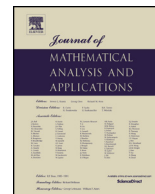




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Convergences for a virus-like evolving population driven by mutually-exciting Hawkes processes

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ABSTRACT

This paper presents a stochastic model motivated by the study of a virus-like evolving population with different mutation rates. This is a continuous time birth-death model: the birth processes are mutually-exciting Hawkes processes and the death process is also a Hawkes process. This structure for the births and the deaths does not allow, in general, to get the Markov property of the processes involved. But considering the couple given by the Hawkes processes and their intensities we are able to deduce the necessary and sufficient conditions for the Markov property of the couple. This property is the main tool to get the convergence results describing the behaviour of the population, and the existence of a phase transition at a critical fitness level.

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1. Introduction

Inspired by the discrete time evolution model studied in Roy and Tanemura [17], in the present paper, we develop a continuous time stochastic model for the evolution and survival of species subject to death and birth of individuals subject to different mutation levels. The model initially introduced by Guiol et al. [12] consists of either a birth or a death taking place independently at each discrete time point with probability p or $1 - p$, respectively. Each individual at birth is assigned a fitness which is chosen uniformly at random from $[0, 1]$ and is independent of all other processes. When a death occurs the individual with the smallest fitness is eliminated, and in case there is no individual present in the population then a death does not affect the system. In the modification of this model, Ben-Ari and Schinazi [3] and Roy and Tanemura [17] assume that

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- (a) a birth or a death takes place at a discrete time point with probability p or $1 - p$, respectively; at birth an individual is either a mutant with probability r or an individual of one of the existing subspecies with probability $1 - r$.
- (b) In case the individual at birth is a mutant, then it is assigned a value, chosen uniformly at random from $[0, 1]$, corresponding to the so-called fitness level.
- (c) In case the individual belongs to one of the existing subspecies, in the Ben-Ari and Schinazi [3] model, the individual is assigned a fitness uniformly chosen from the existing fitness levels, while, in the Roy and Tanemura [17] model, the individual is assigned a fitness f with a probability proportional to the number of individuals with fitness f present in the population.

Note that, even if, in several papers, the authors assume that the births and deaths are driven by Poisson processes, in many real-world problems, these assumptions are not valid. Often, there are situations in which the intensity function for the births cannot be considered as a constant and, moreover, the rate of new birth events increases or decreases with the occurrence of each event, and also new death excites further the death process, see Eick et al. [6], Daw and Pender [5], Koops et al. [15]. Hawkes processes have been studied extensively in connection with modelling earthquakes, societal aberrations and social media trend. In these problems, the future evolution of the models is influenced by the past events. The aforementioned empirical observations motivate the development of models in the financial market, capable of incorporating some kind of contagion effect, see Ait-Sahalia et al. [1]. Similarly, many authors have proposed various stochastic models using Hawkes processes for the temporal growth and migration of COVID-19 to incorporate contagion effect in the disease infection (see Garretto et al. [10] and references therein).

This is the reason why, in the present paper, to incorporate the Darwinian theory of the survival of the fittest, we assume that the births occur according to mutually-exciting Hawkes processes, while the deaths occur according to a self-exciting Hawkes process. Unlike the Poisson process, a Hawkes process is non-Markovian, in general, because the arrival of a particle excites the process such that the chances of the next arrival increases or decreases for some time period subsequent to the arrival. An important property of Hawkes process, the self-exciting property, is well known for its efficiency in reproducing jump clusters due to the dependence of the conditional birth rate, or intensity, on the counting process itself. These processes have also recently been used in problems related to financial markets, for instance, Embrechts et al. [7] discusses the multivariate Hawkes process and their applications to financial data.

The contribution of this work is twofold: on one hand, the necessary and sufficient conditions are derived for the Markov property of the couple given by general mutually-exciting Hawkes processes and their intensities. These conditions apply, in particular, to the continuous time model given by Hawkes births, with mutant and non-mutant individuals, respectively, N^1 and N^2 , and Hawkes deaths N^3 , and their intensities. The specificity of the Hawkes processes N^1 , N^2 , and N^3 , and the queue process $N = N^1 + N^2 - N^3$ is the dependence of the intensity, on the entire size of the population, N , and not only on the size of N^3 . On the other hand, some convergence results for the size of the population are reached, identifying the fitness sites where the population is concentrated.

2. The population model

We consider a population of individuals having different fitness levels: an individual may have a fitness which already exists in the population, or it is a mutant. The fitness level for each individual is a parameter f , taking values uniformly in $[0, 1]$ and independently of other individuals. When the death process rings, an individual who has its fitness closest to the value 0 dies.

This population dynamics is described by three stochastic processes: N^1 counting the births of new mutants, N^2 counting the births of non-mutants and N^3 counting the deaths of mutant and non-mutant individuals. Each process N^i , $i = 1, 2, 3$, is characterized by its own sequence of jump times, we consider

$\{\tau_k\}_{k \geq 0}$ the sequence of jump times obtained by merging these three sequences of jump times. By construction, we assume that initially the population is empty, therefore,

- $\tau_0 = 0$, there is no individual at first.
- τ_1 is a jump time of the process N^1 , since there can be neither a birth of a non-mutant, having a pre-existing fitness level in the population, nor a death.
- The following jump times can be a jump time from any process. The dynamics is given as below:
 - If τ_k is a jump time of N^1 , this means that a new mutant is born, and its fitness level is chosen uniformly at random in $[0, 1]$.
 - If τ_k is a jump time of N^2 , this means that a non-mutant is born, and its fitness level is chosen from the set of all fitness levels already existing in the living population, with a probability proportional to the number of individuals with this fitness level.
 - If τ_k is a jump time of N^3 , the smallest fitness level is chosen so that an individual with this fitness level dies.

Inspired by the previous considerations, this article adopts that the dynamics of the birth processes are mutually-exciting Hawkes processes, while the death process is governed by a self-exciting Hawkes process. The uncertainty is assumed to be described by $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$, a filtered probability space, where the filtration $\mathcal{F}_t$ satisfies the usual conditions of completeness and right continuity, and on which all the stochastic processes till time t are defined.

Definition 2.1. A *self-exciting Hawkes process* is a point process $H := \{H(t) : t \geq 0\}$ on $[0, \infty)$ and $\mathcal{F}_t^H := \sigma\{H(s) : s \leq t\}$ is the filtration generated by H . In the linear case the intensity of H is given by $\lambda(t) = \lambda_0 + \int_0^t \varphi(t-u) dH(u)$, where λ_0 is a positive real constant called the background intensity. The excitation function, $\varphi : (0, \infty) \rightarrow [0, \infty)$, is L^1 -integrable, and, to avoid the explosion of the process, must satisfy the condition $\|\varphi\|_{L^1} < 1$. Note that, when $\varphi(s) = 0$ for all s , the Hawkes process reduces to a homogeneous Poisson process with parameter λ_0 .

Definition 2.2. The processes $N^1 := \{N^1(t) : t \geq 0\}$ and $N^2 := \{N^2(t) : t \geq 0\}$, are *mutually-exciting Hawkes processes* if, given the background intensities λ_0^1 and λ_0^2 , and the mutually excitation functions, φ_{ji} , $i, j \in \{1, 2\}$, their intensities at time t are given by

$$\lambda^i(t) = \lambda_0^i + \sum_{j=1,2} \int_0^t \varphi_{ji}(t-u) dN^j(u), \quad \text{for } i = 1, 2. \quad (1)$$

Definition 2.3. Let $N^3 := \{N^3(t) : t \geq 0\}$ be another Hawkes process, independent of N^1 and N^2 , with background intensity λ_0^3 , excitation function ψ , and intensity given by

$$\lambda^3(t) = \lambda_0^3 + \int_0^t \psi(t-u) dN^3(u). \quad (2)$$

Remark 2.4. We are considering a model which is a simple queueing process $N(t)$ adapted to $\mathcal{F}_t$ and with input process N^1 and N^2 and output driven by N^3 . Hence, at time t , the size of the population is

$$N(t) := N^1(t) + N^2(t) - N^3(t), \quad (3)$$

and only if $N(t)$ is positive there is the possibility of a death at time t . Therefore, Brémaud [4] (Chapter II Section 3), N^3 also admits the $\mathcal{F}_t$ -intensity $\lambda^3(t) \mathbf{1}_{N(t-) > 0}$. By the linearity of the mean value and if N^1 ,

N^2 and N^3 have the $\mathcal{F}_t$ -intensities $\lambda^1(t)$, $\lambda^2(t)$ and $\lambda^3(t)\mathbf{1}_{N(t-)>0}$, respectively, we are able to assert that N has $\mathcal{F}_t$ -parameters $\lambda^1(t)$, $\lambda^2(t)$ and $\lambda^3(t)$, when $\lambda^1(t)$ and $\lambda^2(t)$ are related to the birth intensities and $\lambda^3(t)$ is related to the death intensity.

In Section 3 we derive various properties of the Hawkes processes, like the mean values of all the intensities, the expected number of births and the expected number of deaths. Section 4 and Section 5, are devoted to prove the Markov property of the couple given by the processes and their intensities. Section 6 investigates the stability of the model by studying the expectations of the intensities. Finally, Section 7 describes in details the model given by a virus-like evolving population and in Section 8 there are some convergence results for the behaviour of the population, identifying the region where the population is concentrated. Specifically in Theorem 8.1 and Corollary 8.4 we prove that there exists a phase transition at a critical fitness level.

3. Expected number of births and deaths and expected intensities

From all the assumptions made so far on (1), (2) and (3), taking into account that $\{\tau_i\}_{i \in \mathbb{N} \cup \{0\}}$ is the sequence of jump times of the process (N^1, N^2, N^3) , and setting as usual $\Delta N^j(t) = N^j(t) - N^j(t-)$, for $j = 1, 2, 3$, the intensities admit the representations

$$\lambda^i(t) = \lambda_0^i + \sum_{j=1,2} \sum_{k=1}^{\infty} \varphi_{ji}(t - \tau_k) \Delta N^j(\tau_k) \mathbf{1}_{\tau_k \leq t}, \quad \text{for } i = 1, 2,$$

and

$$\lambda^3(t) = \lambda_0^3 \mathbf{1}_{N(t-)>0} + \sum_{k=1}^{\infty} \psi(t - \tau_k) \Delta N^3(\tau_k) \mathbf{1}_{\tau_k \leq t} \mathbf{1}_{N(t-)>0}.$$

Note that, for $j = 1, 2, 3$, $\Delta N^j(\tau_n)$ is not null only if N^j has a jump in τ_n . In this case, the intensity has an increment due to this jump. For instance, if N^1 has a jump at time τ_8 , then $\Delta N^1(\tau_8) = 1$ and $\Delta N^2(\tau_8) = 0$, which implies that $\sum_{j=1,2} \varphi_{ji}(t - \tau_8) \Delta N^j(\tau_8) \mathbf{1}_{\tau_8 \leq t} = \varphi_{1i}(t - \tau_8) \mathbf{1}_{\tau_8 \leq t}$.

