



Analysis of a Multiple Dual-Stage Vacation Queueing System with Disaster and Repairable Server

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Received: 8 December 2020 / Revised: 14 September 2021 / Accepted: 30 December 2021 /
Published online: 4 March 2022

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Abstract

An $M/M/1$ dual-stage vacation queueing system with a system disaster is analysed in this paper. This work is motivated by the dual-stage sleep mode mechanism in the MAC (Media Access Control) and the physical layer of a WiMAX system with Denial of Service. In this model, the server switches to two types of vacation called stage-I and stage-II vacation. Disaster can occur at any stage of the system and subsequent repair is considered. For this system, explicit expressions for transient and steady-state probabilities are computed. Further, performance indices like the mean, variance of the system size, the probability that the server is on vacation mode and busy mode, and system throughput have been computed. Finally, numerical illustrations are presented to understand the effect of the parameters on the system performance.

Keywords Single server · WiMAX networks · Dual-stage vacation · Disaster · Repairable server · Transient probabilities

1 Introduction

Queueing systems with vacation have gained enormous importance in the system modeling and performance analysis of power saving mechanism in the WiMAX system. WiMAX is a battery-powered system that provides high-speed data over a wide area. To control the power consumption and to lengthen the battery life of the WiMAX systems, the IEEE 802.16 has introduced three types of power-saving classes (PSCs) (Chen et al. 2009). The

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first and the second type of PSCs are used for non-real-time traffic and real-time traffic variable-rate connection respectively. The third type of PSC is suitable for multi-cast connections and management processes. In queueing models, these PSCs are considered as vacations and hence queueing models with vacations play an essential role in the performance analysis of WiMAX systems.

In the classical queueing systems with vacation, the server switches to a vacation mode from operational mode if no jobs are waiting for service; see Doshi (1986). This vacation period can be considered as proper utilization of idle time during which the server may involve in maintenance work or any other work in the queueing system. There are three main vacation policies adopted by the server in the queueing system they are single, multiple and working vacations (WV). These vacation policies offer flexibility to optimal system design and system control. In the single vacation policy, the server switches to a vacation mode if no jobs are waiting for service. After the expiry of the vacation period, the server restarts its service to cater the waiting jobs (if any), else it stays idle; see Altman and Yechiali (2006); Tian et al. (2008), and references therein. In the multiple vacation policy, at the end of the vacation period if the server finds no customer is waiting for service, then the server takes another vacation with the same probability distribution; see Ammar (2015); Ke et al. (2010). In the WV policy, the server offers service both in busy and vacation mode. This WV policy reduces customers' impatience in the queue during vacation; see Servi and Finn (2002); Li and Tian (2007); Lin and Ke (2009); Sudhesh et al. (2017), and references therein.

This paper is motivated by the DoS (Denial of Service) attack in the WiMAX. The DoS is a type of security attack in the WiMAX systems which causes a break in the flow of service Lee et al. (2009). WiMAX with DoS can be compared with a queue with negative customers. G-Queues or queues with negative customers have been investigated extensively by many researchers in recent times due to their far-reaching applications in the field of computer science, cyber-security and communication systems. In the queueing system, an incoming customer can either be negative or positive where the negative customer eliminates one or all the positive customers and obliterate the entire task assigned to the system. This negative arrival can be considered as a virus in a computer system, inhibitory signals in circuits, operation mistakes in manufacturing machines and vulnerabilities in WiMAX systems. The loss of all jobs due to negative arrivals or any technical failures that force all customers to disperse a service system is called catastrophes or disaster; see Di Crescenzo et al. (2003). Queueing system subject to a disaster was first studied by Jain and Sigman (1996), further, it was studied by Chen and Renshaw (1997) in the name of mass exodus. Artalejo and Gomez-Corral (1998) studied the queueing system with disaster in the name of a stochastic clearing. Several authors discussed the queueing system with disaster; see Boxma (2001); Yang and Chae (2001); Anisimov (2001) and references therein. Subsequently, researchers have also discussed discrete times queueing models subject to disaster. Atencia and Moreno (2004) analyzed the effect of negative customers and disaster in a *Geo/Geo/1* queueing model. Sudhesh et al. (2017) presented transient analysis to the disastrous *Geo/Geo/1* queueing model.

In practice, if a system is affected by disaster it breakdown the functioning of the system and service will be terminated for the waiting jobs. Hence the server has to undergo a repair process to accept the new incoming job. Several authors have discussed the disastrous queueing system with a repairable server. Paz and Yechiali (2014) analysed a disastrous *M/M/1* queueing model in a multi-phase environment with a repairable server. Jain and Rakesh (2007) introduced a catastrophic queueing model with restoration time. Kalidass et al. (2014) analysed a disastrous *M/M/1* queueing system with multiple WV further they extended the same

concept to a repairable server with multiple WV. Yang et al. (2019) analysed a heterogeneous $M/M/2$ queuing system with vacation subject to reliable and unreliable servers.

Throughout the literature study on multiple vacation queueing systems it was observed that, the server switches to vacation mode with an identical probability distribution. The vacation protocol defined in WiMAX differs from the above-mentioned vacation mechanism. The WiMAX takes vacations in two stages with a different probability distribution. This type of vacation scheme is called a dual-stage vacation or differentiated vacation (DV). This vacation policy has two stages, stage-I which has a longer period while stage-II is of a shorter duration. The long vacation is taken after serving all the customers in the system. At the end of long vacation, if the server finds that the system is empty, it immediately takes another vacation which is of a shorter duration. The steady-state solution DV model was first studied by Ibe and Isijola (2014). The time-dependent analysis of an $M/M/1/DV$ was studied by Vijayashree and Janani (2018). $M/M/1/DV$ with customers' impatience was studied by Sampath and Liu (2018).

The real-time application of the system under consideration is as follows. In automated service systems like a WiMAX communications system, after serving all the incoming data frames, the server switches to sleep mode (stage-I vacation) for some duration of time. After the expiry of sleep mode, the server checks whether any data frame is waiting for service. If no data frame is waiting for service, then the server switches to the listening state (stage-II vacation) which is also a type of power-saving mode. The PSCs of type-I and type-II take a vacation in two stages. Both sleep and listening states are power-saving modes in WiMAX systems. The disaster in this model can be considered as any security attack in WiMAX like vulnerabilities in both physical and MAC layers, jamming attack, scrambling attack, water torture attack and DoS. These types of attacks particularly the DoS attack slow down the network performance and force shutdown.

The objective of this paper is to study the time-dependent analysis of a DV queueing system subject to disaster and repair. In many real-time queueing theory applications, the system experiences a change and such changes can be measured by the transient analysis. The load fluctuations, congestion control and signal loss in the communication system can be analysed by the time-dependent analysis. For example, the steady-state probability of a DV discussed by Ibe and Isijola (2014) cannot be used to determine the number of jobs waiting in the queue during the busy state or in the vacation state at some instant t . Hence the transient analysis will be more realistic to analyse the performance of the WiMAX system. This motivates us to examine the transient analysis of the proposed model. Throughout the literature survey, no research work has been performed to derive the time-dependent solution of a DV queueing model subject to disaster and repair.

The rest of the paper is organized as follows. The model description is presented in Section 2. The transient probabilities of the idle state, busy state, stage-I vacation and stage-II vacation are discussed in Section 3. The time-dependent mean and variance of the system size are discussed in Section 4. The steady-state probabilities are derived explicitly in Section 5. The steady-state mean is discussed in Section 6. The results obtained in Sections 3, 4 and 5 are illustrated in Section 7. The conclusion and future work are given in Section 8.

2 Description of the Model

This model considers an $M/M/1$ disastrous queueing system with a dual-stage vacation and a repairable server.

1. The data frames (jobs) arrive according to a Poisson process with the rate λ and the service times are exponentially distributed with a rate μ .
2. If all the data frames are transmitted in the busy state b , then the sever switches to stage-I vacation v_1 .
3. After the expiry of the stage-I vacation period, if the server finds any data frame waiting for service then the sever resumes service otherwise the server switches to another vacation called stage-II vacation v_2 . The vacation times of the server in stage-I and stage-II vacation follow an exponential distribution with parameters γ_1 and γ_2 respectively.
4. At the end of the stage-II vacation period, the server switches to the idle state if no data frame is waiting for service. If jobs are found waiting in the queue, the server switches to the busy state to process them. The waiting time distribution of the server in the idle state follows an exponential distribution with a rate γ . At the end of this period if no data frame arrives then the server switches to the stage-I vacation.
5. In this model, the disaster may occur at any stage of the system which follows a Poisson process of rate ζ . Whenever the disaster occurs, all the data frames are removed from the system and the server becomes inactive. A repair process starts forthwith and the repair time of the system is exponentially distributed with mean η^{-1} .

