Markov regenerative credit rating model

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Abstract

Purpose – Credit ratings serve as an important input in several applications in risk management of the financial firms. The level of credit rating changes from time to time because of random credit risk and, thus, can be modeled by an appropriate stochastic process. Markov chain models have been widely used in the literature to generate credit migration matrices; however, emergent empirical evidences suggest that the Markov property is not appropriate for credit rating dynamics. The purpose of this article is to address the non-Markov behavior of the rating dynamics.

Design/methodology/approach – This paper proposes a model based on Markov regenerative process (MRGP) with subordinated semi-Markov process (SMP) to obtain the estimates of rating migration probability matrices and default probabilities. Numerical example is given to illustrate the applicability of the proposed model with the help of historical Standard & Poor’s (S&P) credit rating data.

Findings – The proposed model implies that rating of a firm in the future not only depends on its present rating, but also on its previous ratings. If a firm gets a rating lower than its previous ratings, there are higher chances of further downgrades, and the issue is called the rating momentum. The model also addresses the ageing problem of credit rating evolution.

Originality/value – The contribution of this paper is a more general approach to study the rating dynamics and overcome the issues of inappropriateness of Markov process applied in rating dynamics.

Keywords Credit rating, Default distribution, Markov regenerative process, Semi-Markov process

Paper type Research paper

1. Introduction

The financial institutions face different types of financial risks, in particular market risk, operational risk and credit risk. The focus of this paper is credit risk, also known as default risk or counter party risk. Basel II report published in 2004 stressed the need of credit ratings and it became an important tool in financial institutions and plays a key role in the study of risk management. It is the risk of lender that may arise from a borrower not being able to meet its debt obligations. For example, a corporate bond issued by a company has a credit risk associated with it. The bond issuer, a company, may default in future for some reason or other. As a result, it may not be able to pay back the par value of the bond at maturity. Thus, holders of the bond are exposed to credit risk for the time period starting from the issuance of bond till its maturity. Losses can result from both default and decline in the market value due to deterioration of credit quality of an issuer or counter party.

Credit risk analysis consists of finding the likelihood of default of an obligor going into debt. Credit risk models are basically divided into reduced-form models and firm value models (also known as structural models). Firm value models considered the model in Merton (1974) as the base model, which gives a mechanism of default in terms of the relation...
between liabilities and assets at maturity time $T$. This basic model has been extended by incorporating other factors like stochastic interest rates, default at any time, etc. On the other hand, reduced-form models do not specify the actual mechanism of default, but model it as a non-negative random variable with distribution depending on the economic co-variables. Interested readers can refer to the paper by Duffie and Singleton (2003).

Recent financial crisis has emphasized the significance of correlations in the financial market. In this regard, the study of the default risk of the counter party, in any financial agreement, has become vital in the credit risk. It is the most profoundly studied topic in modern financial world. There are many parameters associated with bond issuer or bond itself, which quantify the credit risk associated with it. Credit rating is one of the important parameters. Credit rating of a credit risky bond, issued by a company, is an evaluation of its likelihood of default and ability to pay back the loan. Better the credit rating of a bond, safer it is. Credit rating plays a very significant role in evaluating the probability of default and, thus, quantifying the credit risk. Banks and the firms that issue bonds are most concerned to quantify the default risk. International organizations like Standard & Poor’s (S&P) issue ratings to the companies that issue bonds to evaluate the credit risk. A credit rating is given to each company that issues a bond, specifying its capacity to repay the debt. Clearly, a higher interest rate is expected from a firm whose rating is lower.

The level of rating changes from time to time because of random credit risk and, thus, can be modeled by an appropriate stochastic process. In credit risk modeling, one of the important tasks for banks is to estimate the credit migration probabilities. It serves as an important input in several applications in risk management, such as portfolio optimization, pricing of credit derivatives, modeling the term structure of credit risk premium, etc. and, hence, their accurate estimation is critical. Transition matrices are important in modern credit risk management. Credit portfolio models, such as CreditMetrics (Gupton et al., 1997), use these matrices to obtain the distribution of a portfolio of risky assets.