Remark 3.1. At t , set $n := N^1(t) + N^2(t) + N^3(t)$, which is the number of jump times of $(N^1(t), N^2(t), N^3(t))$. Therefore, the intensities for $\tau_n \leq t < \tau_{n+1}$ are given by

$$\lambda^i(t) = \lambda_0^i + \sum_{j=1,2} \sum_{k=1}^n \varphi_{ji}(t - \tau_k) \Delta N^j(\tau_k), \quad i = 1, 2, \quad (4)$$

$$\lambda^3(t) = \lambda_0^3 \mathbf{1}_{N(\tau_n)>0} + \sum_{k=1}^n \psi(t - \tau_k) \Delta N^3(\tau_k) \mathbf{1}_{N(\tau_n)>0}. \quad (5)$$

Thus, at the jump time τ_n the intensities are such that

$$\lambda^i(\tau_n) - \lambda^i(\tau_n-) = \sum_{j=1,2} \varphi_{ji}(0) \Delta N^j(\tau_n), \quad i = 1, 2, \quad (6)$$

$$\lambda^3(\tau_n) - \lambda^3(\tau_n-) = \psi(0) \Delta N^3(\tau_n). \quad (7)$$

Note that the sudden increase in τ_n implies a higher probability of having a jump of the same kind immediately after τ_n .

The Hawkes processes are particular cases of non-homogeneous point processes. Their intensities are stochastic, since they explicitly depend on the occurrence of previous events, followed by a progressive normalisation, driven by the memory kernels.

Proposition 3.2. *For $i = 1, 2, 3$, by (1) and (2), $(\lambda^i(t), N^i(t))$ are such that the expected intensities $\mathbb{E}[\lambda^i(t)]$ satisfy the differential equations:*

$$d\mathbb{E}[\lambda^i(t)] = \sum_{j=1,2} \left(\int_0^t \varphi'_{ji}(t-u) \mathbb{E}[\lambda^j(u)] du + \varphi_{ji}(0) \mathbb{E}[\lambda^j(t)] \right) dt, \quad i = 1, 2 \quad (8)$$

$$d\mathbb{E}[\lambda^3(t)] = \int_0^t \psi'(t-u) \mathbb{E}[\lambda^3(u)] du dt + \psi(0) \mathbb{E}[\lambda^3(t)] dt. \quad (9)$$

Proof. For $n > 0$, at $t = \tau_n$, a jump time of the process (N^1, N^2, N^3) , the intensities λ^1 , λ^2 , and λ^3 have a jump given by (6) and (7), respectively, hence,

$$d\lambda^i(t) = \sum_{j=1,2} \varphi_{ji}(0) dN^j(t), \quad \text{for } i = 1, 2,$$

and

$$d\lambda^3(t) = \psi(0) dN^3(t).$$

In between the jump times, for $\tau_{n-1} \leq t < \tau_n$, given (4) and (5), we have that

$$d\lambda^i(t) = \sum_{j=1,2} \int_0^t \varphi'_{ji}(t-u) dN^j(u) dt,$$

and

$$d\lambda^3(t) = \int_0^t \psi'(t-u) dN^3(u) dt.$$

This means that, for all t ,

$$d\lambda^i(t) = \sum_{j=1,2} \left\{ \int_0^t \varphi'_{ji}(t-u) dN^j(u) dt + \varphi_{ji}(0) dN^j(t) \right\},$$

for $i = 1, 2$, and

$$d\lambda^3(t) = \int_0^t \psi'(t-u) dN^3(u) dt + \psi(0) dN^3(t).$$

Recall that, (see the definition of intensity given by Brémaud [4], Chapter II Section 3), the counting process N^i admits intensity $\lambda^i(t)$, if, for all non negative predictable process $C(t)$, the equality $\mathbb{E}[\int_0^\infty C(s) dN^i(s)] = \mathbb{E}[\int_0^\infty C(s) \lambda^i(s) ds]$ holds. Hence, by the Dominated Convergence Theorem, $\mathbb{E}[d\lambda^i(t)] = d\mathbb{E}[\lambda^i(t)]$, and

$$d\mathbb{E}[\lambda^i(t)] = \sum_{j=1,2} \left(\int_0^t \varphi'_{ji}(t-u) \mathbb{E}[\lambda^j(u)] du \, dt + \varphi_{ji}(0) \mathbb{E}[\lambda^j(t)] dt \right),$$

for $i = 1, 2$, and an analogous procedure allows us to deduce (9). $\square$

Theorem 3.3. Let $(\lambda^i(t), N^i(t))$, $i = 1, 2$, be Hawkes processes with dynamics of the intensities given by (1). Solving (8), for $i = 1, 2$, $j = 1, 2$ and $i \neq j$, we get

$$\begin{aligned} \mathbb{E}[\lambda^i(t)] &= \lambda_0^i \int_0^t \int_0^s [\varphi_{ij}(s-u) \varphi_{ji}(u) - \varphi_{ii}(s-u) \varphi_{jj}(u)] \, du \, ds \\ &\quad + \lambda_0^i + \lambda_0^i \int_0^t \varphi_{ii}(s) \, ds + \lambda_0^j \int_0^t \varphi_{ji}(s) \, ds. \end{aligned}$$

Furthermore, taking into account that $\mathbb{E}[N^i(t)] = \int_0^t \mathbb{E}[\lambda^i(v)] \, dv$

$$\begin{aligned} \mathbb{E}[N^i(t)] &= \lambda_0^i \int_0^t \int_0^v \int_0^s [\varphi_{ij}(s-u) \varphi_{ji}(u) - \varphi_{ii}(s-u) \varphi_{jj}(u)] \, du \, ds \, dv \\ &\quad + \lambda_0^i t + \lambda_0^i \int_0^t \int_0^v \varphi_{ii}(s) \, ds \, dv + \lambda_0^j \int_0^t \int_0^v \varphi_{ji}(s) \, ds \, dv. \end{aligned}$$

The proof of Theorem 3.3 is in Appendix.

Remark 3.4. If $(\lambda(t), N(t))$ is a linear Hawkes process with exponential excitation function, namely, $\lambda(t) = \lambda_0 + \int_0^t \alpha e^{-\beta(t-u)} dN(u)$, with $\alpha, \beta > 0$, (8) reduces to

$$d\mathbb{E}[\lambda(t)] = \left(\alpha \mathbb{E}[\lambda_t] - \beta \int_0^t \alpha e^{-\beta(t-u)} \mathbb{E}[\lambda(u)] du \right) dt. \quad (10)$$

Taking the Laplace Transform on both sides, and setting as usual $Y(s) = \mathcal{L}(\mathbb{E}[\lambda(t)])(s) = \int_0^\infty e^{-st} \mathbb{E}[\lambda(t)] dt$, and rearranging, we get

$$Y(s) \left(1 + (\beta - \alpha) \frac{1}{s} \right) = \lambda_0 \left(\frac{1}{s} + \beta \frac{1}{s^2} \right).$$

Taking the Inverse Laplace Transform and then the derivative of both sides, we have

$$d\mathbb{E}[\lambda(t)] + (\beta - \alpha) \mathbb{E}[\lambda(t)] = \beta \lambda_0. \quad (11)$$

Alternately, noting that, in this case, $\mathbb{E}[\lambda(t)] = \lambda_0 + \int_0^t \alpha e^{-\beta(t-u)} \mathbb{E}[\lambda(u)] du$, immediately, (10) reduces to (11). Then, integrating, we get $\mathbb{E}[\lambda(t)] = \frac{\beta \lambda_0}{\alpha - \beta} (e^{(\alpha - \beta)t} - 1) + \lambda_0 e^{(\alpha - \beta)t}$, and, given $N(0) = N_0$, $\mathbb{E}[N(t)] = N_0 + \frac{\alpha \lambda_0}{(\alpha - \beta)^2} (e^{(\alpha - \beta)t} - 1) - \frac{\beta \lambda_0}{\alpha - \beta} t$.

Moreover, note that Theorem 3.3 applies even to the special case of Poisson scenario, in which all the excitation functions are null.

Theorem 3.5. *Given the process $(\lambda^3(t), N^3(t))$ with the structure of the intensity process given by (2), and solving (9), we have $\mathbb{E}[\lambda^3(t)] = \lambda_0^3 + \lambda_0^3 \int_0^t \psi(v) dv$ and $\mathbb{E}[N^3(t)] = \int_0^t \mathbb{E}[\lambda^3(v)] dv = \lambda_0^3 t + \lambda_0^3 \int_0^t \int_0^s \psi(v) dv ds$.*

The proof of Theorem 3.5 is in Appendix.

Remark 3.6. Recall that, in queuing theory, *stable* describes a system where the number of customers or jobs in the queue remains finite and does not grow indefinitely over time, indicating a balance between arrival and service rates, which, in our model, requires that

$$\mathbb{E}[\lambda^1(t)] + \mathbb{E}[\lambda^2(t)] < \mathbb{E}[\lambda^3(t)],$$

see also Remark 5.7 later. Looking at Theorem 3.3 and Theorem 3.5, we get that

$$\begin{aligned} \mathbb{E}[\lambda^1(t)] + \mathbb{E}[\lambda^2(t)] &= \lambda_0^1 + \lambda_0^1 \int_0^t (\varphi_{11}(s) + \varphi_{12}(s)) ds + \lambda_0^2 + \lambda_0^2 \int_0^t (\varphi_{21}(s) + \varphi_{22}(s)) ds \\ &\quad + (\lambda_0^1 + \lambda_0^2) \int_0^t \int_0^s [\varphi_{12}(s-u)\varphi_{21}(u) - \varphi_{11}(s-u)\varphi_{22}(u)] du ds, \end{aligned}$$

and $\mathbb{E}[\lambda^3(t)] = \lambda_0^3 + \lambda_0^3 \int_0^t \psi(v) dv$. Therefore, the stability condition is given by

$$\begin{aligned} \lambda_0^3 + \lambda_0^3 \int_0^t \psi(v) dv &> \lambda_0^1 \int_0^t (\varphi_{11}(s) + \varphi_{12}(s)) ds + \lambda_0^2 \int_0^t (\varphi_{21}(s) + \varphi_{22}(s)) ds \\ &\quad + (\lambda_0^1 + \lambda_0^2) \left(1 + \int_0^t \int_0^s [\varphi_{12}(s-u)\varphi_{21}(u) - \varphi_{11}(s-u)\varphi_{22}(u)] du ds \right). \end{aligned} \quad (12)$$

Note that, when the Hawkes process reduces to a Poisson process, the excitation functions are null, and, $\mathbb{E}[\lambda^i(t)] \equiv \lambda_0^i$. Thus, the stability condition turns to be, as usual, $\lambda_0^3 > \lambda_0^1 + \lambda_0^2$.

4. Representation for the intensities

By Errais et al. [8], we know that the Markov property holds for the couple, Hawkes process and its intensity, having exponential kernel. We are going to discuss this approach investigating in this section and in the next one the necessary and sufficient conditions for the Markov property of our model.