Let $\{M(t), t \geq 0\}$ represent the status of the system at any time t and let $N(t)$ denote the number of data frames in the system at any time t .

$$M(t) = \begin{cases} b, & \text{the WiMAX server is in operational mode and operates with the rate } \mu \\ v_1, & \text{the WiMAX server is in sleep mode (stage-I vacation)} \\ v_2, & \text{the WiMAX server is in listening (stage-II vacation)} \\ F, & \text{the WiMAX server is in failure state} \end{cases}$$

Then, $X(t) = \{M(t), N(t), t \geq 0\}$ represents a continuous time Markov chain with state space $S = \{(i, n), n = 0, 1, 2, \dots; i = b, v_1, v_2, F\}$.

Figure 1 presents the state transition diagram of the proposed model.

Let $P_{i,n}(t) = P\{M(t) = i, N(t) = n\}$, $i = b, v_1, v_2, F; n = 0, 1, 2, \dots$ be the probability that the server is in state ' i ' and there are ' n ' number of jobs in the system at time ' t '. Then $P_{i,n}(t)$ satisfies the following forward Kolmogorov equation.

$$P'_F(t) = -\eta P_F(t) + \zeta [1 - P_F(t)] \quad (1)$$

$$P'_{b,0}(t) = -(\lambda + \zeta + \gamma)P_{b,0}(t) + \gamma_2 P_{v_2,0}(t) + \eta P_F(t) \quad (2)$$

$$P'_{b,n}(t) = -(\lambda + \mu + \zeta)P_{b,n}(t) + \lambda P_{b,n-1}(t) + \mu P_{b,n+1}(t) + \gamma_1 P_{v_1,n}(t) + \gamma_2 P_{v_2,n}(t), n = 1, 2, 3, \dots \quad (3)$$

$$P'_{v_1,0}(t) = -(\lambda + \zeta + \gamma_1)P_{v_1,0}(t) + \gamma P_{b,0}(t) + \mu P_{b,1}(t) \quad (4)$$

$$P'_{v_1,n}(t) = -(\lambda + \zeta + \gamma_1)P_{v_1,n}(t) + \lambda P_{v_1,n-1}(t), n = 1, 2, 3, \dots \quad (5)$$

$$P'_{v_2,0}(t) = -(\lambda + \zeta + \gamma_2)P_{v_2,0}(t) + \gamma_1 P_{v_1,0}(t) \quad (6)$$

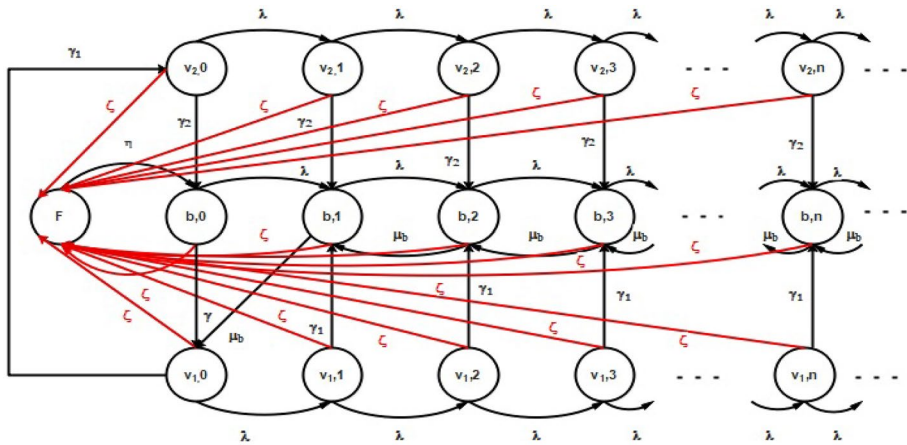


Fig. 1 State transition diagram of the proposed model

$$P'_{v_2,n}(t) = -(\lambda + \zeta + \gamma_2)P_{v_2,n}(t) + \lambda P_{v_2,n-1}(t), n = 1, 2, 3, \dots \quad (7)$$

To set-up initial conditions, we assume that the server is in the stage-I vacation. Therefore, $P_{v_1,0}(0) = 1$, $P_{i,n}(0) = 0$ for $n = 1, 2, \dots$; $i = b, v_1, v_2$ and $P_{j,0}(0) = 0$ for $j = b, v_2$.

3 Transient Analysis

This section presents the transient probabilities of vacation states, idle state, busy state and failure state.

3.1 Evaluation of $P_{v_1,n}(t)$ and $P_{v_2,n}(t)$

Let $\hat{P}_{i,n}(s)$ denote the Laplace transform of $P_{i,n}(t)$; $i = v_1, v_2, b$ and $n = 0, 1, 2, \dots$. Applying Laplace transform on Equations 1, 2, 4 and 6, we get

$$\hat{P}_F(s) = \frac{\zeta}{s(s + \eta + \zeta)}, \quad (8)$$

$$\hat{P}_{b,0}(s) = \frac{\zeta\eta}{s(s + \eta + \zeta)(s + \lambda + \zeta + \gamma)} + \frac{\gamma_2\hat{P}_{v_2,0}(s)}{(s + \lambda + \zeta + \gamma)}, \quad (9)$$

$$\hat{P}_{v_1,0}(s) = \frac{\gamma\hat{P}_{b,0}(s) + \mu_b\hat{P}_{b,1}(s) + 1}{(s + \lambda + \zeta + \gamma_1)}, \quad (10)$$

$$\hat{P}_{v_2,0}(s) = \frac{\gamma_1}{s + \lambda + \zeta + \gamma_2}\hat{P}_{v_1,0}(s). \quad (11)$$

Substituting Equation 11 in Equation 9, we get

$$\hat{P}_{b,0}(s) = \frac{1}{(s + \lambda + \zeta + \gamma)} \left[\frac{\zeta \eta}{s(s + \eta + \zeta)} + \frac{\gamma_1 \gamma_2}{(s + \lambda + \zeta + \gamma_2)} \hat{P}_{v_1,0}(s) \right]. \quad (12)$$

Substituting Equation 12 in Equation 10, we get

$$\begin{aligned} \hat{P}_{v_1,0}(s) &= \frac{1}{(s + \lambda + \zeta + \gamma_1)} [A_0(s) + \mu \hat{P}_{b,1}(s) + 1] \\ &\times \frac{(s + \lambda + \zeta + \gamma)(s + \lambda + \zeta + \gamma_1)(s + \lambda + \zeta + \gamma_2)}{(s + \lambda + \zeta + \gamma)(s + \lambda + \zeta + \gamma_1)(s + \lambda + \zeta + \gamma_2) - \gamma \gamma_1 \gamma_2}. \end{aligned}$$

where

$$A_0(s) = \frac{\eta \zeta \gamma}{s(s + \eta + \zeta)(s + \lambda + \zeta + \gamma)}.$$

After some manipulation, we get

$$\hat{P}_{v_1,0}(s) = \frac{1}{(s + \lambda + \zeta + \gamma_1)} [A_0(s) + \mu \hat{P}_{b,1}(s) + 1] \left[1 + \frac{\gamma \gamma_1 \gamma_2}{(s + c_3)(s^2 + c_1 s + c_2)} \right]. \quad (13)$$

Substituting Equation 13 in Equation 11, we get

$$\begin{aligned} \hat{P}_{v_2,0}(s) &= \frac{\gamma_1}{(s + \lambda + \zeta + \gamma_1)(s + \lambda + \zeta + \gamma_2)} [A_0(s) + \mu \hat{P}_{b,1}(s) + 1] \\ &\times \left[1 + \frac{\gamma \gamma_1 \gamma_2}{(s + c_3)(s^2 + c_1 s + c_2)} \right]. \end{aligned} \quad (14)$$

Taking Laplace transform on Equation 5, we get

$$\hat{P}_{v_1,n}(s) = \frac{\lambda}{s + \lambda + \zeta + \gamma_1} \hat{P}_{v_1,n-1}(s); n = 1, 2, 3, \dots$$

The above equation recursively yields

$$\hat{P}_{v_1,n}(s) = \left(\frac{\lambda}{s + \lambda + \zeta + \gamma_1} \right)^n \hat{P}_{v_1,0}(s); n = 0, 1, 2, \dots \quad (15)$$