In Jarrow et al. (1997), they applied for the first time Markov processes to capture the time evolution of credit ratings. These models are called “migration models”. One of the drawbacks of this model is that it gives, in small time interval, zero probability of default to bonds with high credit ratings. Other papers (Hu et al., 2002; Nickell et al., 2000; Baillo, A. and Fernández, 2007; Grimshaw and Alexander, 2011) followed the same approach to generate the transition matrix. In the papers by Nickell et al. (2000), Kavvathas (2001), Lando and Skodeberg (2002), inappropriateness of Markov process for the description of accurate rating dynamics was addressed. Carty and Fons (1994) and Nickell et al. (2000) proved that the current rating of a company depends not only on its last rating, but on all the previous ones, the effect called rating momentum. Thus, the evolution of credit rating dynamics is non-Markov. Carty and Fons (1994) and Duffie and Singleton (2003) showed that a complete information of the time spent inside the states is of major interest in the credit risk problem. The credit migration probability depends on the time spent by a company in a particular rating. In a continuous-time homogeneous Markov chain, durations in states follow an exponential distribution. An exponential distribution has a constant hazard function. But for credit rating dynamics, the hazard function is not constant. Hence, Markov model for credit rating is not appropriate (Frydmana and Schuermann, 2008). The other issue is time dependence. It means that in general, transition probabilities tend to change with the state of the economy, being low during periods of economic expansion and high during recession. Rating evaluation at two different points in time is different (Nickell et al., 2000) and, hence, the process describing the evolution of rating dynamics is time non-homogeneous.

The literature proposing models that addresses the non-Markov behavior of credit rating dynamics is of recent origin. Giampri et al. (2005) proposed a hidden Markov model to predict
default. Christensen et al. (2004) addressed the serial correlation of rating changes by considering the possibility of excited states for certain downgrades. Korolkiewicz and Elliott (2008) proposed a hidden Markov model assuming that the Markov chain governing the true credit quality evolution is hidden in noisy or incomplete observations about credit ratings. In the paper by D’Amico et al. (2016), they have considered the credit risk problem as a reliability problem. They applied time-homogeneous, semi-Markov processes (SMP) to solve the first issue. The second issue has been solved by extending the state space by the same authors D’Amico et al. (2016) and many further developments have been made in their model (D’Amico et al., 2010, 2011, 2012).

The first issue, as shown by Carty and Fons (1994), exists in speculative bonds but not in investment grade bonds. Empirical studies document that prior rating changes may carry information to predict the direction of movement of rating in future, called rating momentum. If a firm gets a rating lower than its previous rating, the chances that its next rating will also be a downgrade are high. Therefore, the Markov property is not satisfied, as the present rating does not completely determine the transition intensities. But, when rating of a firm moves upward, this anomaly does not hold. Motivated by the empirical evidences of non-suitability of Markov process for rating dynamics, this paper proposes a non-Markov model, which is a non-trivial generalization of a continuous time homogeneous Markov chain and semi-Markov process. In, semi-Markov process, sojourn times are generally distributed, but it lacks the ability to capture local behaviors between two successive regeneration points. Markov regenerative processes (MRGPs) provide a natural generalization of SMPs that account for this local behavior. In this paper, we address the issues described above, by the use of MRGP with embedded SMP (Vaidyanathan, 2002). As rating momentum exists only in downward moving rating, i.e. prior rating changes carry information for further changes, we can say Markov property is satisfied only when there is a migration from a given rating to a better rating. This behavior of the ratings can be modeled by MRGP. Also, the time spent in a state is a general distribution in MRGP. Hence, MRGP models both the non-Markovian behavior of the states and the randomness of time in the transition between two states. The fact that in the proposed MRGP model, the embedded process is SMP, makes it unique. The model produces a probability distribution of true credit quality, given the observed history of credit ratings.

The rest of the paper is as follows: Section 2 gives a short description of MRGP and its applicability to the credit risk problem. Section 3 discusses the proposed model and solution methodology in both time-dependent and steady state conditions. Section 4 discusses some numerical results and Section 5 concludes the paper with future work.

2. Markov regenerative process and credit rating dynamics

In this section, an MRGP is described briefly (Kulkarni, 1995). First, we give the definition of a Markov renewal sequence (MRS). A sequence of the random variables \((X_n, T_n), n = 0, 1, \ldots\) is called a Markov renewal sequence if:

\[
T_0 = 0, \quad T_{n+1} \geq T_n; \quad X_n \in \Omega = \{0, 1, 2, \ldots\}
\]

\[\forall n \geq 0,\]

\[
P\{X_{n+1} = j, T_{n+1} - T_n \leq t \mid X_n = i, T_m, X_{n-1}, \ldots, X_0, T_0\} = P\{X_{n+1} = j, T_{n+1} - T_n \leq t \mid X_n = i\} \quad \text{(Markov property)}
\]