First of all, due to the presence of the birth processes, the death process and the process counting the number of individuals alive, we cannot be sure that if there is a martingale defined with respect to a filtration, still it remains a martingale with respect to a greater filtration. More specifically, when there are two filtrations $\mathcal{G}^1$ and $\mathcal{G}^2$, such that $\mathcal{G}^1 \subset \mathcal{G}^2$, as is well known, a $\mathcal{G}^2$ -martingale implies a $\mathcal{G}^1$ -martingale. But the converse is not true, namely, a $\mathcal{G}^1$ -martingale does not always remain a martingale with respect to the greater filtration $\mathcal{G}^2$. This is the classical problem of the enlargement of filtrations. A usual assumption in this frame is the so-called Immersion property, namely, any $\mathcal{G}^1$ -local martingale must also be a $\mathcal{G}^2$ -local martingale, see Tardelli [19], Selvamuthu and Tardelli [18] and for further details Jeanblanc et al. [14]. Consequently, following the existing literature, the Immersion property is assumed in the present paper. Note that this is an essential tool to prove the Markov property of the model.

For $i = 1, 2, 3$, the behaviour of each λ^i is completely determined by a function of the history of the processes N^i , up to time t , namely, by the so-called *history process* defined by $H_t^i = \{\phi^i = (t_k^i)_{k \in \mathbb{N}} : t_k^i \leq t\}$, for $t > 0$, and $H_0^i = \emptyset$, otherwise. These are pure jump processes having the same jump times of N^i and taking values in the space of the history sets $\mathcal{H}^i = \{\emptyset\} \cup \{\cup_{j \geq 1} \mathcal{H}_j^i\}$, where

$$\mathcal{H}_j^i = \{h = (t_1^i, \dots, t_j^i) : 0 < t_k^i < t_{k+1}^i, k = 1, 2, \dots, j\}.$$

By defining a suitable metric d , $(\mathcal{H}^i, d)$ is a complete and separable metric space, and $\mathcal{B}(\mathcal{H}^i)$ are the Borel subsets of $\mathcal{H}^i$. Given a $h \in \mathcal{H}^i$, let j be the number of elements of h , then $h \in \mathcal{H}_j^i$, and $\sum_{k=1}^j \mathbf{1}_{t_k^i \leq t}$, the corresponding trajectory, is the realization of N^i . Thus, H_t^i is a Markov process. Basic properties on the history process are in Gerardi and Tardelli [11], and further details in Arjas et al. [2].

Unfortunately, the history processes are not really easy to handle. Hence, we want to avoid the use of them, which necessitates the introduction of some heavy technicalities, but, at the same time, we must consider that the vector of processes (N^1, N^2, N^3) still depends on its own whole history. To overcome this difficulty, we introduce the *shot noise processes*, in Definition 4.1 below.

Definition 4.1. For $t > 0$, let the *shot noise processes* be $(\xi_t^1, \xi_t^2, \xi_t^3)$ with

$$\xi_t^i := \sum_{j=1,2} \int_0^t \varphi_{ji}(t-u) dN^j(u) \quad i = 1, 2, \quad \xi_t^3 = \int_0^t \psi(t-u) dN^3(u). \quad (13)$$

Thus, (ξ^1, ξ^2, ξ^3) contains all the information relating to the history of (N^1, N^2, N^3) and of interest for λ^i , $i = 1, 2, 3$, given in (1) and (2). Consequently,

$$\lambda^i(t) = \lambda_0^i + \xi_t^i, \quad i = 1, 2, \quad \text{and} \quad \lambda^3(t) = (\lambda_0^3 + \xi_t^3) \mathbf{1}_{N(t-) > 0}, \quad (14)$$

where, $N(t)$ is the cardinality of the population at time t , as given in (3). Note that, (14) implies that, $\lambda^i(t)$ are deterministic measurable functions of (ξ^1, ξ^2, ξ^3) at t .

Theorem 4.2. Fix a t and set $n := N^1(t) + N^2(t) + N^3(t)$, at the jump time τ_n ,

$$\lambda^i(\tau_n) = \lambda^i(\tau_n-) + \sum_{j=1,2} \varphi_{ji}(0) \Delta N^j(\tau_n), \quad i = 1, 2, \quad (15)$$

$$\lambda^3(\tau_n) = (\lambda^3(\tau_n-) + \psi(0) \Delta N^3(\tau_n)) \mathbf{1}_{N(\tau_n-) > 0}, \quad (16)$$

while, for $\tau_n \leq s < t \leq \tau_{n+1}$ and $i = 1, 2$,

$$\begin{aligned} \lambda^i(t) &= \lambda_0^i + \xi_s^i + \sum_{j=1,2} \sum_{k=1}^n (\varphi_{ji}(t - \tau_k) - \varphi_{ji}(s - \tau_k)) \Delta N^j(\tau_k), \\ \lambda^3(t) &= \left(\lambda_0^3 + \xi_s^3 + \sum_{k=1}^n (\psi(t - \tau_k) - \psi(s - \tau_k)) \Delta N^3(\tau_k) \right) \mathbf{1}_{N(\tau_n) > 0}. \end{aligned}$$

Proof. At the jump time τ_n , (15) and (16) immediately follow by (14), and by (13). For $\tau_n \leq s < t < \tau_{n+1}$, since

$$\xi_t^i - \xi_s^i = \sum_{j=1,2} \sum_{k=1}^n (\varphi_{ji}(t - \tau_k) - \varphi_{ji}(s - \tau_k)) \Delta N^j(\tau_k), \quad i = 1, 2,$$

$$\xi_t^3 - \xi_s^3 = \int_0^s (\psi(t-u) - \psi(s-u)) dN^3(u) = \sum_{k=1}^n (\psi(t-\tau_k) - \psi(s-\tau_k)) \Delta N^3(\tau_k),$$

the representation for intensity processes is achieved as required. $\square$

Corollary 4.3. For all $0 < s < t$, and for $i = 1, 2$, by (13), and (14), the representation for the intensity processes are given by

$$\begin{aligned} \lambda^i(t) &= \lambda_0^i + \xi_s^i + \sum_{j=1,2} \left(\int_0^s (\varphi_{ji}(t-u) - \varphi_{ji}(s-u)) dN^j(u) + \int_s^t \varphi_{ji}(t-u) dN^j(u) \right), \\ \lambda^3(t) &= \left(\lambda_0^3 + \xi_s^3 + \int_0^s (\psi(t-u) - \psi(s-u)) dN^3(u) + \int_s^t \psi(t-u) dN^3(u) \right) \mathbf{1}_{N(t-)>0}. \end{aligned}$$

5. Discussion on the Markov property

For $i = 1, 2, 3$, the representations of the shot noise ξ^i and of the intensity λ^i , obtained in Theorem 4.2, are the key to deduce the assumptions necessary to guarantee the Markov property of this model.

Definition 5.1. For $0 < s < t$, let us assume that there exist $f_i(s, t)$, $i = 1, 2, 3$, deterministic functions, such that the quantities

$$\varphi_{ji}(t - \tau_k) - f_i(s, t)\varphi_{ji}(s - \tau_k) \quad i, j = 1, 2, \quad \text{and} \quad \psi(t - \tau_k) - f_3(s, t)\psi(s - \tau_k),$$

do not depend of the jump time τ_k , for any $\tau_k < s$. Then, we define

$$\begin{aligned} \gamma_j(i, s, t) &:= \varphi_{ji}(t - \tau_k) - f_i(s, t)\varphi_{ji}(s - \tau_k) \quad i, j = 1, 2, \\ \gamma(3, s, t) &:= \psi(t - \tau_k) - f_3(s, t)\psi(s - \tau_k). \end{aligned}$$

Theorem 5.2. Under Definition 5.1, $(N^1, \lambda^1, N^2, \lambda^2, N^3, \lambda^3)$, the Hawkes processes and their intensities, satisfy the Markov property.

Proof. Note that

$$\xi_t^i - f_i(s, t)\xi_s^i = \sum_{j=1,2} \left(\gamma_j(i, s, t)N^j(s) + \int_s^t \varphi_{ji}(t-u) dN^j(u) \right), \quad \text{for } i = 1, 2, \quad (17)$$

and

$$\xi_t^3 - f_3(s, t)\xi_s^3 = \gamma(3, s, t)N^3(s) + \int_s^t \psi(t-u) dN^3(u). \quad (18)$$

Consequently,

$$\lambda^i(t) = \lambda_0^i + f_i(s, t)\xi_s^i + \sum_{j=1,2} \left(\gamma_j(i, s, t)N^j(s) + \int_s^t \varphi_{ji}(t-u) dN^j(u) \right), \quad \text{for } i = 1, 2,$$

and

$$\lambda^3(t) = \left(\lambda_0^3 + f_3(s, t) \xi_s^3 + \gamma(3, s, t) N^3(s) + \int_s^t \psi(t-u) dN^3(u) \right) \mathbf{1}_{N(t-) > 0}.$$

Hence, for $i = 1, 2, 3$, we are able to represent ξ^i and λ^i , at time t , in term of themselves at time s , $s < t$, and of some functions of s and t . This structure guarantees the claim. $\square$

Proposition 5.3. *Under Definition 5.1, $f_i(s, t)$ must be necessarily an exponential function, namely, there exists a positive real constant β_i such that $f_i(s, t) = e^{-\beta_i(t-s)}$.*

Proof. For $0 < s < t$, let τ_k and τ_l be jump times of (N^1, N^2, N^3) , which are also jump times of N^j , $j = 1, 2$, such that $\tau_k \vee \tau_l < s$. Then, taking into account Definition 5.1,

$$\gamma_j(i, s, t) = \varphi_{ji}(t - \tau_k) - f_i(s, t) \varphi_{ji}(s - \tau_k) = \varphi_{ji}(t - \tau_l) - f_i(s, t) \varphi_{ji}(s - \tau_l), \quad (19)$$

for $i = 1, 2$. Hence,

$$f_i(s, t) = \frac{\varphi_{ji}(t - \tau_k) - \varphi_{ji}(t - \tau_l)}{\varphi_{ji}(s - \tau_k) - \varphi_{ji}(s - \tau_l)}, \quad (20)$$

and for each r such that $s \leq r \leq t$, $f_i(s, t) = f_i(s, r) \cdot f_i(r, t)$. Considering the sequence of times $\{\tau'_j\}$, where $\tau'_j = \tau_j - s + 1$, then

$$f_i(s, t) = \frac{\varphi_{ji}(t - s + 1 - \tau'_k) - \varphi_{ji}(t - s + 1 - \tau'_l)}{\varphi_{ji}(1 - \tau'_k) - \varphi_{ji}(1 - \tau'_l)} = \tilde{f}_i(t - s),$$

and setting $u := t - r$ and $v := r - s$, we get the Cauchy functional identity $\tilde{f}_i(u + v) = \tilde{f}_i(u) \cdot \tilde{f}_i(v)$. As well known, by the continuity of f , which is guaranteed by the continuity of φ_{ji} , (20), this Cauchy functional identity implies that $\tilde{f}_i(0) = 1$ and $\tilde{f}_i(u) = e^{-\beta_i u}$, for a $\beta_i \in \mathbb{R}$. Analogous procedure allows us to achieve the same result for $f_3(s, t)$. $\square$

We have to highlight the results of Theorem 5.4 below, since (21) establishes an important restriction on the family of memory kernels of Hawkes processes that satisfy the Markov property.

