P(\tau_x > T) + \sum_{t=0}^{T-1} (X(0, t+1)E^{(r)}[C_{pp}(t+1)]P(t < \tau_x \leq t+1)) = 1. \quad (34)$$

The term $C_{pp}(t)$ in above equation is computed using Algorithm 2. Note that, the term α will denote the break-even participation rate afterwards.

The default parameter values and values used for analysis are mentioned in [Table 1](#). Note that, if the volatility of interest rate is assumed to be zero, then it will correspond to deterministic interest rates. Hence, deterministic interest rates model is also a sub-case of the proposed model. The mortality of the annuitant is assumed to follow China Life Insurance Mortality Table (2010–2013). The Hawkes parameters in all the considered models for analysis are chosen such that the expected number of jumps, i.e., $\frac{\lambda_0}{1-\frac{\theta}{\delta}}$ takes constant value 0.5.

The break-even participation rates α for the point-to-point design of EIAs under some scenarios are shown in [Table 2](#). [Table 2](#) shows that with an increase in the term T of the contract, the break-even participation rate increases. Because of the longer-term greater risk for the policyholders is involved, thus higher participation rates are required to cover possible losses. Similar results are seen from [Fig. 1](#). From [Table 2](#) and [Fig. 1](#) it is evident that, with an increase in ρ values (from negative to positive), the participation rate also increases.

With respect to the volatility of either stock or interest rate or both, [Table 2](#) shows that the increase in volatility forces the break-even participation rate to decline. Increase in volatility implies a higher risk for the insurer, and hence a lower participation level is required to compensate the insurer's risk. Similar results for interest rate volatility can be inferred from [Fig. 2](#). This figure depicts the behaviour of break-even participation rate with respect to changes in interest rate volatility, i.e., σ_2 values and maturity time T . [Fig. 2](#) also shows an increasing gap between the α values for the three σ_2 values with increasing maturity. [Fig. 2](#) also shows the relationship between the values of α and Term T corresponding to

Table 1
Parameters value considered in Analysis.

Parameters	Default Values	Range for Analysis
a	0.85837	0.85837
b	0.089102	0.089102
ρ	0.1	(0.1, 0, -0.1)
θ	1	$< \delta$
δ	2	$> \theta$
λ_0	0.25	≥ 0
λ_∞	λ_0	λ_0
σ_1	0.2	(5%, 10%, 15%)
σ_2	0.1	(20%, 30%)
m	0.05	0.05
σ	0.03	0.03
T	5	1–10 years
Age	50	40–70 years
g	0.03	0.03
γ	0.2	0.2

Table 2
Point-to-point design break-even participation rate.

ρ	Age	T	$\sigma_1 = 0.2$			$\sigma_1 = 0.3$		
			$\sigma_2 = 0.05$	$\sigma_2 = 0.1$	$\sigma_2 = 0.15$	$\sigma_2 = 0.05$	$\sigma_2 = 0.1$	$\sigma_2 = 0.15$
0.1	40	3	0.7936	0.7551	0.6944	0.7111	0.6775	0.6233
		5	0.9177	0.8788	0.8116	0.8700	0.8321	0.7656
		7	0.9760	0.9421	0.8811	0.9494	0.9157	0.8516
	50	3	0.7926	0.7541	0.6939	0.7105	0.6766	0.6227
		5	0.9160	0.8767	0.8103	0.8684	0.8302	0.7638
		7	0.9739	0.9404	0.8788	0.9473	0.9131	0.8492
	60	3	0.7911	0.7530	0.6922	0.7080	0.6750	0.6208
		5	0.9133	0.8746	0.8077	0.8648	0.8267	0.7605
		7	0.9696	0.9361	0.8752	0.9416	0.9079	0.8440
	40	3	0.7979	0.7625	0.7044	0.7135	0.6817	0.6293
		5	0.9207	0.8849	0.8202	0.8708	0.8341	0.7692
		7	0.9783	0.9476	0.8878	0.9496	0.9159	0.8532
0	50	3	0.7970	0.7620	0.7038	0.7120	0.6807	0.6283
		5	0.9197	0.8834	0.8189	0.8692	0.8322	0.7677
		7	0.9768	0.9454	0.8857	0.9470	0.9134	0.8509
	60	3	0.7956	0.7605	0.7024	0.7100	0.6789	0.6267
		5	0.9170	0.8803	0.8164	0.8653	0.8287	0.7642
		7	0.9725	0.9414	0.8817	0.9415	0.9077	0.8454
	40	3	0.8025	0.7714	0.7160	0.7153	0.6861	0.6357
		5	0.9247	0.8919	0.8298	0.8719	0.8367	0.7741
		7	0.9813	0.9532	0.8958	0.9497	0.9163	0.8558
	50	3	0.8016	0.7704	0.7150	0.7143	0.6851	0.6347
		5	0.9231	0.8906	0.8288	0.8700	0.8349	0.7723
		7	0.9792	0.9510	0.8939	0.9470	0.9140	0.8531
-0.1	60	3	0.7999	0.7687	0.7134	0.7121	0.6831	0.6331
		5	0.9202	0.8873	0.8260	0.8665	0.8314	0.7688
		7	0.9752	0.9469	0.8895	0.9415	0.9085	0.8479