\[
P\{X_1 = j, T_1 - T_0 \leq t \mid X_0 = i\} \quad \text{(time homogeneity)}
\]
Now, we define MRGP. A stochastic process \( \{Z(t), t \geq 0\} \) on \( \Omega \) is called an MRGP if there exists an MRS \( \{(X_n, T_n), n = 0, 1, \ldots\} \) such that all conditional finite dimensional distributions of \( \{Z(T_n + t), t \geq 0\} \) given \( \{Z(u), 0 \leq u \leq T_n, X_n = i\} \) are same as those of \( \{Z(t), t \geq 0\} \) given \( X_0 = i, i \in \Omega' \subset \Omega \). This implies that in this case \( \{Z(T_n^+), n = 0,1,\ldots\} \) or \( \{Z(T_n^-), n = 0,1,\ldots\} \) is an embedded discrete time Markov chain (DTMC) and also that \( T_n \)'s are regeneration points of \( \{Z(t), t \geq 0\} \). From the above definition, it can easily be observed that every semi-Markov process is an MRGP. Like SMPs, MRGP allows non-exponentially distributed sojourn times unlike the Markov environment, where it has to be an exponential distribution. Therefore, by using MRGP in the credit risk problem, we can address the issue of the dependence of transition probabilities on the time spent inside a particular rating.

Figure 1(a) and (b) show the sample paths of an MRGP and SMP corresponding to a credit rating dynamics problem. The main difference between SMP and MRGP can be seen by observing their sample paths, and comparing the sequence of regeneration time points \( S_n \) with the sequence obtained from the state transition instants of the process. In other words, every state transition is a regeneration time epoch in an SMP; however, this is not the case for an MRGP. The sample path of an SMP is right continuous and piecewise constant, and \( S_n \) is the time when the \( n \)th jump occurs. But for MRGP, the stochastic process between \( S_n \) and \( S_{n+1} \) could be any continuous-time stochastic process, such as continuous time Markov chain (CTMC), semi-Markov process. Hence, state is allowed to change between two regeneration epochs and the sample paths are no longer piecewise constant – local behaviors exist between two consecutive Markov regenerative points. In queuing theory, for example, consider an \( M/G/1 \) queue, and observe the process \( \{Z(0)\} \) denoting the number of customers in the system. The times at which a customer enters the system are in general not the regeneration points, and the next state of the process depends on the remaining service time of the customer being served. Hence, during service time of that customer, more customers can enter the system and change the state of system.

**Motivation:** The rating issued to a firm by a rating agency gives its capacity to repay the debt. In practice, there are the following ratings given by S&P:

\[
\Omega = \{AAA, AA, A, BBB, BB, B, CCC, D\}
\]

The bonds having rating above BB are investment grade bonds, whereas those having BB or below BB are speculative bonds, and state D corresponds to default. Let us assume that these ratings are \( \Omega = \{1, 2, \ldots, 8\} \) in the same order as above.

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**Figure 1.** Sample path

**Notes:** (a) Semi-Markov process; (b) Markov regenerative process
Consider a firm that starts in some rating \( i \in \Omega \). As the level of rating changes over time, we can model it by using an appropriate stochastic process. Let \( \{ Z(t), t \geq 0 \} \) be the stochastic process, where \( Z(t) \) represents the rating of the firm at time \( t \). We can observe that this is a discrete state continuous time stochastic process. The process \( \{ Z(t), t \geq 0 \} \) is not a time-homogeneous CTMC as the sojourn times in all states are not exponentially distributed, but follow some non-exponential distribution (Carty and Fons, 1994). Furthermore, because whenever a firm gets a bad rating than the previous rating, there is a high probability that its further ratings will be worse, and the effect is called rating momentum (Kavvathas, 2001). Hence, we can observe that the underlying process \( \{ Z(t), t \geq 0 \} \) satisfies the Markovian property only at the time instants when it gets a better rating than the previous one, whereas the time epoch of a downgrade cannot be considered as a regeneration time epoch. It can be seen from the above conditions that the regeneration instances exactly correspond to times of entering state \( i \) from a state \( j \) such that \( j < i \). Therefore, the process \( \{ Z(t), t \geq 0 \} \) is not even an SMP. The rating of a firm can change between two generations (i.e. upgrades) because of downgrades.

Consider the sequence of epochs \( \{ T_n, n = 0, 1, \ldots \} \) at which the process \( \{ Z(t), t \geq 0 \} \) gets a better rating than the previous one. The state of the process can change between two generations (i.e. upgrades) because of downgrades.

Let \( P = [p_{ij}]_{i,j \in \Omega} \) be the one-step transition probability matrix of the embedded DTMC with state space \( \Omega' \). It follows that:

\[
P_{ij}(t) = \lim_{\Delta t \to 0} K_{ij}(t), \quad i, j \in \Omega'
\]

**Local kernel:** The local kernel \( E(t) = [E_{ij}(t)]_{i,j \in \Omega'} \) is given by

\[
E_{ij}(t) = P(Z(t) = j, T_1 > t | Z(0) = i), \quad i \in \Omega', j \in \Omega, t \geq 0
\]

As their names suggest, global kernel \( K(t) \) describes the behavior of the process immediately after the next regenerative point, whereas the local kernel \( E(t) \) describes the behavior of the process between two regeneration epochs i.e. the local behavior.