Theorem 5.4. *The family of memory kernels satisfying Definition 5.1 is*

$$\varphi_{ji}(t) = \delta_{ji} + \alpha_{ji} e^{-\beta_i t}, \quad i, j = 1, 2, \quad \text{and} \quad \psi(t) = \delta_3 + \alpha_3 e^{-\beta_3 t}, \quad (21)$$

where α_{ji} , δ_{ji} , $i, j = 1, 2$, β_i , $i = 1, 2, 3$, α_3 , and δ_3 are positive real constants.

Proof. Taking into account that, under Definition 5.1, $f_i(s, t) = e^{-\beta_i(t-s)}$, and choosing τ_k and τ_l , jump times of (N^1, N^2, N^3) , by (19), we have that

$$\varphi_{ji}(t - \tau_k) - \varphi_{ji}(t - \tau_l) = e^{-\beta_i(t-s)} [\varphi_{ji}(s - \tau_k) - \varphi_{ji}(s - \tau_l)].$$

Choosing again $\tau'_k = \tau_k - s + 1$,

$$\varphi_{ji}(t - s + 1 - \tau'_k) - \varphi_{ji}(t - s + 1 - \tau'_l) = e^{-\beta_i(t-s)} [\varphi_{ji}(1 - \tau'_k) - \varphi_{ji}(1 - \tau'_l)],$$

for all k , which allows us to deduce that there exist the constants α_{ji} , β_i and $\delta_{ji} \in \mathbb{R}$, such that $\varphi_{ji}(t)$ assumes the form given in (21). Analogously, we are able to get the constants α_3 , β_3 and $\delta_3 \in \mathbb{R}$. $\square$

Corollary 5.5. *Given $f_i(s, t) = e^{-\beta_i(t-s)}$ and the results achieved in Theorem 5.4, immediately, we get that, for $i, j = 1, 2$, $\gamma_j(i, s, t) = \delta_{ji} [1 - e^{-\beta_i(t-s)}]$ and $\gamma(3, s, t) = \delta_3 [1 - e^{-\beta_3(t-s)}]$.*

Definition 5.1 is the fundamental tool to get the Markov property. Proposition 5.3 and Theorem 5.4 assert that the Hawkes processes that satisfies Definition 5.1 must necessarily have memory kernels with the structure (21). This allows us to assure that the class of memory kernels of Hawkes processes satisfying the Markov property cannot be larger than that having the structure given in (21). An analogous procedure generalises this result to a finite number of Hawkes processes, Theorem 5.6.

Theorem 5.6. *The mutually exciting Hawkes processes and their intensity processes*

$$(N^1, \lambda^1, N^2, \lambda^2, \dots, N^m, \lambda^m)$$

satisfy the Markov property if and only if the memory kernels assume the form given in (21).

Remark 5.7. Note that from Lima and Choi [16] and Hawkes [13] on the assumption that $\int_0^{+\infty} \varphi_{ji}(t) dt \leq 1$ and $\int_0^{+\infty} \psi(t) dt \leq 1$, we are assured that the corresponding processes will attain wide-sense stationary behaviours. For the present model, these assumptions allow us to write that $\mathbb{E}[\lambda^i(t)] < 4(\lambda_0^1 \vee \lambda_0^2)$, for $i = 1, 2$, and $\mathbb{E}[\lambda^3(t)] < 2\lambda_0^3$. Note that, under Definition 5.1, namely in the Markov case, if we want non-explosion of the processes, N^1 , N^2 , and N^3 , we must assume that, for $i, j = 1, 2$,

$$\delta_{ij} = 0, \quad \delta_3 = 0, \quad (22)$$

$$\alpha_{ji} \leq \beta_i \quad \text{and} \quad \alpha_3 \leq \beta_3. \quad (23)$$

5.1. Dynamical behaviour of the intensities

Under Definition 5.1, we are going to give the representations of the behaviours of λ^i and ξ^i , for $i = 1, 2, 3$. Then, we write the infinitesimal generator for $(N^1, \lambda^1, N^2, \lambda^2, N^3, \lambda^3)$, given by the mutually exciting Hawkes processes, the self-exciting Hawkes process and their intensities.

Proposition 5.8. *Under Definition 5.1, for $i = 1, 2, 3$, we have that*

$$\begin{aligned} d\lambda^i(t) &= d\xi_t^i = \beta_i \left(\sum_{j=1,2} \delta_{ji} N^j(t-) - \lambda^i(t) + \lambda_0^i \right) dt + \sum_{j=1,2} \alpha_{ji} dN^j(t), \quad i = 1, 2, \\ d\lambda^3(t) &= d\xi_t^3 = \beta_3 (\delta_3 N^3(t-) - \lambda^3(t) + \lambda_0^3) dt + \alpha_3 dN^3(t). \end{aligned}$$

Proof. Fix a $t > 0$ and a positive $n = N^1(t) + N^2(t) + N^3(t)$. For h small enough such that $\tau_n \leq t < t+h < \tau_{n+1}$, by (17) and (18), we get that

$$d\lambda^i(t) = d\xi_t^i = \beta_i \left(\sum_{j=1,2} \delta_{ji} N^j(t) - \xi_t^i \right) = \beta_i \left(\sum_{j=1,2} \delta_{ji} N^j(t) - \lambda_t^i + \lambda_0^i \right), \quad i = 1, 2,$$

and

$$d\lambda^3(t) = d\xi_t^3 = \beta_3 (\delta_3 N^3(t) - \xi_t^3) = \beta_3 (\delta_3 N^3(t) - \lambda^3(t) + \lambda_0^3).$$

By (15) and (16), we get the claim by noting that, for $i = 1, 2$,

$$d\lambda^i(\tau_n) = d\xi_{\tau_n}^i = \sum_{j=1,2} \alpha_{ji} dN^j(\tau_n), \quad \text{and} \quad d\lambda^3(\tau_n) = d\xi_{\tau_n}^3 = \alpha_3 dN^3(\tau_n). \quad \square$$

Theorem 5.9. Under Definition 5.1, let $\zeta := (n_1, l_1, n_2, l_2, n_3, l_3)$. Let μ_i be the transition probability if an increment for N^i , $i = 1, 2, 3$, takes place. For a suitable choice of $F(\zeta)$ (differentiable with respect to l_i), a representation for the generator $\mathcal{G}$ of the process $(N^1, \lambda^1, N^2, \lambda^2, N^3, \lambda^3)$ is given by

$$\begin{aligned} \mathcal{G}F(\zeta) = & \sum_{i=1,2} \left\{ \mathcal{G}^i F(\zeta) + l_i [F(\zeta + \Delta_i \zeta) - F(\zeta)] \mu_i(\zeta) \right\} \\ & + \mathcal{G}^3 F(\zeta) + l_3 \mathbf{1}_{n_1+n_2-n_3>0} [F(\zeta + \Delta_3 \zeta) - F(\zeta)] \mu_3(\zeta), \end{aligned} \quad (24)$$

where, for $i = 1, 2$,

$$\mathcal{G}^i F(\zeta) = \beta_i \left(\sum_{j=1,2} \delta_{ji} n_j - l_i + \lambda_0^i \right) \frac{\partial F(\zeta)}{\partial l_i}, \quad \mathcal{G}^3 F(\zeta) = \beta_3 (\delta_3 n_3 - l_3 + \lambda_0^3) \frac{\partial F(\zeta)}{\partial l_3},$$

and

$$\Delta_1 \zeta = (1, \alpha_{11}, 0, \alpha_{12}, 0, 0), \Delta_2 \zeta = (0, \alpha_{21}, 1, \alpha_{22}, 0, 0), \Delta_3 \zeta = (0, 0, 0, 0, 1, \alpha_3). \quad (25)$$

Hence, $(N^1, \lambda^1, N^2, \lambda^2, N^3, \lambda^3)$ is a Markov process which solves the Martingale problem for $\mathcal{G}$, $MgP(\mathcal{G})$.

The proof follows by standard procedure, see for instance Theorem 7.3 in Ethier and Kurtz [9].

Corollary 5.10. Under Definition 5.1 and the conditions given in Remark 5.7, the representation for the generator $\mathcal{G}$ of the process $(N^1, \lambda^1, N^2, \lambda^2, N^3, \lambda^3)$ given in Theorem 5.9 reduces to (24), where $\mathcal{G}^i F(\zeta) = \beta_i (\lambda_0^i - l^i) \frac{\partial F(\zeta)}{\partial l^i}$, $i = 1, 2, 3$, and the sizes of the jumps are given by (25). Hence, again, $(N^1, \lambda^1, N^2, \lambda^2, N^3, \lambda^3)$ is a Markov process which solves the Martingale problem for $\mathcal{G}$, $MgP(\mathcal{G})$.

6. Stability

Recall that, the population model introduced in Section 2 is a simple queueing process $N(t)$, and its size is given by $N(t) = N^1(t) + N^2(t) - N^3(t)$, a process having intensity $\lambda^1(t) + \lambda^2(t) - \lambda^3(t)$, and taking non-negative integer values. This process $\{N(t) : t \geq 0\}$ is the continuous time interpolation of the discrete time process $\{N(\tau_n) : n \geq 0\}$. Before giving the proof of Theorem 8.1, some preliminaries are needed.

Lemma 6.1. Under Definition 5.1, which implies the Markov property of the system, and taking into account the observations of Remark 5.7, we get $\varphi_{ij}(t) = \alpha_{ij} e^{-\beta_i t}$, for $i, j = 1, 2$, and $\psi(t) = \alpha_3 e^{-\beta_3 t}$. For $i = 1, 2$ and $i \neq j$, setting

$$A(i, j) = \frac{\lambda_0^i}{\beta_i} \cdot \frac{\alpha_{ij} \alpha_{ji} - \alpha_{ii} \alpha_{jj}}{\beta_i - \beta_j} - \frac{\lambda_0^i}{\beta_i} \alpha_{ii} - \frac{\lambda_0^j}{\beta_i} \alpha_{ji}, \quad B(i, j) = \frac{\lambda_0^i}{\beta_j} \cdot \frac{\alpha_{ii} \alpha_{jj} - \alpha_{ij} \alpha_{ji}}{\beta_i - \beta_j}, \quad (26)$$

and

$$C(i, j) = \lambda_0^i - A(i, j) - B(i, j),$$

by Theorem 3.3 and Theorem 3.5, the expected number of births, the expected number of deaths and the corresponding expected number of intensities reduce to

$$\mathbb{E}[\lambda^i(t)] = C(i, j) + A(i, j)e^{-\beta_i t} + B(i, j)e^{-\beta_j t}, \quad (27)$$

$$\begin{aligned} \mathbb{E}[N^i(t)] &= \int_0^t \mathbb{E}[\lambda^i(s)] ds = C(i, j)t + A(i, j)\frac{1 - e^{-\beta_i t}}{\beta_i} + B(i, j)\frac{1 - e^{-\beta_j t}}{\beta_j}, \\ \mathbb{E}[\lambda^3(t)] &= \lambda_0^3 + \lambda_0^3(1 - e^{-\beta_3 t})\frac{\alpha_3}{\beta_3}, \\ \mathbb{E}[N^3(t)] &= \lambda_0^3 \left(1 + \frac{\alpha_3}{\beta_3}\right)t + \lambda_0^3(e^{-\beta_3 t} - 1)\frac{\alpha_3}{(\beta_3)^2}. \end{aligned} \quad (28)$$