Substituting Equation 13 in Equation 15, we get

$$\hat{P}_{v_1,n}(s) = \frac{\lambda^n}{(s + \lambda + \zeta + \gamma_1)^{n+1}} [A_0(s) + \mu \hat{P}_{b,1}(s) + 1] \left[1 + \frac{\gamma \gamma_1 \gamma_2}{(s + c_3)(s^2 + c_1 s + c_2)} \right]. \quad (16)$$

Taking Laplace transform on Equation 7, we get

$$\hat{P}_{v_2,n}(s) = \frac{\lambda}{s + \lambda + \gamma_2 + \zeta} \hat{P}_{v_2,n-1}(s); n = 1, 2, 3, \dots$$

The above equation recursively yields

$$\hat{P}_{v_2,n}(s) = \left(\frac{\lambda}{s + \lambda + \zeta + \gamma_2} \right)^n \hat{P}_{v_2,0}(s); \quad n = 0, 1, 2, 3 \dots \quad (17)$$

Substituting Equation 14 in Equation 17, we get

$$\begin{aligned} \hat{P}_{v_2,n}(s) = & \left[A_0(s) + \mu \hat{P}_{b,1}(s) + 1 \right] \left[\frac{\lambda^n \gamma_1}{(s + \lambda + \zeta + \gamma_1)(s + \lambda + \zeta + \gamma_2)^{n+1}} \right] \\ & \times \left[1 + \frac{\gamma \gamma_1 \gamma_2}{(s + c_3)(s^2 + c_1 s + c_2)} \right]. \end{aligned} \quad (18)$$

Applying inversion on Equations 16 and 18 respectively and using the identity [Abramowitz and Stegun (1970), P-443, 10.2.13 and 10.2.14] yields

$$P_{v_1,n}(t) = \left[A_0(t) + \mu P_{b,1}(t) + \delta(t) \right] * \frac{\lambda^n t^n \exp \{ -(\lambda + \zeta + \gamma_1)t \}}{n!} * H(t), \quad (19)$$

$$\begin{aligned} P_{v_2,n}(t) = & \left[\frac{(-1)^{n+1} \exp \{ -(\lambda + \zeta + \gamma_1)t \}}{(\gamma_1 - \gamma_2)^{n+1}} + \sum_{k=1}^{n+1} \frac{(-1)^{n+k+1} t^{k-1} \exp \{ -(\lambda + \zeta + \gamma_2)t \}}{(k-1)!(\gamma_1 - \gamma_1)^{n+2-k}} \right] \\ & * \lambda^n \gamma_1 \left[A_0(t) + \mu P_{b,1}(t) + \delta(t) \right] * H(t), \end{aligned} \quad (20)$$

where “*” represents convolution and $\delta(t)$ represents Dirac delta function,

$$\begin{aligned} A_0(t) = & \eta \zeta \gamma \left[\frac{1}{(\eta + \zeta)(\lambda + \zeta + \gamma)} - \frac{\exp \{ -(\lambda + \zeta + \gamma)t \}}{(\lambda + \zeta + \gamma)(\eta - \lambda - \gamma)} + \frac{\exp \{ -(\eta + \zeta)t \}}{(\eta + \zeta)(\eta - \lambda - \gamma)} \right], \\ H(t) = & \left[\delta(t) + \frac{\gamma \gamma_1 \gamma_2}{(c_3^2 - c_3 c_1 + c_2) \sqrt{(c_1)^2 - 4c_2}} \frac{\exp \left\{ -\frac{c_1}{2} t \right\}}{2} \sqrt{\frac{\pi \left\{ (c_1)^2 - 4c_2 \right\} t}{2}} \right. \\ & \times \left. \left\{ \left(\sqrt{(c_1)^2 - 4c_2} \right) I_{-\frac{1}{2}} \left(\sqrt{\frac{(c_1)^2 - 4c_2}{4}} \right) (t) + (c_1 - 2c_3) I_{\frac{1}{2}} \left(\sqrt{\frac{(c_1)^2 - 4c_2}{4}} \right) (t) \right\} \right], \end{aligned}$$

$c_1 = 2c_3 + (\gamma + \gamma_1 + \gamma_2)$, $c_2 = c_3(\gamma + \gamma_1 + \gamma_2) + (\gamma \gamma_1 + \gamma_1 \gamma_2 + \gamma \gamma_2)$ and $c_3 = (\lambda + \zeta)$.

3.2 Evaluation of $P_{b,0}(t)$

Substituting Equation 13 in Equation 12, after some simplification

$$\begin{aligned} \hat{P}_{b,0}(s) = & \frac{\eta \zeta}{s(s + \eta + \zeta)(s + \lambda + \zeta + \gamma)} + \frac{\gamma_1 \gamma_2}{(s + \lambda + \zeta + \gamma)(s + \lambda + \zeta + \gamma_1)(s + \lambda + \zeta + \gamma_2)} \\ & \times \left[\left\{ A_0(s) + \mu \hat{P}_{b,1}(s) + 1 \right\} \left\{ 1 + \frac{\gamma \gamma_1 \gamma_2}{(s + c_3)(s^2 + c_1 s + c_2)} \right\} \right]. \end{aligned} \quad (21)$$

On inversion, we get

$$P_{b,0}(t) = \frac{A_0(t)}{\gamma} + \frac{\exp\{-(\lambda + \zeta + \gamma)t\}}{(\gamma - \gamma_1)(\gamma - \gamma_2)} + \frac{\exp\{-(\lambda + \zeta + \gamma_1)t\}}{(\gamma_1 - \gamma_2)(\gamma_1 - \gamma)} + \frac{\exp\{-(\lambda + \zeta + \gamma_2)t\}}{(\gamma_2 - \gamma)(\gamma_2 - \gamma_1)} \\ * \left[\{A_0(t) + \mu P_{b,1}(t) + \delta(t)\} * H(t) \right]. \quad (22)$$

3.3 Evaluation of $P_{b,n}(t)$

The probability $P_{b,n}(t), n = 1, 2, 3, \dots$ is obtained by applying generating function on Equation 3. Let

$$G(z, t) = \sum_{n=1}^{\infty} P_{b,n}(t) z^n, G(z, 0) = 0$$

Using Equation 3, after some simple algebra

$$\frac{\partial}{\partial t} G(z, t) - \left[-(\lambda + \mu + \zeta) + \left(\lambda z + \frac{\mu}{z} \right) \right] G(z, t) = \gamma_1 \sum_{n=1}^{\infty} P_{v_1,n}(t) z^n + \gamma_2 \sum_{n=1}^{\infty} P_{v_2,n}(t) z^n \\ + \lambda z P_{b,0}(t) - \mu P_{b,1}(t). \quad (23)$$

On Integration, we get

$$G(z, t) = \gamma_1 \int_0^t \sum_{n=1}^{\infty} P_{v_1,n}(w) z^n \exp\{-(\beta_1) + (\lambda z + \mu z^{-1})\}(t-w) dw \\ + \gamma_2 \int_0^t \sum_{n=1}^{\infty} P_{v_2,n}(w) z^n \exp\{-(\beta_1) + (\lambda z + \mu z^{-1})\}(t-w) dw \\ + \lambda z \int_0^t P_{b,0}(w) \exp\{-(\beta_1) + (\lambda z + \mu z^{-1})\}(t-w) dw \\ - \mu \int_0^t P_{b,1}(w) \exp\{-(\beta_1) + (\lambda z + \mu z^{-1})\}(t-w) dw, \quad (24)$$

where $\beta_1 = \lambda + \mu + \zeta$. Let $\kappa = 2\sqrt{\lambda\mu}$ and $v = \sqrt{\lambda\mu^{-1}}$, then

$$\exp\left[\left(\lambda z + \frac{\mu}{z}\right)(t-w)\right] = \sum_{n=-\infty}^{\infty} (vz)^n I_n(\kappa(t-w)).$$

Using the above expression in Equation 24 and equating the coefficient of z^n on both sides for $n = 1, 2, 3, \dots$, we get

$$\begin{aligned}
P_{b,n}(t) = & \gamma_1 \int_0^t \sum_{m=1}^{\infty} P_{v_1,m}(w) \exp \{-(\beta_1)(t-w)\} v^{n-m} I_{n-m}(\cdot) dw \\
& + \gamma_2 \int_0^t \sum_{m=1}^{\infty} P_{v_2,m}(w) \exp \{-(\beta_1)(t-w)\} v^{n-m} I_{n-m}(\cdot) dw \\
& + \lambda \int_0^t P_{b,0}(w) \exp \{-(\beta_1)(t-w)\} v^{n-1} I_{n-1}(\cdot) dw \\
& - \mu \int_0^t P_{b,1}(w) \exp \{-(\beta_1)(t-w)\} v^n I_n(\cdot) dw,
\end{aligned} \tag{25}$$

where $I_n(\cdot) = I_n(\kappa(t-w))$.