The transition probabilities for \( \{ Z(t), t \geq 0 \} \) are defined by:

\[
V_{ij}(t) = P(Z(t) = j | Z(0) = i), \quad i \in \Omega', j \in \Omega, t \geq 0
\]

These can be obtained by solving the generalized Markov renewal equation (Kulkarni, 1995):

\[
V_{ij}(t) = E_{ij}(t) + \sum_{y \in \Omega'} \int_0^t V_{ij}(t - y)dK_{ij}(y), \quad i \in \Omega', j \in \Omega.
\]

In the matrix form, it can be written as:
$V(t) = E(t) + K(t) * V(t)$

In the credit risk environment, the first part of the above equation can be interpreted as the probability that firm has a lower rating $j$ at time $t$, before the next regeneration time point that is before getting a better rating, given the firm was at state $i$ at time 0. In second part of the above equation, $K_{ij}(t)$ represents the probability that the firm will get a better rating $\gamma$ in time $y$, and then the firm will migrate to rating $j$ in time $(t - y)$ following one of the possible paths. Let $\tilde{A}(s)$ denote the Laplace transform of $A(\theta)$, then the solution of above equation in transform domain is given by:

$$\tilde{V}(s) = [I - \tilde{K}(s)]^{-1} \times \tilde{E}(s)$$

Note that both SMP and MRGP satisfy the same generalized Markov renewal equation. The only difference is the matrix $E(t)$, which is diagonal for the SMP (as every transition epoch is not a regeneration epoch) and non-diagonal for the MRGP. In the following subsections, the methodology to determine the elements of $K(t)$ and $E(t)$ is described, and time-dependent and steady state analysis of the proposed model is presented.

### 3.1 Determining the global kernel

Let $p_{ij}$ be the one-step transition probabilities for the embedded Markov chain. Then, for $i, j \in \Omega'$:

$$p_{ij} = \lim_{t \to \infty} K_{ij}(t)$$

Let $F_{ij}(t); i \in \{1, 2, \ldots, 7\}$ and $j \in \Omega$ represent the distribution of migrating from state $i$ to state $j$. It basically represents the distribution function of the waiting time in each state $i$, given that next state $j$ is known:

$$F_{ij}(t) = P(T_{i+1} \leq t | X(T_i) = i, X(T_{i+1}) = j), \quad t \geq 0$$

The global kernel $K(t) = [K_{ij}(t)]_{i,j\in\Omega'}$ can be obtained as follows:

$$K_{ij}(t) = P(Z(T_i) = j; T_i \leq t | Z(0) = i) = \begin{cases} F_{ij}(t) p_{ij} & \text{if } p_{ij} \neq 0 \\ 0 & \text{if } p_{ij} = 0 \end{cases}$$

### 3.2 Determining the local kernel

Now, we need to determine $E_{ij}(t), i \in \Omega'$ and $j \in \Omega$, the probability of migrating from $i$ to $j$ without any regeneration (upward movement). Clearly, we have:

$$E_{ij}(t) = 0 \text{ if } i > j, \quad t \geq 0, \quad i \in \Omega', j \in \Omega$$

as a transition from $i$ to $j$ when $i > j$, i.e. from a lower rating category to a higher-rating category is not possible without upgrade (regeneration).

For a transition from $i$ to $j$ when $i \leq j$, let $\Omega^{(i)}$ for each $i \in \Omega'$ be the set of all states reachable from state $i$ by downward movement, that is $\Omega^{(i)} = \{i, i + 1 \ldots, 8\}$. Let $\{M^{(i)}(t), t \geq 0\}, i \in \Omega'$ be the subordinated SMP (as the sojourn time in each state has general distribution in which there are only downward movements from state $i$, but no upward movement). Here, by choosing parameters such that probability of downward movement is high, the problem of rating momentum can be addressed. The transition probabilities of this SMP for each $i \in \Omega'$ will give us $E_{ij}(t) \forall j$, local kernel of MRGP.
Given that the system is in state $i$ at time 0, the transition probabilities of subordinated SMP’s can be obtained by solving SMP, say $\{M^0(t), t \geq 0\}$ with state space $\Omega^0$ and with only downward movement possible from state $j$ to state $k (k > j), j, k \in \Omega^0 = \{i, i + 1 \ldots, 8\}$. Let $(Y_n^{(0)}, S_n^{(0)})$ be the Markov renewal sequence, where $Y_n^{(0)}$ represents the state at the $n$th transition and $S_n^{(0)}$, $n \geq 1$ with state space equal to $\mathbb{R}^+ \rightarrow \mathbb{R}^+$ represents the time of $n$th transition. The kernel $Q^{(0)}(t) = [Q_{jk}^{(0)}(t)], j, k \in \Omega^0$ of the subordinated SMP is defined by:

\[
Q_{jk}^{(0)}(t) = P(Y_{n+1}^{(0)} = k; S_{n+1}^{(0)} - S_n^{(0)} \leq t \mid Y_n^{(0)} = j), j, k \in \Omega^{(0)}
\]

and it follows that:

\[
q_{jk}^{(0)} = \lim_{t \to \infty} Q_{jk}^{(0)}(t), \quad j, k \in \Omega^{(0)}
\]

are the transition probabilities of the embedded Markov chain in the process. Therefore, for $j, k \in \Omega^{(0)}$:

\[
G_{jk}^{(0)}(t) = P(S_{n+1}^{(0)} - S_n^{(0)} \leq t \mid Y_n^{(0)} = j, Y_{n+1}^{(0)} = k) = \begin{cases} Q_{jk}^{(0)}(t) & \text{if } q_{jk}^{(0)} \neq 0 \\ 1 & \text{if } q_{jk}^{(0)} = 0 \end{cases}
\]

Clearly, $G_{jk}^{(0)}(t)$ is the same as $F_{jk}(t)$ defined in Section 3.1. Furthermore, the probability that the process will leave state $j$ before time $t$ is given by:

\[
H_j^{(0)}(t) = P(S_{n+1}^{(0)} - S_n^{(0)} \leq t \mid Y_n^{(0)} = j), j \in \Omega^{(0)}
\]

We can observe that:

\[
H_j^{(0)}(t) = \sum_{k=j}^{8} Q_{jk}^{(0)}(t)
\]

Now, the transition probabilities of the SMP $\{M^{(0)}(t), t \geq 0\}$ are defined by:

\[
\phi_{jk}^{(0)}(t) = P(M^{(0)}(t) = k \mid M^{(0)}(0) = j), j, k \in \Omega^{(0)}
\]

and are obtained by solving the renewal equation:

\[
\phi_{jk}^{(0)}(t) = \delta_{jk}(1 - H_j^{(0)}(t)) + \sum_{\beta=k}^{8} \int_0^t \phi_{jk}(t - y)dQ_{jk}^{(0)}(y)
\]

We observe that:

\[
\phi_{jk}^{(0)}(t) = 0 \quad \text{if } j > k
\]

In the matrix form, it can be written as:

\[
\phi^{(0)}(t) = H^{(0)}(t) + Q^{(0)}(t) \ast \phi^{(0)}(t)
\]
where \( H^{(i)}(t) \) is the diagonal matrix with \( j \)th diagonal entry \( 1-H_{i1}^{(i)} \). \( \phi^{(i)}(t) \) will be an upper triangular matrix.

Therefore, for the given initial state \( i \in \Omega' \), the local kernel of MRGP \( E_{ij}(t) \forall j \neq i \) can be obtained as follows:

\[
E_{ij}(t) = P(\mathbf{Z}(t) = j, T_1 > t | Z_0 = i) \\
= P(\mathbf{Z}(t) = j | T_1 > t, Z_0 = i) P(T_1 > t | Z_0 = i) \\
= \phi^{(i)}(t) \times (1 - P(T_1 \leq t | Z_0 = i)) \\
= \phi^{(i)}(t) \times \left(1 - \sum_{k \in \Omega'} K_{ik}(t)\right) .
\]

For each \( i \in \Omega' \), \( E_{ij}(t), j \in \Omega \) of the MRGP describes the behavior of rating evolution between two regeneration epochs as to how the rating moves to a lower rating before going to upper rating and is given by:

\[
E_{ij}(t) = \left\{ \begin{array}{ll}
0 & \text{if } i > j \\
\phi^{(i)}(t) \times \left(1 - \sum_{k \in \Omega'} K_{ik}(t)\right) & \text{if } i \leq j
\end{array} \right.
\]

Hence, by taking \( F_{ij}(t) \) with different intensities, increasing or decreasing, we can take care of downward momentum using local kernel.

### 3.3 Time-dependent solution of MRGP

The time-dependent solution of the proposed model is given by:

\[
V_{ij}(t) = E_{ij}(t) + \sum_{y \in \Omega'} \int_0^t V_{ij}(t-y)dK_{ij}(y); i \in \Omega', j \in \Omega
\]

where \( E(t) \) and \( K(t) \) are given in subsections 3.2 and 3.3.