Proof. Recalling that, in the general case,

$$\begin{aligned} \mathbb{E}[\lambda^i(t)] &= \lambda_0^i \int_0^t \int_0^s [\varphi_{ij}(s-u)\varphi_{ji}(u) - \varphi_{ii}(s-u)\varphi_{jj}(u)] du ds \\ &\quad + \lambda_0^i + \lambda_0^i \int_0^t \varphi_{ii}(s) ds + \lambda_0^j \int_0^t \varphi_{ji}(s) ds, \end{aligned}$$

substituting, successively we get that

$$\varphi_{ij}(s-u)\varphi_{ji}(u) - \varphi_{ii}(s-u)\varphi_{jj}(u) = \alpha_{ij}\alpha_{ji}e^{-\beta_j s}e^{(\beta_j - \beta_i)u} - \alpha_{ii}\alpha_{jj}e^{-\beta_i s}e^{(\beta_i - \beta_j)u},$$

and

$$\begin{aligned} &\int_0^s [\varphi_{ij}(s-u)\varphi_{ji}(u) - \varphi_{ii}(s-u)\varphi_{jj}(u)] du \\ &= \alpha_{ij}\alpha_{ji}e^{-\beta_j s}\frac{e^{(\beta_j - \beta_i)s} - 1}{\beta_j - \beta_i} - \alpha_{ii}\alpha_{jj}e^{-\beta_i s}\frac{e^{(\beta_i - \beta_j)s} - 1}{\beta_i - \beta_j} \\ &= \frac{\alpha_{ij}\alpha_{ji} - \alpha_{ii}\alpha_{jj}}{\beta_j - \beta_i}e^{-\beta_i s} + \frac{\alpha_{ii}\alpha_{jj} - \alpha_{ij}\alpha_{ji}}{\beta_j - \beta_i}e^{-\beta_j s}. \end{aligned}$$

Therefore,

$$\begin{aligned} \mathbb{E}[\lambda^i(t)] &= \lambda_0^i \int_0^t \left[\frac{\alpha_{ij}\alpha_{ji} - \alpha_{ii}\alpha_{jj}}{\beta_j - \beta_i}e^{-\beta_i s} + \frac{\alpha_{ii}\alpha_{jj} - \alpha_{ij}\alpha_{ji}}{\beta_j - \beta_i}e^{-\beta_j s} \right] ds \\ &\quad + \lambda_0^i + \lambda_0^i \int_0^t \alpha_{ii}e^{-\beta_i s} ds + \lambda_0^j \int_0^t \alpha_{ji}e^{-\beta_j s} ds \\ &= \lambda_0^i + \left(\lambda_0^i \left(\alpha_{ii} + \frac{\alpha_{ij}\alpha_{ji} - \alpha_{ii}\alpha_{jj}}{\beta_j - \beta_i} \right) + \lambda_0^j \alpha_{ji} \right) \frac{1 - e^{-\beta_i t}}{\beta_i} + \lambda_0^i \frac{\alpha_{ii}\alpha_{jj} - \alpha_{ij}\alpha_{ji}}{\beta_j - \beta_i} \frac{1 - e^{-\beta_j t}}{\beta_j}. \quad \square \end{aligned}$$

Note that in the Poisson case ($\alpha_{ij} = 0$ and $\alpha_3 = 0$), $\mathbb{E}[\lambda^i(t)] = \lambda_0^i$, for $i = 1, 2, 3$.

Remark 6.2. The stability of the system (see Remark 3.6) implies that

$$\mathbb{E}[\lambda^3(t)] > \mathbb{E}[\lambda^1(t)] + \mathbb{E}[\lambda^2(t)],$$

and, in the general case, this condition is given by (12). In the Markov case, under the hypothesis of Lemma 6.1, and under (22) ($\delta_{ij} = 0$, $i, j = 1, 2$, and $\delta_3 = 0$), the stability condition requires that

$$\lambda_0^3 + \lambda_0^3(1 - e^{-\beta_3 t}) \frac{\alpha_3}{\beta_3} > C(1, 2) + C(2, 1) + [A(1, 2) + B(2, 1)]e^{-\beta_1 t} + [A(2, 1) + B(1, 2)]e^{-\beta_2 t},$$

where

$$\begin{aligned} A(1, 2) + B(2, 1) &= \frac{\lambda_0^1}{\beta_1} \cdot \frac{\alpha_{12}\alpha_{21} - \alpha_{11}\alpha_{22}}{\beta_1 - \beta_2} - \frac{\lambda_0^1}{\beta_1}\alpha_{11} - \frac{\lambda_0^2}{\beta_1}\alpha_{21} + \frac{\lambda_0^2}{\beta_1} \cdot \frac{\alpha_{22}\alpha_{11} - \alpha_{21}\alpha_{12}}{\beta_2 - \beta_1} \\ &= \frac{(\lambda_0^1 + \lambda_0^2)(\alpha_{12}\alpha_{21} - \alpha_{11}\alpha_{22})}{\beta_1(\beta_1 - \beta_2)} - \frac{\lambda_0^1\alpha_{11}}{\beta_1} - \frac{\lambda_0^2\alpha_{21}}{\beta_1}, \\ A(2, 1) + B(1, 2) &= \frac{(\lambda_0^1 + \lambda_0^2)(\alpha_{12}\alpha_{21} - \alpha_{11}\alpha_{22})}{\beta_2(\beta_2 - \beta_1)} - \frac{\lambda_0^2\alpha_{22}}{\beta_2} - \frac{\lambda_0^1\alpha_{12}}{\beta_2}, \\ C(1, 2) + C(2, 1) &= \lambda_0^1 - A(1, 2) - B(1, 2) + \lambda_0^2 - A(2, 1) - B(2, 1) \\ &= \lambda_0^1 + \lambda_0^2 - \frac{(\lambda_0^1 + \lambda_0^2)(\alpha_{12}\alpha_{21} - \alpha_{11}\alpha_{22})}{\beta_1(\beta_1 - \beta_2)} + \frac{\lambda_0^1\alpha_{11} + \lambda_0^2\alpha_{21}}{\beta_1} \\ &\quad - \frac{(\lambda_0^1 + \lambda_0^2)(\alpha_{12}\alpha_{21} - \alpha_{11}\alpha_{22})}{\beta_2(\beta_2 - \beta_1)} + \frac{\lambda_0^2\alpha_{22} + \lambda_0^1\alpha_{12}}{\beta_2}. \end{aligned}$$

7. A virus-like evolving population

Recall that $\{\tau_n : n \geq 0\}$ is the set of jump times of (N^1, N^2, N^3) , obtained by merging the jump times of each N^i , $i = 1, 2, 3$, where N^1 counts the number of births of the mutants, N^2 counts the number of births of the non-mutants, and N^3 counts the number of deaths. At $\tau_0 = 0$, there is no individual at site 0. Also, at τ_1 , there is necessarily a birth of a mutant. At the jump time τ_n , $n \geq 2$, there is either a birth of a mutant, a birth of a non-mutant or a death of an individual from the existing population.

In case of a birth, there are two possibilities: a *mutant* is born, which means that there exists a value k such that $\tau_n = t_k^1$, namely the n -th jump time of N corresponds to the k -th birth time of a mutant. Its fitness parameter f is chosen uniformly at random from $[0, 1]$. Otherwise, a *non mutant* is born, which means that there exists a value k such that $\tau_n = t_k^2$, namely the n -th jump time of N corresponds to the k -th birth time of a non mutant individual. Its fitness parameter f is chosen with a probability proportional to the number of individuals with fitness f among the entire population present at that time. Here we have a caveat that, if there is no individual present at the time of birth, then the fitness of the individual is sampled uniformly in $[0, 1]$.

In case of a death, an individual from the population at the site closest to 0 is eliminated.

For a mutant, the fitness parameter cannot match, almost surely, any of the fitness parameters already existing in the population. Here and henceforth, a site represents a fitness level. The total population at time t is divided in exactly l_t sites $x_1, \dots, x_{l_t}$, with the size of the population at site x_i being exactly k_t^i . To describe the partition of the population at time t , let $X_t = \{(k_t^i, x_i) : k_t^i \geq 1, x_i \in [0, 1], i = 1, \dots, l_t\}$. The couple (k_t^i, x_i) belongs to the partition X_t only if $k_t^i \geq 0$. For example in Fig. 1, there is an empty circle corresponding to the fitness parameter x_5 , which means that all the individuals belonging to that specie have died by time t , therefore the couple $(0, x_5) \notin X_t$. In case there is no individual present at time t in the entire population, we take $X_t = \emptyset$. Hence, X_t takes values on the state space $\mathbb{X} = \{\emptyset\} \cup \{(k, x)_{x \in \Lambda} : (k, x) \in \mathbb{N} \times [0, 1], \# \Lambda < \infty\}$.

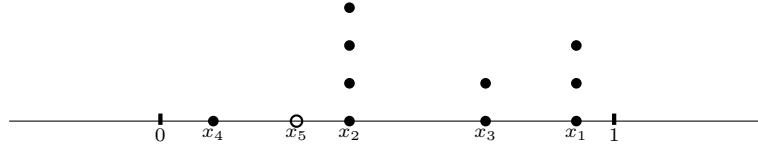


Fig. 1. A partition of the population at time t is $X_t = \{(1, x_4), (4, x_2), (2, x_3), (3, x_1)\}$.

For a given constant $f \in (0, 1)$, at time t , let $L^f(t)$ be the size of the population with fitness smaller than f , and let $R^f(t)$ denote the size of the population with fitness larger than f , respectively having the representations given by

$$L^f(t) := \sum_{i=1}^{l_t} k_t^i \mathbf{1}_{x_i \in [0, f]} \mathbf{1}_{(k_t^i, x_i) \in X_t}, \quad R^f(t) := \sum_{i=1}^{l_t} k_t^i \mathbf{1}_{x_i \in (f, 1]} \mathbf{1}_{(k_t^i, x_i) \in X_t}. \quad (29)$$

Clearly, $L^f(t) + R^f(t) := N(t)$ almost surely, where $N(t)$ denotes the size of the entire population at time t . The couple $(L^f(t), R^f(t))$ takes values on $\mathbb{Z}^+ \times \mathbb{Z}^+$, ($\mathbb{Z}^+ = \mathbb{N} \cup \{0\}$), and its dynamics is driven by $(N^1, \lambda^1, N^2, \lambda^2, N^3, \lambda^3)$. Hence, the model given by $(L^f(t), R^f(t))$ admits jumps at times $\{\tau_n\}_{n>0}$, while, between the jump times, it remains constant, i.e. $(L^f(t), R^f(t)) = (L^f(\tau_n), R^f(\tau_n))$, for $\tau_n \leq t < \tau_{n+1}$.