different values of σ_2 . From this figure and Table 2, it is evident that the α values decrease with an increase in maturity term.

With respect to the age at inception of the contract, Table 2 shows that the participation rates decrease with an increase in the age. As age at inception increases, the expected time for which the contract remains in force decreases. Hence the claim amount is discounted for lesser duration resulting in a comparatively higher present value. Since, the initial premium is constant, therefore, to compensate this gain α reduces as age increases.

Figs. 3 and 4, show comparison between Hawkes model with different parameters, and with GBM and Poisson JD model. For consistency of the expected number of jumps in unit time, the Hawkes parameters ($\lambda_0, \theta, \delta$) in all the Hawkes models considered in Figs. 3 and 4 are such that the expected number of jumps in unit time of the models, i.e., $\frac{\lambda_0}{1-\frac{\theta}{\delta}}$, takes constant value 0.5. Note that the model $HW(.25, 1, 2)$ has the default parameter values and is same as the Hawkes model considered in Fig. 3. Additionally, the intensity of the Poisson JD model in Fig. 3 is considered to be 0.5, same as the value of $\frac{\lambda_0}{1-\frac{\theta}{\delta}}$.

Algorithm 2 Computation of $C_{pp}(t)$ values with parameters as given in Table 1.

Require: Expected claim amount of a point-to-point design at time t for a single unit of EIA as a function of alpha

- 1: **Set** parameters as per the values in Table 1 and $N_{sim} = 100000$,
- 2: Initialize $E = 0$,
- 3: **for** $i = 1 : 100000$ **do**
- 4: Generate a path of Hawkes process till time t using Algorithm 1. This results in n jump times (say) $t_1, t_2, \dots, t_n$,
- 5: Compute $\Lambda_{0,t}$ using Eq. (16) by simple calculation of the definite integrals $\int_0^t \frac{\sigma_1 \sigma_2}{a} \rho M(u, T, a) du$ and $\int_0^t \frac{\sigma_2^2}{a^2} M(u, t, a) M(u, T, a) du$; and $int_\lambda = \int_0^t \lambda_s ds$ by the following steps
- 6: **Initialize** $int_\lambda = \lambda_\infty * t + (\lambda_0 - \lambda_\infty) * (1 - \exp(-\delta t)) / \delta$
- 7: **if** n is non-zero
- 8: **for** $k = 1 : n$ **do**
- 9: $int_\lambda = int_\lambda + \theta * (1 - \exp(-\delta * (t - t_k))) / \delta$
- 10: **end**
- 11: **end**
- 12: Compute M_t^r and V_t^r from Eqs. (17) and (18) respectively by using value of $\Lambda_{0,t}$ and solving definite integrals $\int_0^t \frac{\sigma_2^2}{a^2} M^2(u, t, a) du$ and $\int_0^t \frac{2\sigma_1 \sigma_2}{a} M(u, t, a) du + n\sigma^2$
- 13: Obtain the value

$$C_{pp} = e^{g(t)} \Phi\left(\frac{\frac{g(t)}{\alpha} - M_t^r + \ln(S_0)}{\sqrt{V_t^r}}\right) + e^{\alpha M_t^r + \frac{\alpha^2}{2} V_t^r} \left(\Phi\left(\frac{\frac{\gamma(t)}{\alpha} - M_t^r + \ln(S_0) - \alpha V_t^r}{\sqrt{V_t^r}}\right) - \Phi\left(\frac{\frac{g(t)}{\alpha} - M_t^r + \ln(S_0) - \alpha V_t^r}{\sqrt{V_t^r}}\right) \right) + e^{\gamma(t)} \Phi\left(-\frac{\frac{\gamma(t)}{\alpha} - M_t^r + \ln(S_0)}{\sqrt{V_t^r}}\right)$$

where $\Phi(\cdot)$ is the cumulative density function of standard Normal distribution