The above Markov renewal equation represents a Volterra integral equation of second kind. It is a very particular case of indexed Markov renewal equations. This equation can be solved numerically using discretization to numerically evaluate the integrals (D’ Amico, 2011; German et al., 1995). Let \( h \) be the step size of discretization, then we have the countable linear system given by:

\[
V_{ij}^h(kh) = E_{ij}(kh) + \sum_{y \in \Omega'} \sum_{\tau=1}^{k} d_{ij}(\tau h)V_{ij}^h((k - \tau)h), \quad k = 0, 1, \ldots
\]

where:

\[
d_{ij}(kh) = \begin{cases}
K_{ij}(kh) - K_{ij}((k - 1)h) & \text{if } k > 0 \\
0 & \text{if } k = 0
\end{cases}
\]

\( E_{ij}(kh) \) is given by the numerical solution of SMP:

\[
E_{ij}(kh) = g_{ij}(kh) + \sum_{l=\max(i,0)}^{8} \sum_{\tau=1}^{\min(k,1)} q_{ij}(\tau h)E_{ij}((k - \tau)h)
\]
where:

\[ q_{ij}(kh) = \begin{cases} Q_{ij}(k) - Q_{ij}(k-1) & \text{if } k > 0 \\ 0 & \text{if } k = 0 \end{cases} \]

and

\[ g_{ij}(kh) = \begin{cases} 1 - H_i(k) & \text{if } i = j \\ 0 & \text{if } k \neq j \end{cases} \]

In the Laplace transform domain the solution can be obtained as:

\[ \tilde{V}(s) = [I - \tilde{K}(s)]^{-1}\tilde{E}(s) \]

where \( \tilde{E}(s) \) is given by the transform solution of subordinated SMP:

\[ \tilde{E}(s) = [I - \tilde{Q}(s)]^{-1}\tilde{H}(s) \]

where, \( Q(t) \) and \( H(t) \) are described in subsection 3.1.

On solving the above system of equations, the following measures can be obtained:

- \( V_{ij}(t) \), which represents the probability of being in state \( j \) after a time \( t \), given that initially it was in state \( i \) at time 0. This takes into account the time spent in a particular state (ageing).
- \( \Sigma_{j=1} V_{ij}(t) V_{ij}(t) \), which represents the probability that the system will always be in non-default states in time interval \( (0, t) \).

Note that in the credit risk model discussed, the process is reducible with one absorbing class. As there is only one absorbing state in absorbing class, we get the steady state solution as:

\[ \lim_{t \to \infty} V_{ij}(t) = \prod_{j} = \begin{cases} 0 & \text{if } j \neq 8 \\ 1 & \text{if } j = 8 \end{cases} \]

As \( \lim_{t \to \infty} V_{is}(t) = 1 \ \forall i \in \Omega \), \( V_{is}(t) \), \( t \geq 0 \) gives the distribution function of the first time of default for a firm, given that the firm has rating \( i \) initially at time 0.

4. Numerical illustration

In this section, we illustrate the applicability of the proposed model with the help of a numerical example.

4.1 Data and methodology

The quarterly data of long-term issuer ratings for companies that S&P has rated in USA since 1985 to 2015 has been used. To implement the model, the required parameters \( F_{ij}(t) \) and \( p_{ij}, i, j \in \Omega \), where \( F_{ij}(t) \) is the distribution of time spent in a rating, given that the next rating and \( p_{ij} \) is the transition probability of embedded Markov chain has been estimated as follows. Within the given period, credit rating changes in consecutive quarters are collected and assigned to a frequency matrix. The transition from and to the not rated (NR) status are eliminated from the sampled data. The probability \( p_{ij} \) for the given \( i \) and \( j \) is estimated by dividing the number of transitions from \( i \) to \( j \) by total number of transitions from state \( i \).
To estimate the distribution functions $F_{ij}(t)$, those transitions are identified in which initial rating is $i$ and next rating is $j$. For all the identified transitions, number of quarters for which rating $i$ is held before transition to $j$ are collected. By plotting the histogram of time spent in a rating given next rating, best-fit distribution is estimated using R software. After finding the best-fit distribution, its parameters can be obtained by maximum likelihood estimation method with the help of R software. This is the same approach as that followed by Carty and Fons (1994). The maximum likelihood estimator for the parameter $\lambda$ given $k$ is:

$$\hat{\lambda} = \frac{1}{n} \sum_{i=1}^{n} x_i$$

The maximum likelihood estimator for $k$ is:

$$\hat{k}^{-1} = \frac{\sum_{i=1}^{n} x_i^k \ln x_i}{\sum_{i=1}^{n} x_i^k} - \frac{1}{n} \sum_{i=1}^{n} \ln x_i$$

4.2 Results
Following the procedure mentioned above, the estimated one-step transition probability matrix is given in Matrix 1.