Equating the coefficients of z^{-n} for $n=1,2,3,\dots$ and applying $I_{-n}(\cdot) = I_n(\cdot)$, we get

$$\begin{aligned}
0 = & \gamma_1 \int_0^t \sum_{m=1}^{\infty} P_{v_1,m}(w) \exp \{-(\beta_1)(t-w)\} v^{-n-m} I_{n+m}(\cdot) dw \\
& + \gamma_2 \int_0^t \sum_{m=1}^{\infty} P_{v_2,m}(w) \exp \{-(\beta_1)(t-w)\} v^{-n-m} I_{n+m}(\cdot) dw \\
& + \lambda \int_0^t P_{b,0}(w) \exp \{-(\beta_1)(t-w)\} v^{-n-1} I_{n+1}(\cdot) dw \\
& - \mu \int_0^t P_{b,1}(w) \exp \{-(\beta_1)(t-w)\} v^{-n} I_n(\cdot) dw.
\end{aligned} \tag{26}$$

Multiplying v^{2n} on both sides of Equation 26 and subtracting from Equation 25, for $n=1,2,3,\dots$

$$\begin{aligned}
P_{b,n}(t) = & \gamma_1 \int_0^t \sum_{m=1}^{\infty} P_{v_1,m}(w) \exp \{-(\beta_1)(t-w)\} v^{n-m} [I_{n-m}(\cdot) - I_{n+m}(\cdot)] dw \\
& + \gamma_2 \int_0^t \sum_{m=1}^{\infty} P_{v_2,m}(w) \exp \{-(\beta_1)(t-w)\} v^{n-m} [I_{n-m}(\cdot) - I_{n+m}(\cdot)] dw \\
& + \lambda \int_0^t P_{b,0}(w) \exp \{-(\beta_1)(t-w)\} v^{n-1} [I_{n-1}(\cdot) - I_{n+1}(\cdot)] dw.
\end{aligned} \tag{27}$$

Taking Laplace Transform on Equation 27, we get

$$\begin{aligned}\hat{P}_{b,n}(s) = & \frac{1}{\sqrt{d^2 - \kappa^2}} \sum_{m=1}^{\infty} [\{\gamma_1 \hat{P}_{v_1,m}(s) + \gamma_2 \hat{P}_{v_2,m}(s)\} v^{n-m} \{\hat{\psi}(s)^{n-m} - \hat{\psi}(s)^{n+m}\}] \\ & + \frac{2\lambda \hat{P}_{b,0}(s)}{\kappa} v^{n-1} \hat{\psi}(s)^n,\end{aligned}\quad (28)$$

where

$$\hat{\psi}(s) = \frac{d - \sqrt{d^2 - \kappa^2}}{\kappa} \text{ and } d = s + \lambda + \mu + \zeta.$$

Substituting Equations 16, 18 and 21 in the Equation 28, we get

$$\begin{aligned}\hat{P}_{b,n}(s) = & \frac{\gamma_1}{(s + \lambda + \zeta + \gamma_1)} \left[A_0(s) + \mu \hat{P}_{b,1}(s) + 1 \right] \left[1 + \frac{\gamma \gamma_1 \gamma_2}{(s + c_3)(s^2 + c_1 s + c_2)} \right] \\ & \times \left[\frac{\gamma_1}{\sqrt{d^2 - \kappa^2}} \times \sum_{m=1}^{\infty} \left\{ \frac{\lambda^m v^{n-m}}{(s + \lambda + \zeta + \gamma_1)^m} + \frac{\gamma_2 \lambda^m v^{n-m}}{(s + \lambda + \zeta + \gamma_2)^{m+1}} \right\} \right. \\ & \times \left. \{\hat{\psi}(s)^{n-m} - \hat{\psi}(s)^{n+m}\} \right] + \frac{\gamma_2}{(s + \lambda + \zeta + \gamma)(s + \lambda + \zeta + \gamma_2)} \frac{2\lambda}{\kappa} \hat{\psi}(s)^n v^{n-1} \\ & + \frac{\eta \zeta}{s(s + \eta + \zeta)(s + \lambda + \zeta + \gamma)} \frac{2\lambda}{\kappa} \hat{\psi}(s)^n v^{n-1}.\end{aligned}\quad (29)$$

On inversion, we get

$$\begin{aligned}P_{b,n}(t) = & \gamma_1 \exp\{-(\lambda + \zeta + \gamma_1)t\} * [A(t) + \mu P_{b,1}(t) + \delta(t)] * H(t) \\ & * \left[\sum_{m=1}^{\infty} \lambda^m v^{n-m} \left\{ \frac{t^{m-1} \exp\{-(\lambda + \zeta + \gamma_1)t\}}{(m-1)!} + \frac{\gamma_2 \lambda^m t^m \exp\{-(\lambda + \zeta + \gamma_2)t\}}{m!} \right\} \right. \\ & * \{I_{n-m}(\kappa t) - I_{n+m}(\kappa t)\} \exp\{-(\beta_1)t\} + \left\{ \frac{\gamma_2 \exp\{-(\lambda + \zeta + \gamma)t\}}{(\gamma_2 - \gamma)} \right. \\ & + \left. \frac{\gamma_2 \exp\{-(\lambda + \zeta + \gamma_2)t\}}{(\gamma - \gamma_2)} \right\} * \frac{2n I_n(\kappa t)}{\kappa t} \exp\{-(\beta_1)t\} \Big] \\ & + \frac{A_0(t)}{\gamma} \frac{2n I_n(\kappa t)}{\kappa t} \exp\{-(\beta_1)t\}.\end{aligned}\quad (30)$$

where $I_m(t)$ represents the modified Bessel function of the first kind of order m with $\kappa = 2\sqrt{\lambda\mu}$ and $v = \sqrt{\lambda\mu^{-1}}$.

3.4 Remark

If $\gamma_2 = 0$ and considering $\gamma_1 = \gamma$ then the result 27 coincide with the result (3.8) of Sudhesh et al. (2017).

3.5 Evaluation of $P_{b,1}(t)$

Setting $n = 1$ in Equation 30 and using the identity of modified Bessel function of first kind, we get

$$\begin{aligned}
P_{b,1}(t) &= 2\gamma_1 \int_0^t \sum_{m=1}^{\infty} P_{v_1,m}(w) \exp\{-(\beta_1)(t-w)\} m v^{1-m} \frac{I_m(\kappa(t-w))}{\kappa(t-w)} dw \\
&+ 2\gamma_2 \int_0^t \sum_{m=1}^{\infty} P_{v_2,m}(w) \exp\{-(\beta_1)(t-w)\} m v^{1-m} \frac{I_m(\kappa(t-w))}{\kappa(t-w)} dw \\
&+ 2\lambda \int_0^t P_{b,0}(w) \exp\{-(\beta_1)(t-w)\} \frac{I_1(\kappa(t-w))}{\kappa(t-w)} dw.
\end{aligned} \quad (31)$$

Applying Laplace transform on Equation 31, after some algebraic simplification, we get

$$\hat{P}_{b,1}(s) = \frac{\gamma_1}{\mu} \sum_{m=1}^{\infty} \hat{P}_{v_1,m}(s) \frac{1}{v^m} \hat{\psi}(s)^m + \frac{\gamma_2}{\mu} \sum_{m=1}^{\infty} \hat{P}_{v_2,m}(s) \frac{1}{v^m} \hat{\psi}(s)^m + \hat{P}_{b,0}(s) v \hat{\psi}(s).$$

Substituting Equations 16, 18 and 21 in the above Equation, after some algebraic manipulation, we get

$$\hat{P}_{b,1}(s) = \hat{G}_2(s) \sum_{h=0}^{\infty} [\hat{G}_1(s)]^h, \quad (32)$$

where

$$\begin{aligned}
G_1(s) &= \frac{\gamma_1}{(s + \lambda + \zeta + \gamma_1)} \left\{ 1 + \frac{\gamma_1 \gamma_2}{(s + c_3)(s^2 + c_1 s + c_2)} \right\} \left[\sum_{m=1}^{\infty} \left\{ \frac{\hat{\psi}(s)^m}{(s + \lambda + \zeta + \gamma_1)^m} \right. \right. \\
&\quad \left. \left. + \frac{\gamma_2 v^{1-m}}{(s + \lambda + \zeta + \gamma_2)^{m+1}} \right\} \left(\frac{\lambda}{v} \right)^m \right] + \frac{\mu v \gamma_2 \hat{\psi}(s)}{(s + \lambda + \zeta + \gamma)(s + \lambda + \zeta + \gamma_2)}, \\
\hat{G}_2(s) &= \frac{[\hat{A}_0(s) + 1] \hat{G}_1(s)}{\mu} + \frac{\hat{A}_0(s) v \hat{\psi}(s)}{\gamma}.
\end{aligned}$$