Assuming that the structure of the Hawkes processes $(N^1, \lambda^1, N^2, \lambda^2, N^3, \lambda^3)$ is such that the model $(L^f(t), R^f(t))$ satisfies the Markov property, (see Section 5), at each jump time the transition probabilities are given by

1. If $(L^f(t), R^f(t)) = (0, 0)$

$$(L^f(t+h), R^f(t+h)) = \begin{cases} (1, 0) & \text{w.p. } f\lambda^1(t)h + o(h) \\ (0, 1) & \text{w.p. } (1-f)\lambda^1(t)h + o(h) \\ (0, 0) & \text{w.p. } o(h) \end{cases}$$

2. If $(L^f(t), R^f(t)) \in \{0\} \times \mathbb{N}$

$$(L^f(t+h), R^f(t+h)) = \begin{cases} (1, R^f(t)) & \text{w.p. } f\lambda^1(t)h + o(h) \\ (0, R^f(t) + 1) & \text{w.p. } [(1-f)\lambda^1(t) + \lambda^2(t)]h + o(h) \\ (0, R^f(t) - 1) & \text{w.p. } \lambda^3(t)h + o(h) \end{cases}$$

3. If $(L^f(t), R^f(t)) \in \mathbb{N} \times \{0\}$

$$(L^f(t+h), R^f(t+h)) = \begin{cases} (L^f(t) + 1, 0) & \text{w.p. } [f\lambda^1(t) + \lambda^2(t)]h + o(h) \\ (L^f(t), 1) & \text{w.p. } (1-f)\lambda^1(t)h + o(h) \\ (L^f(t) - 1, 0) & \text{w.p. } \lambda^3(t)h + o(h) \end{cases}$$

4. If $(L^f(t), R^f(t)) \in \mathbb{N} \times \mathbb{N}$

$$(L^f(t+h), R^f(t+h)) = \begin{cases} (L^f(t) + 1, R^f(t)) & \text{w.p. } \left(f\lambda^1(t) + \lambda^2(t) \frac{L^f(t)}{N(t)}\right)h + o(h) \\ (L^f(t), R^f(t) + 1) & \text{w.p. } \left((1-f)\lambda^1(t) + \lambda^2(t) \frac{R^f(t)}{N(t)}\right)h + o(h) \\ (L^f(t) - 1, R^f(t)) & \text{w.p. } \lambda^3(t)h + o(h). \end{cases}$$

8. Convergence results for a virus-like evolving population

Some convergence results for the model introduced in Section 7 are derived below.

Theorem 8.1. *Assuming that*

$$f_c := \lim_{t \rightarrow \infty} f_c(t) := \lim_{t \rightarrow \infty} \frac{\mathbb{E}[\lambda^3(t)]}{\mathbb{E}[\lambda^1(t)]} \quad (30)$$

exists, there is a phase transition at a critical fitness level f_c , in the sense that

1. *In case $\liminf_{t \rightarrow +\infty} \frac{\mathbb{E}[\lambda^3(t)]}{\mathbb{E}[\lambda^1(t) + \lambda^2(t)]} \geq 1$, the probability that a birth happens is less likely than a death, then $P\left(\liminf_{\tau \rightarrow +\infty} \frac{l\{t \in [0, \tau] : N(t) = 0\}}{\tau} > 0\right) = 1$, where l is the Lebesgue measure.*
2. *In case $f_c \leq 1$ (*), $N(t)$ goes to infinity as $t \rightarrow \infty$, and most of the population is distributed at sites in the interval $[f_c, 1]$, in the sense that, for $f > f_c$,*

$$P\left(\lim_{t \rightarrow \infty} \frac{R^{f_c}(t)}{N(t)} = 1\right) = 1 \quad \text{and} \quad P\left(\liminf_{t \rightarrow \infty} \frac{R^{f_c}(t) - R^f(t)}{N(t)} > 0\right) = 1.$$

3. *In case $\limsup_{t \rightarrow +\infty} \frac{\mathbb{E}[\lambda^3(t)]}{\mathbb{E}[\lambda^1(t) + \lambda^2(t)]} < 1$, but $f_c \geq 1$, $N(t)$ goes to infinity as $t \rightarrow \infty$, and most of the population is concentrated at site near 1, in the sense that*

$$P\left(\lim_{t \rightarrow \infty} N(t) = \infty\right) = 1 \quad \text{and} \quad P\left(\lim_{t \rightarrow \infty} \frac{R^{1-\epsilon}(t)}{N(t)} = 1\right) = 1, \quad \text{for } \epsilon > 0.$$

First we need some remarks and then some preliminaries in Lemma 8.5 and Lemma 8.7 below, successively, we are able to prove Theorem 8.1.

Remark 8.2. Recall that in the model developed by Roy and Tanemura [17], $f_c := \frac{1-p}{pr}$, and a birth or a death takes place at a discrete time point with probability p or $1-p$ respectively; an individual is either a mutant with probability $p \cdot r$ or an individual of one of the existing subspecies with probability $p(1-r)$.

In the present paper, $f_c := \lim_{t \rightarrow +\infty} \frac{\mathbb{E}[\lambda^3(t)]}{\mathbb{E}[\lambda^1(t)]}$, and a birth or a death takes place at continuous time, following a Hawkes process with intensity $\lambda^1(t) + \lambda^2(t)$ or $\lambda^3(t)$, respectively; at birth an individual is either a mutant with intensity $\lambda^1(t)$ or an individual of one of the existing subspecies with intensity $\lambda^2(t)$.

Remark 8.3. When the mean birth rate, $\mathbb{E}[\lambda^1(t) + \lambda^2(t)]$, is less than the mean death rate, $\mathbb{E}[\lambda^3(t)]$, this means that $\lim_{t \rightarrow \infty} \frac{\mathbb{E}[\lambda^3(t)]}{\mathbb{E}[\lambda^1(t) + \lambda^2(t)]} > 1$, then for n big enough, $\{N(\tau_n) : n \geq 0\}$ is equivalent to a random walk on $\mathbb{Z}^+$ with non-positive drift and reflection at 0. Thus, $N(\tau_n)$ returns to 0 infinitely often with probability 1.

When the mean birth rate, $\mathbb{E}[\lambda^1(t) + \lambda^2(t)]$, is greater than the mean death rate, $\mathbb{E}[\lambda^3(t)]$, $\{N(\tau_n) : n \geq 0\}$ is equivalent to a random walk on $\mathbb{Z}^+$ with positive drift and, $N(\tau_n) \rightarrow \infty$ as $n \rightarrow \infty$ with probability 1.

Then, we have to study the situation in which $\mathbb{E}[\lambda^1(t) + \lambda^2(t)] > \mathbb{E}[\lambda^3(t)]$.

Corollary 8.4. Let $F_t(f) := \frac{\#\{s \in [0, f] : (k, s) \in X_t, \text{ for some } k \geq 1\}}{\#\{s \in [0, 1] : (k, s) \in X_t, \text{ for some } k \geq 1\}}$ be the empirical distribution of sites at t . If $f_c < 1$, $F_t(f) \rightarrow \frac{\max\{f - f_c, 0\}}{1 - f_c}$ uniformly a.s.

¹ (*) Since $\mathbb{E}[\lambda^2(t)] \geq 0$, the condition $f_c := \lim_{t \rightarrow \infty} \frac{\mathbb{E}[\lambda^3(t)]}{\mathbb{E}[\lambda^1(t)]} \leq 1$, obviously, implies that $\limsup_{t \rightarrow \infty} \frac{\mathbb{E}[\lambda^3(t)]}{\mathbb{E}[\lambda^1(t) + \lambda^2(t)]} \leq 1$.

When the processes attain a wide-sense stationary behaviours, which implies the conditions given in (22) and (23), f_c assumes a closed form expression as given in (31) below.

Lemma 8.5. *Recalling (26), (27) and (28), and assuming the existence of f_c , as defined in (30), under the conditions (22), Remark 5.7, namely, $\delta_{ij} = 0$, $i, j = 1, 2$, and $\delta_3 = 0$, Equation (30) turns to be*

$$\begin{aligned} f_c &:= \lim_{t \rightarrow \infty} \frac{\mathbb{E}[\lambda^3(t)]}{\mathbb{E}[\lambda^1(t)]} \\ &= \lim_{t \rightarrow \infty} \frac{\lambda_0^3 + \lambda_0^3(1 - e^{-\beta_3 t}) \frac{\alpha_3}{\beta_3}}{C(1, 2) + A(1, 2)e^{-\beta_1 t} + B(1, 2)e^{-\beta_2 t}} \\ &= \frac{\lambda_0^3(\alpha_3 + \beta_3)\beta_1\beta_2}{[\lambda_0^1\beta_1\beta_2 - \lambda_0^1(\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}) + \beta_2(\lambda_0^1\alpha_{11} + \lambda_0^2\alpha_{21})]\beta_3}. \end{aligned} \quad (31)$$

Furthermore, assuming also the conditions (23) set in Remark 5.7, namely $\alpha_{ji} \leq \beta_i$, $i, j = 1, 2$, and $\alpha_3 \leq \beta_3$, we have that $\frac{\lambda_0^3}{2\lambda_0^1 + \lambda_0^2} \leq f_c \leq 2\frac{\lambda_0^3}{\lambda_0^1}$.

Remark 8.6. As a special case, assuming that $\alpha_{ij} = \alpha_3 = \alpha$, constant, and $\beta_i = \beta$, for all i, j , then $f_c = \lambda_0^3 \left(\lambda_0^1 + \lambda_0^2 \frac{\alpha}{\alpha + \beta} \right)^{-1} \leq \frac{\lambda_0^3}{\lambda_0^1}$.

Lemma 8.7. *Under all the assumptions made so far, by the markovianity of the model, taking into account that f_c is given by (31), we get that*

1. If $f_c < 1$, we have to consider two situations

(a) For $f < f_c$ and for any $\eta \in (0, 1)$, then

$$P \left(\exists T > 0 : \rho_t^f := \frac{L^f(t)}{N(t)} \leq \eta, \forall t \geq T \right) = 1, \quad \text{and} \quad (32)$$

$$P(L^f(\tau_n) = 0 \text{ infinitely often}) = 1. \quad (33)$$

(b) For $f > f_c$, then

$$P(L^f(\tau_n) = 0 \text{ infinitely often}) = 0. \quad (34)$$

2. If $f_c \geq 1$, then

(a) For $f < 1$ and for $\eta \in (0, 1)$, since $f \in (0, 1)$, we have (32) and (33),

(b) For $f = 1$ we have (34).