Matrix 1. One-year transition probability matrix $P$:

<table>
<thead>
<tr>
<th></th>
<th>AAA</th>
<th>AA</th>
<th>A</th>
<th>BBB</th>
<th>BB</th>
<th>B</th>
<th>CCC</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>AAA</td>
<td>0.9726</td>
<td>0.0259</td>
<td>0.0010</td>
<td>0.0002</td>
<td>0.0002</td>
<td>0.0000</td>
<td>0.0002</td>
<td>0.0000</td>
</tr>
<tr>
<td>AA</td>
<td>0.0013</td>
<td>0.9760</td>
<td>0.0217</td>
<td>0.0008</td>
<td>0.0001</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>A</td>
<td>0.0002</td>
<td>0.0052</td>
<td>0.9814</td>
<td>0.0124</td>
<td>0.0005</td>
<td>0.0001</td>
<td>0.0001</td>
<td>0.0000</td>
</tr>
<tr>
<td>BBB</td>
<td>0.0000</td>
<td>0.0004</td>
<td>0.0088</td>
<td>0.9787</td>
<td>0.0106</td>
<td>0.0010</td>
<td>0.0002</td>
<td>0.0003</td>
</tr>
<tr>
<td>BB</td>
<td>0.0001</td>
<td>0.0003</td>
<td>0.0005</td>
<td>0.0131</td>
<td>0.9612</td>
<td>0.0223</td>
<td>0.0017</td>
<td>0.0008</td>
</tr>
<tr>
<td>B</td>
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<td>0.0001</td>
<td>0.0004</td>
<td>0.0004</td>
<td>0.0146</td>
<td>0.9619</td>
<td>0.0192</td>
<td>0.0034</td>
</tr>
<tr>
<td>CCC</td>
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<td>0.0000</td>
<td>0.0006</td>
<td>0.0009</td>
<td>0.0017</td>
<td>0.0481</td>
<td>0.8780</td>
<td>0.0706</td>
</tr>
<tr>
<td>D</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

The parameters for distribution $F_{ij}(t)$ are estimated using maximum likelihood estimator. For example, $F_{1,1}(t)$ is Weibull (1.378, 2.019), $F_{1,2}(t)$ is Weibull (0.936, 2.898) and $F_{3,4}(t)$ is Weibull (1.245, 1.497). The estimated distribution $F_{ij}(t)$, $i, j \in \Omega$ combined with $p_{ij}$ are then used to obtain the transition probabilities of MRGP. We have generated the rating migration matrices at different time instants. The transition matrices constructed by the proposed model at times five and seven years are given in Matrix 2 and 3, respectively. In the Matrix, the entry 0.0144 in row BBB and in column BB of Matrix 3 represents the probability that a firm will have a rating BB at times 7, given that it initially started with rating BBB.
Matrix 2. Five-year transition probability matrix, $V(5)$:

\[
\begin{array}{cccccccc}
\text{AAA} & \text{AA} & A & \text{BBB} & BB & B & \text{CCC} & D \\
\text{AAA} & 0.9454 & 0.0495 & 0.0034 & 0.0006 & 0.0005 & 0.0000 & 0.0005 & 0.0000 \\
\text{AA} & 0.0013 & 0.9760 & 0.0210 & 0.0014 & 0.0002 & 0.0000 & 0.0000 & 0.0002 \\
A & 0.0002 & 0.0036 & 0.9851 & 0.0100 & 0.0007 & 0.0002 & 0.0001 & 0.0000 \\
\text{BBB} & 0.0000 & 0.0004 & 0.0057 & 0.9824 & 0.0093 & 0.0014 & 0.0003 & 0.0005 \\
BB & 0.0001 & 0.0003 & 0.0006 & 0.0088 & 0.9562 & 0.0209 & 0.0026 & 0.0016 \\
B & 0.0000 & 0.0001 & 0.0004 & 0.0005 & 0.0002 & 0.0000 & 0.0000 & 0.0000 \\
\end{array}
\]

Matrix 3. Seven-year transition probability matrix, $V(7)$:

\[
\begin{array}{cccccccc}
\text{AAA} & \text{AA} & A & \text{BBB} & BB & B & \text{CCC} & D \\
\text{AAA} & 0.9285 & 0.0789 & 0.0088 & 0.0014 & 0.0012 & 0.0000 & 0.0010 & 0.0002 \\
\text{AA} & 0.0012 & 0.9600 & 0.0362 & 0.0022 & 0.0003 & 0.0000 & 0.0000 & 0.0002 \\
A & 0.0002 & 0.0045 & 0.9778 & 0.0160 & 0.0010 & 0.0002 & 0.0002 & 0.0000 \\
\text{BBB} & 0.0000 & 0.0004 & 0.0074 & 0.9748 & 0.0144 & 0.0020 & 0.0004 & 0.0007 \\
BB & 0.0001 & 0.0003 & 0.0006 & 0.0113 & 0.9486 & 0.0330 & 0.0035 & 0.0025 \\
B & 0.0000 & 0.0001 & 0.0004 & 0.0005 & 0.0116 & 0.9512 & 0.0250 & 0.0111 \\
\end{array}
\]