Applying inversion on Equation 32, we get

$$P_{b,1}(t) = G_2(t) * \sum_{h=0}^{\infty} G_1(t)^{*h}, \quad (33)$$

where

$$\begin{aligned}
G_1(t) &= \mu \gamma_1 \exp\{-(\lambda + \zeta + \gamma_1)t\} * H(t) * \left[\sum_{m=1}^{\infty} \left\{ \frac{t^{m-1} \exp\{-(\lambda + \zeta + \gamma_1)t\}}{(m-1)!} \right. \right. \\
&\quad \left. \left. + \gamma_2 \frac{\lambda^m \exp\{-(\lambda + \zeta + \gamma_2)t\} t^m}{m!} \right\} * v^{1-m} I_{m-1}(\kappa t) - I_{m+1}(\kappa t) \exp\{-\beta_1 t\} \right] \\
&\quad + \frac{[\exp\{-(\lambda + \zeta + \gamma)t\} - \exp\{-(\lambda + \zeta + \gamma_2)t\}]}{(\gamma_2 - \gamma)} * [I_0(\kappa t) - I_1(\kappa t)] \exp\{-\beta_1 t\} \\
G_2(t) &= \frac{\lambda A_0(t)}{\gamma} * [I_0(\kappa t) - I_1(\kappa t) \exp\{-\beta_1 t\}] + \frac{1}{\mu} [A_0(t) + \delta(t)] * G_1(t).
\end{aligned}$$

3.6 Evaluation of $P_F(t)$

Applying Laplace transform on Equation 1, we get

$$\hat{P}_F(s) = \frac{\zeta}{s(s + \zeta + \eta)}. \quad (34)$$

Inversion on Equation 34 gives

$$P_F(t) = \frac{\zeta}{\eta + \zeta} [1 - \exp \{-(\eta + \zeta)t\}]. \quad (35)$$

4 Performance Measures

The time-dependent mean, variance of the system size, probability that the server is on stage-I and stage-II vacation and system throughput are presented in this section.

4.1 Mean

Let $M(t)$ denote the expected system size at time t . For $t > 0$, we have

$$M(t) = E[X(t)] = \sum_{n=1}^{\infty} n P_{b,n}(t) + \sum_{n=1}^{\infty} n \left(\sum_{i=1}^2 P_{v_i,n}(t) \right).$$

Then using Equations 1-7, we get

$$\frac{d}{dt}(M(t)) = \lambda(1 - P_F(t)) - \mu \sum_{n=1}^{\infty} P_{b,n}(t) - \zeta \sum_{n=1}^{\infty} n P_{b,n}(t) - \zeta \sum_{n=1}^{\infty} n \left(\sum_{i=1}^2 P_{v_i,n}(t) \right).$$

The above equation gives,

$$\begin{aligned} M(t) = & \lambda t - \lambda \int_0^t P_F(y) dy - \mu \sum_{n=1}^{\infty} \int_0^t P_{b,n}(y) dy \\ & - \zeta \sum_{n=1}^{\infty} n \left[\int_0^t P_{b,n}(y) dy + \sum_{i=1}^2 \left(\int_0^t P_{v_i,n}(y) dy \right) \right]. \end{aligned} \quad (36)$$

4.2 Variance

Let $V(t)$ denote the variance system size at time t . For $t > 0$,

$$V(t) = E[X^2(t)] - (E[X(t)])^2,$$

where

$$E[X^2(t)] = \sum_{n=1}^{\infty} n^2 P_{b,n}(t) + \sum_{n=1}^{\infty} n^2 \left(\sum_{i=1}^2 P_{v_i,n}(t) \right).$$

Using Equations 1-7, we obtain

$$\begin{aligned} \frac{d}{dt}(E[X^2(t)]) &= 2\lambda M(t) + \lambda(1 - P_F(t)) - \mu \sum_{n=1}^{\infty} (2n-1)P_{b,n}(t) - \zeta \sum_{n=1}^{\infty} n^2 P_{b,n}(t) \\ &\quad - \zeta \sum_{n=1}^{\infty} n^2 \left(\sum_{i=1}^2 P_{v_i,n}(t) \right). \end{aligned}$$

Then

$$\begin{aligned} E[X^2(t)] &= \lambda t + 2\lambda \int_0^t M(y)dy - \int_0^t P_F(y)dy - \mu \sum_{n=1}^{\infty} (2n-1) \int_0^t P_{b,n}(y)dy \\ &\quad - \zeta \sum_{n=1}^{\infty} n^2 \left[\int_0^t P_{b,n}(y)dy + \sum_{i=1}^2 \left(\int_0^t P_{v_i,n}(y)dy \right) \right]. \end{aligned} \quad (37)$$

The results 19, 20 and 30 gives the probabilities of $P_{v_1,n}(t)$, $P_{v_2,n}(t)$ and $P_{b,n}(t)$ respectively.

4.3 Probability That the Server is On Vacation

Let $P_{v_1,\bullet}(t)$ and $P_{v_2,\bullet}(t)$ denote the probability that the server is on the stage-I and stage-II vacation respectively, then

$$\hat{P}_{v_i,\bullet}(s) = \sum_{n=0}^{\infty} \hat{P}_{v_i,n}(s), i = 1, 2.$$

Using Equations 16 and 18 in the above expression and taking inversion, we obtain

$$\begin{aligned} P_{v_1,\bullet}(t) &= \exp\{-(\lambda + \zeta + \gamma_1)t\} * \left\{ A_0(t) + \mu P_{b,1}(t) + \delta(t) \right\} * \left\{ \delta(t) + \lambda \exp\{-(\zeta + \gamma_1)t\} \right\} \\ &\quad * H(t), \end{aligned}$$

$$\begin{aligned} P_{v_2,\bullet}(t) &= \frac{\gamma_1}{\gamma_1 - \gamma_2} \left[\exp\{-(\lambda + \zeta + \gamma_2)t\} - \exp\{-(\lambda + \zeta + \gamma_1)t\} \right] * H(t) \\ &\quad * \left\{ A_0(t) + \mu P_{b,1}(t) + \delta(t) \right\} * \left\{ \delta(t) + \lambda \exp\{-(\zeta + \gamma_2)t\} \right\}. \end{aligned}$$

4.4 Probability That the Server is Busy

Let $P_{b,\bullet}(t)$ denote that the server is on busy mode, then

$$\hat{P}_{b,\bullet}(s) = \sum_{n=1}^{\infty} \hat{P}_{b,n}(s).$$

Using Equation 29 in the above expression and taking inverse Laplace transform, we obtain

$$\begin{aligned} P_{b,\bullet}(t) = & \sum_{n=1}^{\infty} \left[\gamma_1 \sum_{m=1}^{\infty} P_{v_1,m}(t) * \exp \{ -\beta_1 t \} v^{n-m} [I_{n-m}(\kappa t) - I_{n+m}(\kappa t)] \right. \\ & + \gamma_2 \sum_{m=1}^{\infty} P_{v_2,m}(t) * \exp \{ -\beta_1 t \} v^{n-m} [I_{n-m}(\kappa t) - I_{n+m}(\kappa t)] \\ & \left. + \lambda P_{b,0}(t) * \exp \{ -\beta_1 t \} v^{n-1} [I_0(\kappa t) - I_2(\kappa t)] \right], \end{aligned}$$

where the expression for $P_{v_1,m}(t)$, $P_{v_2,m}(t)$ and $P_{b,0}(t)$ is given in the results 19, 20 and 22 respectively.