Proof. The idea of the proof is to prove 1. and 2. together. To this end, note that choosing $f < f_c \wedge 1$, we have that $R^f(t)$ is much larger than $L^f(t)$. Hence, the main tool is to study a Markov chain, which is a modification of that given in Section 7 and constructed as follows. This is a procedure which is along the lines of that used in Roy and Tanemura [17] in the discrete case. Given $\epsilon \in [0, 1]$, we construct a modified Markov chain $(L_t^f(\epsilon), R_t^f(\epsilon))$, which admits jumps at times $\{\tau_n\}_{n>0}$, while, between the jump times, remains constant, i.e. $(L_t^f(\epsilon), R_t^f(\epsilon)) = (L_{\tau_n}^f(\epsilon), R_{\tau_n}^f(\epsilon))$ for $\tau_n \leq t < \tau_{n+1}$, and such that at each jump time the stationary transition probabilities are given by

1. If $(L_t^f(\epsilon), R_t^f(\epsilon)) = (0, 0)$

$$(L_{t+h}^f(\epsilon), R_{t+h}^f(\epsilon)) = \begin{cases} (1, 0) & \text{w.p. } f(\lambda^1(t) + \lambda^2(t))h + o(h) \\ (0, 1) & \text{w.p. } (1-f)(\lambda^1(t) + \lambda^2(t))h + o(h) \\ (0, 0) & \text{w.p. } o(h) \end{cases}$$

2. If $(L_t^f(\epsilon), R_t^f(\epsilon)) \in \{0\} \times \mathbb{N}$

$$(L_{t+h}^f(\epsilon), R_{t+h}^f(\epsilon)) = \begin{cases} (1, R_t^f(\epsilon)) & \text{w.p. } f\lambda^1(t)h + o(h) \\ (0, R_t^f(\epsilon) + 1) & \text{w.p. } ((1-f)\lambda^1(t) + \lambda^2(t))h + o(h) \\ (0, R_t^f(\epsilon) - 1) & \text{w.p. } \lambda^3(t)h + o(h) \end{cases}$$

3. If $(L_t^f(\epsilon), R_t^f(\epsilon)) \in \mathbb{N} \times \{0\}$

$$(L_{t+h}^f(\epsilon), R_{t+h}^f(\epsilon)) = \begin{cases} (L_t^f(\epsilon) + 1, 0) & \text{w.p. } (f\lambda^1(t) + \lambda^2(t))h + o(h) \\ (L_t^f(\epsilon), 1) & \text{w.p. } (1-f)\lambda^1(t)h + o(h) \\ (L_t^f(\epsilon) - 1, 0) & \text{w.p. } \lambda^3(t)h + o(h) \end{cases}$$

4. If $(L_t^f(\epsilon), R_t^f(\epsilon)) \in \mathbb{N} \times \mathbb{N}$

$$(L_{t+h}^f(\epsilon), R_{t+h}^f(\epsilon)) = \begin{cases} (L_t^f(\epsilon) + 1, R_t^f(\epsilon)) & \text{w.p. } (f\lambda^1(t) + \epsilon\lambda^2(t))h + o(h) \\ (L_t^f(\epsilon), R_t^f(\epsilon) + 1) & \text{w.p. } ((1-f)\lambda^1(t) + (1-\epsilon)\lambda^2(t))h + o(h) \\ (L_t^f(\epsilon) - 1, R_t^f(\epsilon)) & \text{w.p. } \lambda^3(t)h + o(h). \end{cases}$$

The difference between this Markov chain and that given in Section 7 is in the definition of the step 4, since we replace $\frac{L^f(t)}{N(t)}$ with ϵ , and $\frac{R^f(t)}{N(t)} = \frac{N(t) - L^f(t)}{N(t)} = 1 - \frac{L^f(t)}{N(t)}$ with $1 - \epsilon$. For $\epsilon, \epsilon' \in [0, 1]$, we have that

$$L_t^f(\epsilon) \leq L_t^f(\epsilon') \quad \text{and} \quad R_t^f(\epsilon) \leq R_t^f(\epsilon') \quad \text{for } \epsilon \leq \epsilon' \quad \text{and} \quad t \geq 0. \quad (35)$$

Given $L^f(t), R^f(t)$ and $N(t)$ as in Section 7, Equations (29), $N(t) = L^f(t) + R^f(t)$. Taking $L_t^f(\epsilon), R_t^f(\epsilon)$ and $N_t(\epsilon)$ as above, and choosing $\rho_t^f := \frac{L^f(t)}{N(t)}$, we get

$$\begin{aligned} L^f(t+h) &= L_{t+h}^f(\rho_t^f), & R^f(t+h) &= R_{t+h}^f(\rho_t^f), \\ N_t(\epsilon) &:= L_t^f(\epsilon) + R_t^f(\epsilon) = N(t), & L_t^f(0) &\leq L_t^f \leq L_t^f(1), & R_t^f(1) &\leq R_t^f \leq R_t^f(0). \end{aligned} \quad (36)$$

Setting, as usual, $\Delta L_t^f(\epsilon) = L_t^f(\epsilon) - L_{t-}^f(\epsilon)$ and $\Delta N_t(\epsilon) = N_t(\epsilon) - N_{t-}(\epsilon)$, and recalling that the number of jumps of the chain until time t is $N_t^1(\epsilon) + N_t^2(\epsilon) + N_t^3(\epsilon)$, by the Law of Large Numbers, we get that

$$\begin{aligned} &\lim_{t \rightarrow \infty} \frac{L_t^f(\epsilon)}{N_t^1(\epsilon) + N_t^2(\epsilon) + N_t^3(\epsilon)} = \lim_{t \rightarrow \infty} \mathbb{E} [\Delta L_t^f(\epsilon)] \\ &= f\lambda_0^1 + \epsilon\lambda_0^2 - \lambda_0^3 + f \left(\frac{\lambda_0^1 \alpha_{11}}{\beta_1} + \frac{\lambda_0^2 \alpha_{21}}{\beta_1} \right) + \epsilon \left(\frac{\lambda_0^2 \alpha_{22}}{\beta_2} + \frac{\lambda_0^1 \alpha_{12}}{\beta_2} \right) - \lambda_0^3 \frac{\alpha_3}{\beta_3} - (f\lambda_0^1 + \epsilon\lambda_0^2) \frac{\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}}{\beta_1\beta_2}, \end{aligned}$$

and

$$\begin{aligned} & \lim_{t \rightarrow \infty} \frac{N_t(\epsilon)}{N_t^1(\epsilon) + N_t^2(\epsilon) + N_t^3(\epsilon)} = \lim_{t \rightarrow \infty} \mathbb{E} [\Delta N_t(\epsilon)] \\ &= \lambda_0^1 + \lambda_0^2 - \lambda_0^3 + \left(\frac{\lambda_0^1 \alpha_{11}}{\beta_1} + \frac{\lambda_0^2 \alpha_{21}}{\beta_1} \right) + \left(\frac{\lambda_0^2 \alpha_{22}}{\beta_2} + \frac{\lambda_0^1 \alpha_{12}}{\beta_2} \right) - \lambda_0^3 \frac{\alpha_3}{\beta_3} - (\lambda_0^1 + \lambda_0^2) \frac{\alpha_{11} \alpha_{22} - \alpha_{12} \alpha_{21}}{\beta_1 \beta_2}, \end{aligned}$$

almost surely. Thus, setting $\Lambda_i := \lim_{t \rightarrow \infty} \mathbb{E}[\lambda^i(t)]$, for $i = 1, 2, 3$, and $\rho_t^f(\epsilon) := \frac{L_t^f(\epsilon)}{N_t(\epsilon)}$,

$$\lim_{t \rightarrow \infty} \rho_t^f(\epsilon) := \lim_{t \rightarrow \infty} \frac{L_t^f(\epsilon)}{N_t(\epsilon)} = \lim_{t \rightarrow \infty} \frac{f \mathbb{E}[\lambda^1(t)] + \epsilon \mathbb{E}[\lambda^2(t)] - \mathbb{E}[\lambda^3(t)]}{\mathbb{E}[\lambda^1(t)] + \mathbb{E}[\lambda^2(t)] - \mathbb{E}[\lambda^3(t)]} = \frac{f \Lambda_1 - \Lambda_3}{\Lambda_1 + \Lambda_2 - \Lambda_3} + \frac{\epsilon \Lambda_2}{\Lambda_1 + \Lambda_2 - \Lambda_3}. \quad (37)$$

Let us introduce the linear function $h(x) := \frac{f \Lambda_1 - \Lambda_3}{\Lambda_1 + \Lambda_2 - \Lambda_3} + \frac{\Lambda_2}{\Lambda_1 + \Lambda_2 - \Lambda_3} x$. Since, for $i, j = 1, 2$, and $j \neq i$, $\Lambda_i = \lambda_0^i + \left(\frac{\lambda_0^i \alpha_{ii}}{\beta_i} + \frac{\lambda_0^j \alpha_{ji}}{\beta_j} \right) - \lambda_0^i \frac{\alpha_{ii} \alpha_{jj} - \alpha_{ij} \alpha_{ji}}{\beta_i \beta_j}$, and $\Lambda_3 = \lambda_0^3 + \lambda_0^3 \frac{\alpha_3}{\beta_3}$, without loss of generality, for the present model we assume that $\Lambda_i > 0$, for $i = 1, 2, 3$, and by the stability condition in the Markov case, we have that $\Lambda_1 + \Lambda_2 - \Lambda_3 > 0$. Hence $h(x)$ is an increasing line, and recalling that $f \in (0, 1)$, for $\Lambda_1 < \Lambda_3$,

$$h(0) = \frac{f \Lambda_1 - \Lambda_3}{\Lambda_1 + \Lambda_2 - \Lambda_3} < 0, \quad h(1) := \frac{f \Lambda_1 - \Lambda_3}{\Lambda_1 + \Lambda_2 - \Lambda_3} + \frac{\Lambda_2}{\Lambda_1 + \Lambda_2 - \Lambda_3} < 1.$$

Hence, there exists a real $\delta > 0$ such that $h(x + \delta) < x$, for all $x \in [0, 1]$. Setting $H(\epsilon, \delta) := \left\{ \omega : \exists T = T(\omega) : \forall t \geq T, \rho_t^f(\epsilon) < h(\epsilon + \delta) \right\}$, by (37), for all $\epsilon, \delta \in (0, 1]$, $\lim_{t \rightarrow \infty} \rho_t^f(\epsilon) = h(\epsilon) < h(\epsilon + \delta)$, which implies that

$$P(H(\epsilon, \delta)) = 1. \quad (38)$$

Thus, for $\epsilon_c > 0$, such that $h(\epsilon_c + \delta) = \eta$, since $h(x)$ is an increasing line, for $\epsilon < \epsilon_c$, $h(\epsilon + \delta) < h(\epsilon_c + \delta) = \eta$. But the sequence defined by $x_{n+1} := h(x_n + \delta)$, is a decreasing sequence, since $x_{n+1} = h(x_n + \delta) < x_n$, for $x_n \in [0, 1]$. This means that $\lim_{n \rightarrow \infty} x_n = h(0) = \frac{f \Lambda_1 - \Lambda_3}{\Lambda_1 + \Lambda_2 - \Lambda_3} < 0$. Setting $h(k, x + \delta) := h(2^{-k}([2^k x] + 1) + \delta)$, for $k \in \mathbb{N}$, big enough, there exists $n_c \in \mathbb{N}$ such that the sequence

$$h_n = h_n(k, 1 + \delta) := h(k, h_{n-1} + \delta) \leq \eta, \quad \forall n \geq n_c. \quad (39)$$

Taking into account that by (35) and (36), $\rho_t^f(\epsilon) \geq \rho_t^f(\epsilon')$ for $\epsilon > \epsilon'$, and $\rho_t^f \leq \rho_t^f(1)$, for all $\omega \in \cap_{m \in \mathbb{N}} H(m 2^{-k}, \delta)$, there exists $T_1(\omega)$, such that, $\rho_t^f[\omega] \leq \rho_t^f(1)[\omega] \leq h(k, 1 + \delta)$, for all $t \geq T_1(\omega)$, and there exists $T_2(\omega) \geq T_1(\omega)$, such that, for all $t \geq T_2(\omega)$, $\rho_t^f[\omega] \leq \rho_t^f(h(k, 1 + \delta))[\omega] \leq h(k, h(k, 1 + \delta) + \delta) = h_2(k, 1 + \delta)$. Repeating this procedure, we get that, for any $l \in \mathbb{N}$, there exists $T_l(\omega)$ such that $\rho_t^f[\omega] \leq h_l(k, 1 + \delta)$, for all $t \geq T_l(\omega)$. From (39), we have that $\rho_t^f[\omega] \leq \eta$, for all $t \geq T_{n_c}(\omega)$. Since $P(\cap_{m \in \mathbb{N}} H(m 2^{-k}, \delta)) = 1$, from (38), we have that $\lim_{t \rightarrow \infty} \rho_t^f[\omega] \leq \eta$, a.s., and the condition given in (32) is proven. For the condition (33), L_t^f is the continuous time interpolation of $L_{\tau_n}^f$, and that $L_{\tau_n}^f$ is recurrent for

$$f < f_c - \frac{\Lambda_2}{\Lambda_1} \epsilon.$$

Hence, for $f < f_c \wedge 1$, this condition holds for ϵ small enough, which implies that $L_{\tau_n}^f = 0$ infinitely often with probability 1.