In the Matrix 4, we give the probabilities of no default in time $t$ given that the firm has rating $i$, initially. This can be interpreted as reliability and provides important financial information as discussed in D’Amico et al. (2005). For example, consider the entry 0.9793 in row 2 and column $B$. It gives the probability of the firm being reimbursed after one year given that the firm has initial rating $B$. This value can be used to determine the interest rate that $B$-rated bond with maturity five years must pay. Let us consider the one-year risk free rate to be 5 per cent, then the interest that this bond must pay can be computed as follows:

\[
\frac{1}{0.9793} \times 1.05 - 1 = 7.2 \text{ per cent}
\]

Figure 2 gives the distribution of first time of default on log scale obtained by the proposed model for different initial ratings.

Matrix 4. Probabilities of not going into default in a time horizon of ten years since the initial rating

\[
\begin{array}{cccccccc}
\text{AAA} & \text{AA} & A & \text{BBB} & BB & B \\
1 & 1.0000 & 1.0000 & 1.0000 & 0.9998 & 0.9993 & 0.9945 \\
2 & 1.0000 & 0.9999 & 1.0000 & 0.9994 & 0.9970 & 0.9793 \\
3 & 1.0000 & 0.9998 & 0.9999 & 0.9987 & 0.9933 & 0.9597 \\
4 & 0.9999 & 0.9997 & 0.9998 & 0.9977 & 0.9883 & 0.9376 \\
5 & 0.9999 & 0.9996 & 0.9997 & 0.9966 & 0.9822 & 0.9141 \\
6 & 0.9999 & 0.9995 & 0.9995 & 0.9953 & 0.9751 & 0.8902 \\
7 & 0.9998 & 0.9993 & 0.9993 & 0.9938 & 0.9671 & 0.8663 \\
8 & 0.9997 & 0.9991 & 0.9990 & 0.9921 & 0.9585 & 0.8427 \\
9 & 0.9996 & 0.9989 & 0.9986 & 0.9902 & 0.9491 & 0.8198 \\
10 & 0.9995 & 0.9987 & 0.9982 & 0.9880 & 0.9393 & 0.7977 \\
\end{array}
\]

To compare the proposed model with semi-Markov model and the default probabilities given by S&P for 1981-2014, we consider the average cumulative default rates by ratings given in
“Annual Global Corporate Default Study and Rating Transitions (2014)” by S&P. We started with the average transition matrix (1981-2014) given in the report to find the default distribution for the MRGP and SMP models. We have fixed the distributions $F_t\beta(t)$ and $p_{ij}$ for both semi-Markov and MRGP models. Figure 3 shows the default distribution on the log scale, and we can observe from Figure 3 that the proposed model is better than the SMP model. For instance, if the initial rating is B, the default distribution obtained by the proposed model coincides with the default distribution given by S&P. To show the statistical significance of the results, we performed one-sided t-test on the difference between the errors obtained from the two models, where errors are calculated as a difference between the model and S&P as shown below:

\[ H_0: E_1 \geq E_2 \]
\[ H_1: E_1 < E_2 \]

where $E_1$ and $E_2$ denote the error between the MRGP and S&P, SMP and S&P, respectively. The null hypothesis is rejected at 99 per cent confidence level as the reported $p$-value is 0.0037.

5. Conclusion
In this paper, we introduced and analyzed a credit rating model through MRGP. To the best of our knowledge, the MRGP approach is applied first time to study the credit rating dynamics. In the proposed MRGP model, the subordinated process was SMP, which allowed us to address the issue of rating momentum. Our model is very general and addresses the issues regarding the suitability of Markov process in credit risk environment. The proposed model also discusses the approach to determine the rating transition matrices and some measures that can provide important financial information regarding the firm. The applicability of the proposed model is illustrated with the help of the real credit rating data obtained from Bloomberg. We believe that the proposed model may be of great interest to the financial intermediaries and banks, who are most concerned about evaluating the credit risk.
Figure 3 gives the distribution of first time of default on log scale obtained by the proposed model for different initial ratings (1981-2014).
In general, the process of rating evaluation depends on the time when it is performed. In particular, it depends on the business cycle. Future research will consider the model in non-homogeneous setting that would allow us to address the issue of dependence of rating evaluation on time. Further, it could be extended to multivariate level to capture the default dependency in the portfolio of bonds.

References


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