4.5 System Throughput

Let $H(t)$ denote the throughput of the system at time t , then

$$H(t) = \left[1 - \left(\sum_{n=0}^{\infty} P_{v_1,n}(t) + \sum_{n=0}^{\infty} P_{v_2,n}(t) + \pi_F(t) \right) \right] \mu.$$

5 Steady-State Probabilities

The steady-state probabilities of the system are presented in this section. Let $\lim_{t \rightarrow \infty} P_{in}(t) = \pi_{in}$ and $\lim_{t \rightarrow \infty} P'_{in}(t) = 0$. Hence, Equations 1-7 becomes

$$0 = -\eta\pi_F + \zeta \left(\sum_{n=0}^{\infty} \pi_{b,n} + \sum_{n=0}^{\infty} \pi_{v_1,n} + \sum_{n=0}^{\infty} \pi_{v_2,n} \right) \quad (38)$$

$$0 = -(\lambda + \zeta + \gamma)\pi_{b,0} + \eta\pi_F + \gamma_2\pi_{v_2,0} \quad (39)$$

$$0 = -(\lambda + \mu + \zeta)\pi_{b,n} + \lambda\pi_{b,n-1} + \mu\pi_{b,n+1} + \gamma_1\pi_{v_1,n} + \gamma_2\pi_{v_2,n}, n = 1, 2, 3, \dots \quad (40)$$

$$0 = -(\lambda + \zeta + \gamma_1)\pi_{v_1,0} + \gamma\pi_{b,0} + \mu\pi_{b,1} \quad (41)$$

$$0 = -(\lambda + \zeta + \gamma_1)\pi_{v_1,n} + \lambda\pi_{v_1,n-1}, n = 1, 2, 3, \dots \quad (42)$$

$$0 = -(\lambda + \zeta + \gamma_2)\pi_{v_2,0} + \gamma_1\pi_{v_1,0} \quad (43)$$

$$0 = -(\lambda + \zeta + \gamma_2)\pi_{v_2,n} + \lambda\pi_{v_2,n-1}, n = 1, 2, 3, \dots \quad (44)$$

Using Equations 16, 18 and 21, we get

$$\pi_{v_1,n} = A_1(b_1)^n + A_2(b_1)^n\pi_{b,1}, \quad (45)$$

$$\pi_{v_2,n} = A_3(b_2)^n + A_4(b_2)^n\pi_{b,1}, \quad (46)$$

$$\pi_{b,0} = c_1 + c_2\pi_{b,1}, \quad (47)$$

where

$$A_1 = \frac{\zeta\eta\gamma}{(1-A_5)(\lambda + \zeta + \gamma_1)(\eta + \zeta)(\lambda + \zeta + \gamma)}, A_2 = \frac{\mu}{(1-A_5)(\lambda + \zeta + \gamma_1)},$$

$$A_3 = \frac{A_5\zeta\eta}{(1-A_5)\gamma_2(\eta + \zeta)}, A_4 = \frac{\gamma_1\mu}{(1-A_5)(\lambda + \zeta + \gamma_1)(\lambda + \zeta + \gamma_2)},$$

$$A_5 = \frac{\gamma\gamma_1\gamma_2}{(\lambda + \zeta + \gamma)(\lambda + \zeta + \gamma_1)(\lambda + \zeta + \gamma_2)},$$

$$b_1 = \left(\frac{\lambda}{\lambda + \zeta + \gamma_1}\right), b_2 = \left(\frac{\lambda}{\lambda + \zeta + \gamma_2}\right), c_1 = \frac{\zeta\eta}{(\eta + \zeta)(\lambda + \zeta + \gamma)(1-A_5)}, c_2 = \frac{A_5\mu}{\gamma(1-A_5)}$$

provided $|A_5| < 1$. The steady-state probability $\pi_{b,1}$ can be obtained as follows. Define

$$G_z(n) = \sum_{n=1}^{\infty} \pi_{b,n}z^n, |z| \leq 1.$$

From Equation 40, after some algebraic manipulation, we get

$$(\lambda z^2 - (\lambda + \mu + \zeta)z + \mu)G_z(n) = -\lambda z^2\pi_{b,0} + \mu z\pi_{b,1} - \gamma_1 z \sum_{n=1}^{\infty} \pi_{v_1,n}z^n - \gamma_2 z \sum_{n=1}^{\infty} \pi_{v_2,n}z^n. \quad (48)$$

Applying 45, 46 and 47 in the above Equation,

$$\begin{aligned} ((z - \alpha_1)(z - \alpha_2))G_z(n) = & -\lambda z^2(c_1 + c_2\pi_{b,1}) + \mu z\pi_{b,1} - \gamma_1 z \left(\sum_{n=1}^{\infty} A_1(zb_1)^n + A_2(zb_1)^n\pi_{b,1} \right) \\ & - \gamma_2 z \left(\sum_{n=1}^{\infty} A_3(zb_2)^n + A_4(zb_2)^n\pi_{b,1} \right), \end{aligned} \quad (49)$$

where

$$\alpha_1 = \frac{(\lambda + \mu + \zeta) + \sqrt{(\lambda + \mu + \zeta)^2 - 4\lambda\mu}}{2\lambda}$$

and

$$\alpha_2 = \frac{(\lambda + \mu + \zeta) - \sqrt{(\lambda + \mu + \zeta)^2 - 4\lambda\mu}}{2\lambda}.$$

It is noted that $\alpha_2 < 1$ for any $\lambda > 0$, $\mu > 0$ and $\zeta > 0$. This can be proved by assuming $\alpha_2 > 1$ which will result in a contraction $-4\lambda\zeta > 0$. It can be also seen instinctively as follows: if $\zeta > 0$, the disaster will eventually occur with probability one and prevents the process from escaping to infinity immaterial for any $\lambda > 0$ and $\mu > 0$. Hence at $z = \alpha_2$, Equation 49 gives

$$\pi_{b,1} = \frac{\lambda\alpha_2^2c_1 + \frac{\gamma_1\alpha_2^2b_1A_1}{1-\alpha_2b_1} + \frac{\gamma_2\alpha_2^2b_2A_3}{1-\alpha_2b_2}}{\mu\alpha_2 - \lambda\alpha_2^2c_2 - \frac{\gamma_1\alpha_2^2b_1A_2}{1-\alpha_2b_1} - \frac{\gamma_2\alpha_2^2b_2A_4}{1-\alpha_2b_2}}. \quad (50)$$

Using the results 45-47 in Equation 38, we get

$$\sum_{n=1}^{\infty} \pi_{b,n} = \frac{\eta\pi_F}{\zeta} - c_1 - \frac{A_1}{1-b_1} - \frac{A_3}{1-b_2} - \left(c_2 + \frac{A_2}{1-b_1} + \frac{A_4}{1-b_2}\right)\pi_{b,1} \quad (51)$$

and

$$\pi_F = \frac{\zeta}{\zeta + \eta}. \quad (52)$$

6 Steady-State Performance Measures

The performance measures like mean system size, probability that the server is on stage-I and stage-II vacation, probability that the server is busy, system throughput are derived in this section.

6.1 Mean System Size

Let $E[N_s]$ be the mean number of jobs in the system in the steady-state and let $E[N_b]$, $E[N_{v_1}]$ and $E[N_{v_2}]$ denote mean number of jobs in busy state, stage-I vacation and stage-II vacation respectively. Then,

$$E[N_s] = E[N_b] + E[N_{v_1}] + E[N_{v_2}],$$

where

$$E[N_{v_1}] = \sum_{n=0}^{\infty} n \pi_{v_1,n} = \frac{A_1b_1}{(1-b_1)^2} + \frac{A_2b_1}{(1-b_1)^2}\pi_{b,1}$$

and

$$E[N_{v_2}] = \sum_{n=0}^{\infty} n \pi_{v_2,n} = \frac{A_3 b_2}{(1-b_2)^2} + \frac{A_4 b_2}{(1-b_2)^2} \pi_{b,1}.$$

It is well known that

$$E[N_b] = \lim_{z \rightarrow 1} G'_z(n).$$

Using Equation 48, we get

$$\begin{aligned} E[N_b] &= \frac{1}{\zeta} \left[\lambda(c_1 + c_2 \pi_{b,1}) + \frac{\gamma_1 b_1}{(1-b_1)^2} (A_1 + A_2 \pi_{b,1}) + \frac{\gamma_2 b_2}{(1-b_2)^2} (A_3 + A_4 \pi_{b,1}) \right] + \frac{(\lambda - \mu)}{\zeta^2} \\ &\quad \times \left[\lambda(c_1 + c_2 \pi_{b,1}) - \mu \pi_{b,1} + \frac{\gamma_1 b_1}{(1-b_1)} (A_1 + A_2 \pi_{b,1}) + \frac{\gamma_2 b_2}{(1-b_2)} (A_3 + A_4 \pi_{b,1}) \right]. \end{aligned}$$

Using Little's formula,

$$E[W_s] = \frac{E[N_s]}{\lambda}.$$

The mean number of data frames waiting in the queue during the steady-state is given by

$$E[W_q] = \sum_{n=0}^{\infty} n \pi_{v_1,n} + \sum_{n=0}^{\infty} n \pi_{v_2,n} + \sum_{n=1}^{\infty} (n-1) \pi_{b,n}.$$