To prove (34), let $f_c < 1$, and note that, at t , the number of sites in $[0, f]$, is less than $L_t^f(0)$, and that when $f > f_c$, the number of sites in $[0, f]$ cannot be null with positive probability. This completes the proof of part 1 point b of Lemma 8.7. The proof of part 2 point b of Lemma 8.7 follows since, for $1 \leq f_c < \frac{\Lambda_3}{\Lambda_1}$, the observation given in Remark 6.2 assures that $N_t \rightarrow \infty$ as $t \rightarrow \infty$ with probability 1. $\square$

Summing up, for the proof of Theorem 8.1, first, taking into account the closed form expressions obtained for $\mathbb{E}[\lambda^i(t)]$, $i = 1, 2, 3$, under the Markov property and the wide-sense stationary behaviours, we have that: Theorem 8.1.1 is obtained analogously to queueing theory with mean arrival rates less than mean service rates.

For Theorem 8.1.2, the first statement is derived by Lemma 8.7.1.b and Lemma 8.7.2.b., while the second statement is derived taking into account that, considering $\lambda^1(t)$, the birth rate of mutants, the limiting expected number of them with a fitness between (a, b) , with $f_c < a < b \leq 1$, is $\lambda^1(t)(b - a)$. Thus, by the Strong Law of Large Numbers,

$$\liminf_{t \rightarrow \infty} \frac{R^a(t) - R^b(t)}{N(t)} \geq \frac{b - a}{1 - f_c} \quad \text{a.s..}$$

Theorem 8.1.3 is derived by Lemma 8.7.2. Finally, since the sites are independently and uniformly distributed in $[0, 1]$, Corollary 8.4 follows from Lemma 8.7.

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Appendix A. Proofs

A.1. Proof of Theorem 3.3

Remark 8.8. Recall that, any non-negative $\mathcal{F}_t$ -progressively measurable process $\lambda(t)$ is called $\mathcal{F}_t$ -intensity of a counting process $N(t)$, provided that, for all s, t , with $0 < s < t$, $\mathbb{E}[N(s, t) | \mathcal{F}_s] = \mathbb{E}[N(t) - N(s) | \mathcal{F}_s] = \mathbb{E}\left[\int_s^t \lambda(u) du | \mathcal{F}_s\right]$. This implies that, for $N(0) = 0$, $\mathbb{E}[N(t)] = \mathbb{E}\left[\int_0^t \lambda(u) du\right] = \int_0^t \mathbb{E}[\lambda(u)] du$, and that $d\mathbb{E}[N(t)] = \mathbb{E}[\lambda(t)] dt$, since

$$\begin{aligned} d\mathbb{E}[N(t)] &= \lim_{h \rightarrow 0} \frac{1}{h} \mathbb{E}[N(t+h) - N(t)] dt = \lim_{h \rightarrow 0} \frac{1}{h} \mathbb{E}\left[\int_t^{t+h} \lambda(u) du\right] dt \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \int_t^{t+h} \mathbb{E}[\lambda(u)] du dt = \lim_{h \rightarrow 0} \mathbb{E}[\lambda(t + o(h))] dt = \mathbb{E}[\lambda(t)] dt. \end{aligned}$$

Proof. of Theorem 3.3. Let $y_t^i = \mathbb{E}[\lambda^i(t)]$, with $y_0^i = \mathbb{E}[\lambda^i(0)] = \lambda_0^i$. By the ODE given in (8) we have $(y_t^i)' = \sum_{j=1,2} \left(\int_0^t \varphi'_{ji}(t-u) y_u^j du + \varphi_{ji}(0) y_t^j \right)$. Taking the Laplace Transform on both sides, and setting as usual $Y^i(s) = \mathcal{L}(y_t^i)(s) = \int_0^\infty e^{-st} y_t^i dt$, for the LHS (left hand side) we get that $\mathcal{L}((y_t^i)')(s) = sY^i(s) - y_0^i = sY^i(s) - \lambda_0^i$. For the RHS (right hand side), we have successively

$$\mathcal{L}\left(\sum_{j=1,2} \left(\int_0^t \varphi'_{ji}(t-u) y_u^j du + \varphi_{ji}(0) y_t^j\right)\right)(s)$$

$$= \sum_{j=1,2} [(s\mathcal{L}(\varphi_{ji}(t))(s) - \varphi_{ji}(0))Y^j(s) + \varphi_{ji}(0)Y^j(s)] = s \sum_{j=1,2} \mathcal{L}(\varphi_{ji}(t))(s)Y^j(s).$$

Hence, we get the system

$$\begin{cases} sY^1(s) - \lambda_0^1 = s\mathcal{L}(\varphi_{11}(t))(s)Y^1(s) + s\mathcal{L}(\varphi_{21}(t))(s)Y^2(s) \\ sY^2(s) - \lambda_0^2 = s\mathcal{L}(\varphi_{12}(t))(s)Y^1(s) + s\mathcal{L}(\varphi_{22}(t))(s)Y^2(s) \end{cases}$$

and by Cramer's rule, immediately we get, for $i = 1, 2$, $j = 1, 2$ and $j \neq i$,

$$Y^i(s) = \frac{1}{s} \cdot \frac{\lambda_0^i[1 - \mathcal{L}(\varphi_{jj}(t))(s)] + \lambda_0^j \mathcal{L}(\varphi_{ji}(t))(s)}{[1 - \mathcal{L}(\varphi_{ii}(t))(s)][1 - \mathcal{L}(\varphi_{jj}(t))(s)] - \mathcal{L}(\varphi_{ij}(t))(s)\mathcal{L}(\varphi_{ji}(t))(s)},$$

which implies that

$$\begin{aligned} Y^i(s) \{1 - \mathcal{L}(\varphi_{ii}(t))(s) - \mathcal{L}(\varphi_{jj}(t))(s) + \mathcal{L}(\varphi_{ii}(t))(s)\mathcal{L}(\varphi_{jj}(t))(s) \\ - \mathcal{L}(\varphi_{ij}(t))(s)\mathcal{L}(\varphi_{ji}(t))(s)\} \\ = \lambda_0^i \frac{1}{s} - \lambda_0^i \frac{1}{s} \mathcal{L}(\varphi_{jj}(t))(s) + \lambda_0^j \frac{1}{s} \mathcal{L}(\varphi_{ji}(t))(s). \end{aligned}$$

Recalling that, as well known, $\mathcal{L}(f(t))(s)\mathcal{L}(g(t))(s) = \mathcal{L}\left(\int_0^t f(t-u)g(u) du\right)(s)$, and that $\mathcal{L}(1)(s) = 1/s$, taking the inverse Laplace Transform of LHS, we get

$$\begin{aligned} \mathcal{L}^{-1}(LHS) = y_t^i - \int_0^t y_{t-v}^i [\varphi_{ii}(v) + \varphi_{jj}(v) \\ + \int_0^v [\varphi_{ij}(v-u))\varphi_{ji}(u) - \varphi_{ii}(v-u)\varphi_{jj}(u)] du] dv. \end{aligned}$$

While, taking the inverse Laplace Transform of RHS, we get

$$\mathcal{L}^{-1}(RHS) = \lambda_0^i - \lambda_0^i \int_0^t \varphi_{jj}(v) dv + \lambda_0^j \int_0^t \varphi_{ji}(v) dv.$$

Hence,

$$\begin{aligned} y_t^i = \int_0^t y_{t-v}^i \left[\varphi_{ii}(v) + \varphi_{jj}(v) + \int_0^v [\varphi_{ij}(v-u))\varphi_{ji}(u) - \varphi_{ii}(v-u)\varphi_{jj}(u)] du \right] dv \\ + \lambda_0^i - \lambda_0^i \int_0^t \varphi_{jj}(v) dv + \lambda_0^j \int_0^t \varphi_{ji}(v) dv, \end{aligned}$$

by taking the derivative of both sides and given the initial condition $y_0^i = \lambda_0^i$

$$(y_t^i)' = \lambda_0^i \left[\varphi_{ii}(t) + \int_0^t [\varphi_{ij}(t-u))\varphi_{ji}(u) - \varphi_{ii}(t-u)\varphi_{jj}(u)] du \right] + \lambda_0^j \varphi_{ji}(t).$$

By a standard procedure to solve this Cauchy Problem, we get the required result. $\square$

A.2. Proof of Theorem 3.5

Proof. Let $y_t = \mathbb{E}[\lambda^3(t)]$ and, $y_0 = \mathbb{E}[\lambda^3(0)] = \lambda_0^3$. Then (9) becomes

$$y'_t = \int_0^t \psi'(t-u)y_u du + \psi(0)y_t.$$

Analogously to the proof of Theorem 3.3, taking the Laplace Transform on both sides, and setting as usual $Y(s) = \mathcal{L}(y_t)(s) = \int_0^\infty e^{-st}y_t dt$, for the LHS we get $\mathcal{L}((y_t)')(s) = sY(s) - y_0$, and for the RHS we get

$$\mathcal{L}\left(\int_0^t \psi'(t-u)y_u du + \psi(0)y_t\right)(s) = s\mathcal{L}(\psi(t))(s)Y(s).$$

Therefore, $Y(s)(1 - \mathcal{L}(\psi(t))(s)) = y_0 \frac{1}{s}$ and taking the inverse Laplace Transform, we get that $y_t - \mathcal{L}^{-1}\left(\mathcal{L}\left(\int_0^t y_{t-v}\psi(v) dv\right)\right) = y_0$. Hence, $y_t = \int_0^t y_{t-v}\psi(v) dv + y_0$, and after differentiating both sides, we get $d\mathbb{E}[\lambda^3(t)] = (y_t)' = y_0\psi(t)$. $\square$

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