- 14: $E = E + C_{pp}$

15: **end**

- 16: The expected value of $C_{pp}(t)$ is $\frac{E}{n}$

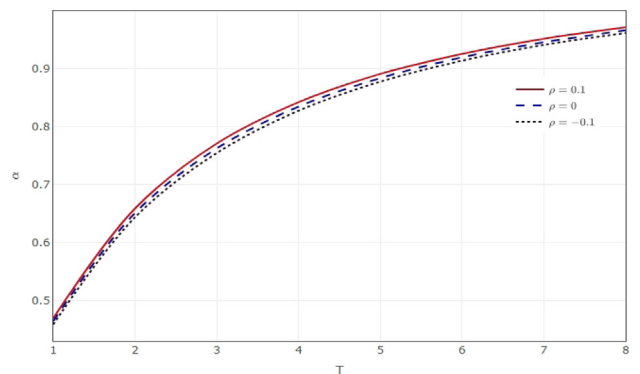


Fig. 1. Break-even participation rate with respect to different values of ρ .

Fig. 3 shows that the GBM model gives the lowest α values compared to the JD model with Poisson jumps and Hawkes JD model. Comparing Poisson and Hawkes models, for the considered default parameters, the break-even value for Hawkes model is slightly lower than that of the Poisson model (see Fig. 3). The difference in the α values for the three models is approximately constant as age increases with the Poisson model having the highest values and GBM model, giving least α values. But as the value of Hawkes parameters θ and δ increases keeping the average intensity constant i.e., (0.5), the break-even value also increases (see Fig. 4). Fig. 4 shows behaviour of different Hawkes models with notation $HW(\lambda_0, \theta, \delta)$ and Poisson Model with intensity 0.5. The initial intensity, λ_0 , the ratio $\frac{\theta}{\delta}$ and the average jumps, $\frac{\lambda_0}{1-\theta}$ are all constant in all the three Hawkes models. This figure shows that as the values of θ and δ increase keeping ratio $\frac{\theta}{\delta}$ constant, the value of α also increases. From this figure, it is observed that for Hawkes parameters (0.25, 3, 6) and (0.25, 5, 10), the α values are marginally higher than those of the Poisson model and GBM model. Further, with the increase in age at inception, the α value decreases for all the four models in Fig. 4 with a reducing in-between gap.

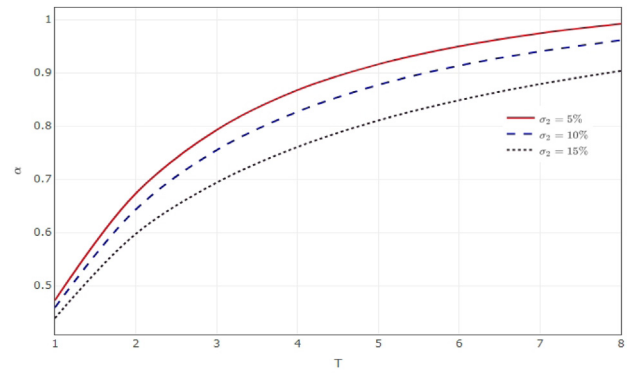


Fig. 2. Relation between α and maturity time T for Hawkes models with different interest rate volatility σ_2 .

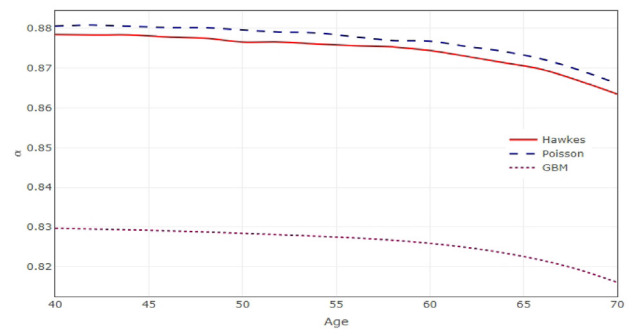


Fig. 3. Effect of age on α for GBM, Poisson and Hawkes models.