6.2 Probability That the Server is on Vacation

Let $\pi_{v_1,\bullet}$ and $\pi_{v_2,\bullet}$ denote the probability that the server is in stage-I and stage-II vacation respectively during the steady state, then

$$\pi_{v_i,\bullet} = \sum_{n=0}^{\infty} \pi_{v_i,n}, i = 1, 2.$$

Using Equations 45 and 46 in the above expression, we get

$$\pi_{v_1,\bullet} = \frac{A_1 + A_2 \pi_{b,1}}{1 - b_1},$$

$$\pi_{v_2,\bullet} = \frac{A_3 + A_4 \pi_{b,1}}{1 - b_2}.$$

6.3 Probability That the Server is Busy

Let $\pi_{b,\bullet}$ denote the probability that the server is in busy mode during the steady state, then

$$\pi_{b,\bullet} = \sum_{n=1}^{\infty} \pi_{b,n}.$$

Applying the results 45-47 in Equation 38, we get

$$\sum_{n=1}^{\infty} \pi_{b,n} = \frac{\eta\pi_F}{\zeta} - c_1 - \frac{A_1}{1-b_1} - \frac{A_3}{1-b_2} - \left(c_2 + \frac{A_2}{1-b_1} + \frac{A_4}{1-b_2} \right) \pi_{b,1}.$$

The expression for $\pi_{b,1}$ is given in Equation 50.

6.4 Probability That Server is Idle State or Vacation State or Failure State

Let $\pi_{D,\bullet}$ denote the probability that the server is in idle state or vacation state or failure state during the steady state, then

$$\pi_{D,\bullet} = \sum_{n=0}^{\infty} \pi_{v_1,n} + \sum_{n=0}^{\infty} \pi_{v_2,n} + \pi_F.$$

Using the results 45-47 and 52, we get

$$\pi_{D,\bullet} = c_1 + \frac{A_1}{1-b_1} + \frac{A_3}{1-b_2} + \frac{\zeta}{\zeta + \eta} + \left(c_2 + \frac{A_2}{1-b_1} + \frac{A_4}{1-b_2} \right) \pi_{b,1}.$$

6.5 System Throughput

Let π_H denote the throughput of the system then

$$\pi_H = [1 - \pi_{D,\bullet}] \mu.$$

7 Numerical Illustrations

This section presents numerical illustration to demonstrate how the proposed system is useful in analysing the effect of DoS attack in WiMAX systems.

7.1 Numerical Illustrations On Transient Probabilities

The graph of $P_{b,n}(t)$, $P_{v_1,n}(t)$, $P_{v_2,n}(t)$, $P_F(t)$, $M(t)$ and $V(t)$ of the system are plotted using MATLAB software and the effect of change in certain parameters over the aforementioned performance measures are studied. To plot the graph, the value of the parameters are chosen as follows $\lambda = 1$, $\mu = 1.25$, $\zeta = 0.01$, $\gamma = 0.8$, $\gamma_1 = 0.4$, $\gamma_2 = 0.7$ and $\eta = 0.1$. Figure 2 presents the behaviour of stage-I vacation $P_{v_1,n}(t)$. Initially, it is assumed that $P_{v_1,0}(0) = 1$ and therefore it is evident from the graph that the probability curve of $P_{v_1,0}(t)$ begins at 1 and then the curve decreases as 't' increases and attains the steady-state. Rest of the curves of $P_{v_1,0}(t)$ starts at zero and reaches the steady-state as

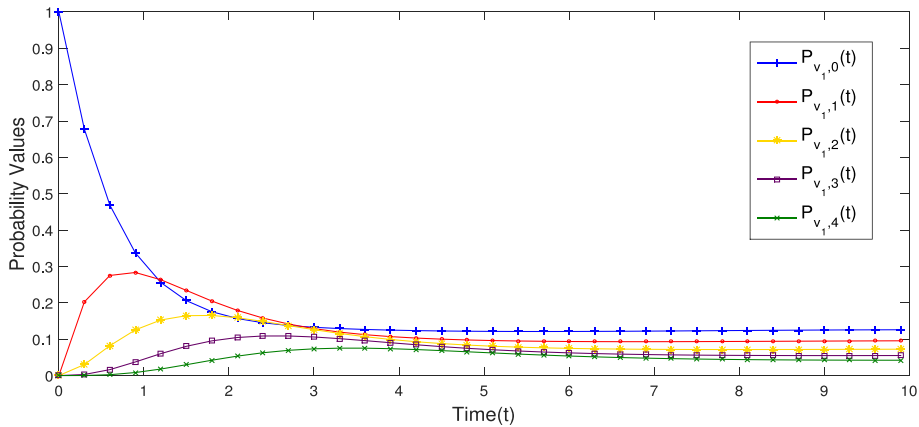


Fig. 2 Probabilities of the stage-I vacation $P_{v1,n}(t)$

' t ' increases. Figure 3 presents the behaviour of stage-II vacation $P_{v2,n}(t)$. All the probability curves of $P_{v2,n}(t)$ starts at zero and reaches the steady-state as ' t ' increases. From Figs. 2 and 3, we observe that the probability of the number of data frames at any time instant ' t ' in stage-I vacation is greater than the stage-II vacation. Since the vacation duration of stage-I vacation is greater than the stage-I vacation, the server in stage-II vacation will quickly turn to busy mode after completing the vacation. Hence the jobs need not wait for more time for service when the server is in stage-II vacation. Figure 4 exemplifies the behaviour of busy state $P_{b,n}(t)$. From this graph, it is observed that all probability curves start at 0 and increases to a certain extent as ' t ' increases and reaches a steady-state. Figure 5 presents the behaviour of failure state F for varying values of disaster rate ζ . We notice that as ζ increases, the probability that the system in failure state is also increases. Figures 6 and 7 present the mean and the variance of the system. It is evident from the graph that, as the disaster rate increases the mean system size decreases. Hence, we conclude that the increase in DoS attack in the WiMAX system leads to signal loss in the system.

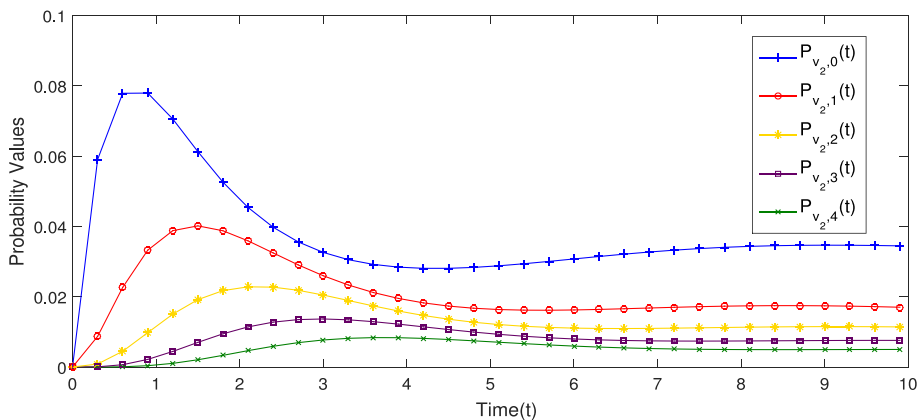


Fig. 3 Probabilities of the stage-II vacation $P_{v2,n}(t)$

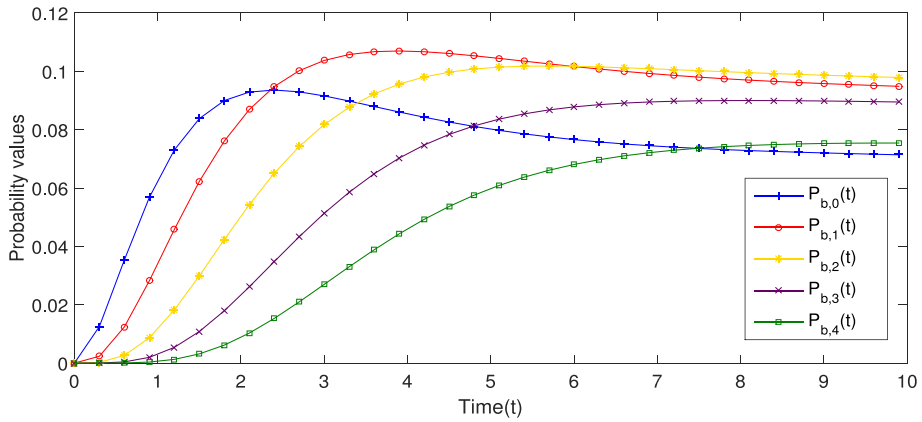


Fig. 4 Probabilities of the busy-state $P_{b,n}(t)$

7.2 Numerical Illustrations On Steady-State Probabilities

To plot the graph, we choose the same parameter values as chosen in Section 7.1. As the result, we get $A_1 = 0.003003886541$, $A_2 = 0.9344903291$, $A_3 = 0.005621308149$, $A_4 = 0.2185942290$, $A_5 = 0.05132798561$, $b_1 = 0.7092198582$, $b_2 = 0.5847953216$, $c_1 = 0.005294350030$, $c_2 = 0.08453920459$.