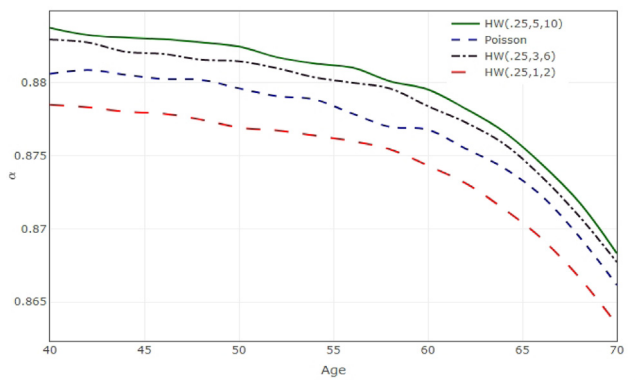


Fig. 4. Effect of age on α for different Hawkes models with parameter set $(\lambda_0, \theta, \delta)$ and Poisson model.

The relation between changes in the Hawkes parameters θ and δ values on the break-even fee is shown in Figs. 5 and 6 respectively. In Fig. 5, as the value of θ changes from 1 to 1.9 fixing $\delta = 2$ constant, the initial intensity λ_0 of Hawkes processes is adjusted correspondingly such that the expected number of jumps is consistent. The equation for initial intensity is $\lambda_0 = 0.5 \left(1 - \frac{\theta}{2}\right)$. Similarly, in Fig. 6, as the value of δ changes from 1.1 to 2 with $\theta = 1$, the initial intensity is $\lambda_0 = 0.5 \left(1 - \frac{1}{\delta}\right)$.

5. Conclusion

In this work, we considered stochastic interest rate, jumps and clustering of jumps risk into the valuation of EIAs. By applying the change in probability measures technique, we obtain the closed-form solutions for the valuation of point-to-point design and annual reset design of EIAs. Numerical experiments and sensitivity analysis with respect to various

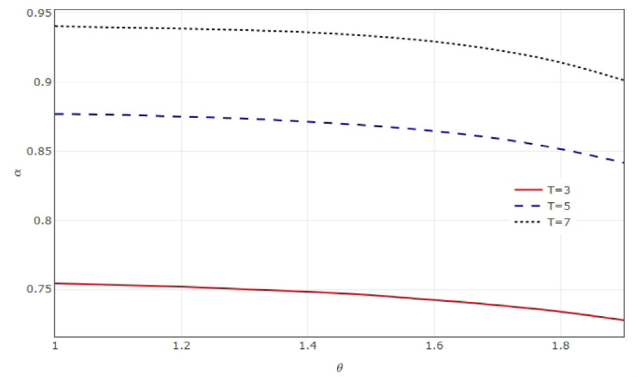


Fig. 5. Effect of θ on α for Hawkes model with varying maturity.

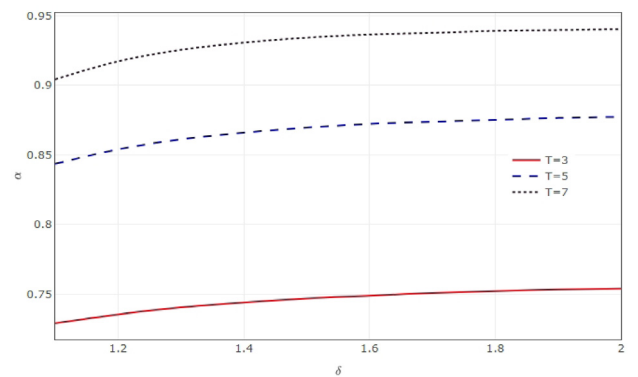


Fig. 6. Effect of δ on α for Hawkes model with varying maturity.

parameters of the proposed model are performed for the point-to-point design of EIA. Our results show that the α values decrease with an increase in age at inception, the correlation coefficient ρ , and the volatility of both stock and interest rate. But with the increasing value of time to maturity, the value of α also increases. As a special case, when $\theta = 0$, there is no clustering of jumps, hence proposed model reduces to a Poisson JD model. In addition, if the jump intensity $\lambda_0 = 0$, the model becomes a simple geometric Brownian motion model. Similarly, changing the parameters of the intensity model, we can have constant and independent intensity sub-cases. Comparison with α values obtained through the GBM model and the Poisson JD model is also performed. For the considered parameters, the α values for the proposed model lie between those of Poisson and GBM models with GBM having the least values.

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