Substituting these values in Equations 45-47, 50-52 we get

$$\pi_{v_1,n} = 0.0030038865(0.7092198582)^n + 0.0766823533(0.7092198582)^n,$$

$$\pi_{v_2,n} = 0.0056213085(0.5847953216)^n + 0.01793739260(0.5847953216)^n,$$

$$P_{b,0} = 0.0057287935,$$

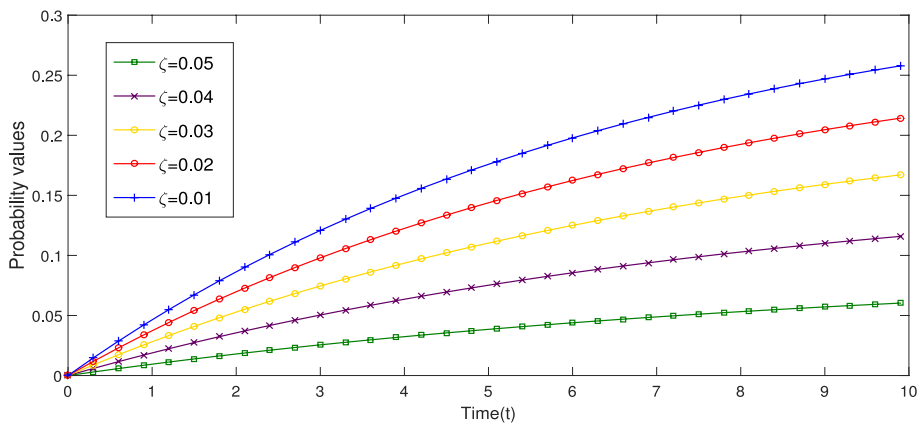


Fig. 5 Probabilities of the failure state $P_F(t)$

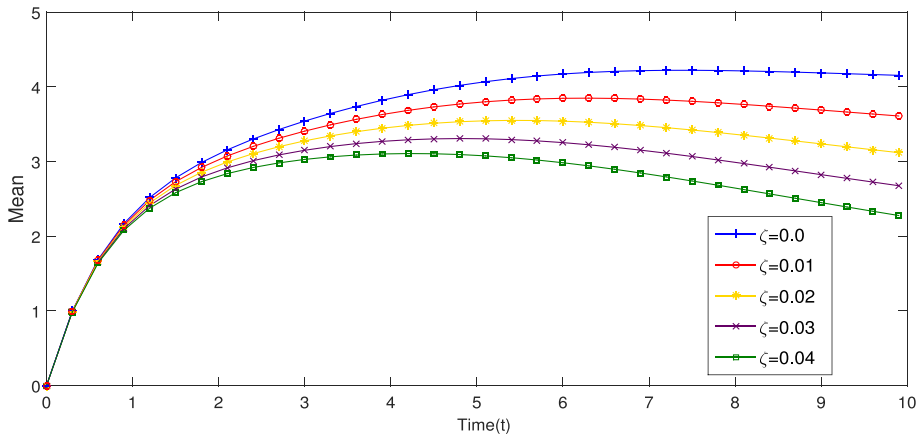


Fig. 6 Mean system size for varying disaster rates ζ

$$\sum_{n=1}^{\infty} \pi_{b,n} = 0.5725792235,$$

$$\pi_F = 0.0909090909.$$

Figures 8 and 9 present the graph of $\pi_{v_1,n}$ and $\pi_{v_2,n}$ respectively for varying values of disaster rate ζ . It is evident from the graph that, increase in disaster rate leads to a decrease in system size and finally converges to zero. The rate of convergence of stage II vacation $\pi_{v_2,n}$ is faster than the stage-I vacation $\pi_{v_1,n}$. The steady-state probability of $\pi_{b,1}$ against λ for varying values of ζ is plotted in Fig. 10. Figure 11 reveals that the system size decreases with an increase in disaster rate. Hence from the numerical illustrations, we observe that the size of the data frame in the WiMAX system decreases as the rate of DoS attack increases.

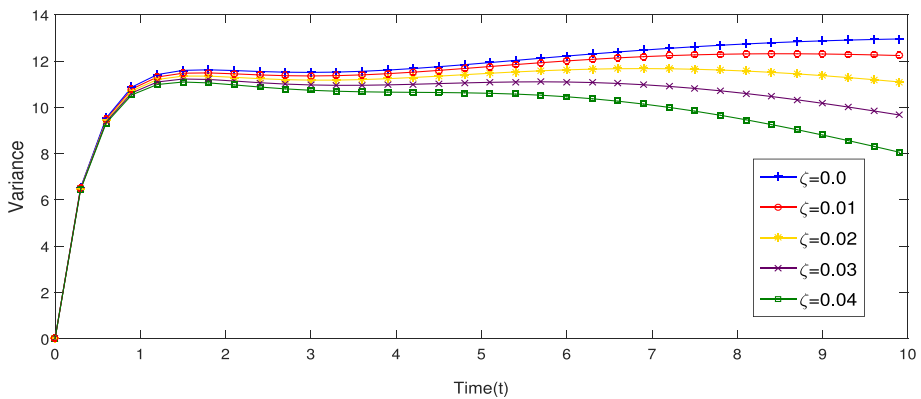


Fig. 7 Variance of the system for varying disaster rates ζ

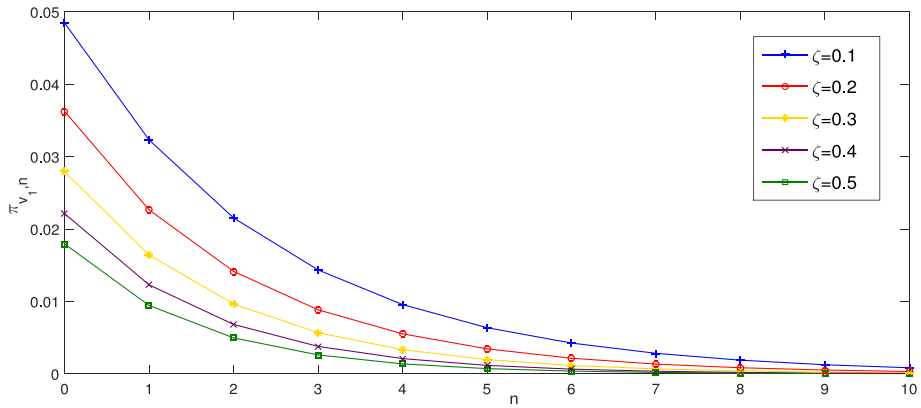


Fig. 8 Steady-state probability of stage-I vacation varying disaster rates ζ

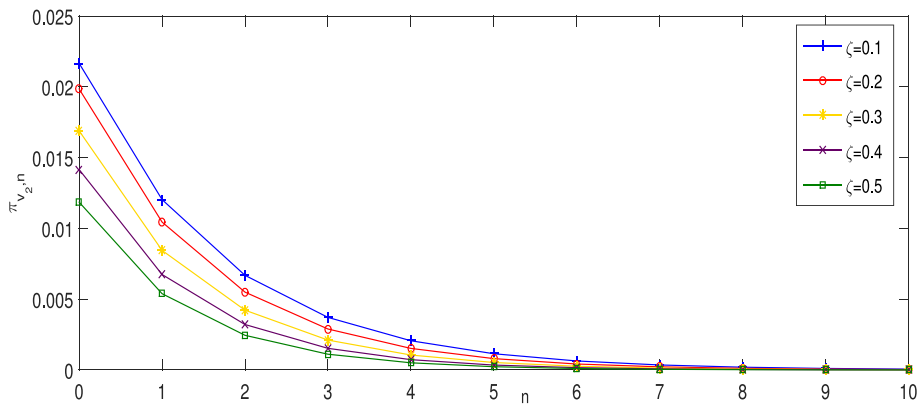


Fig. 9 Steady-state probability of stage-II vacation varying disaster rates ζ

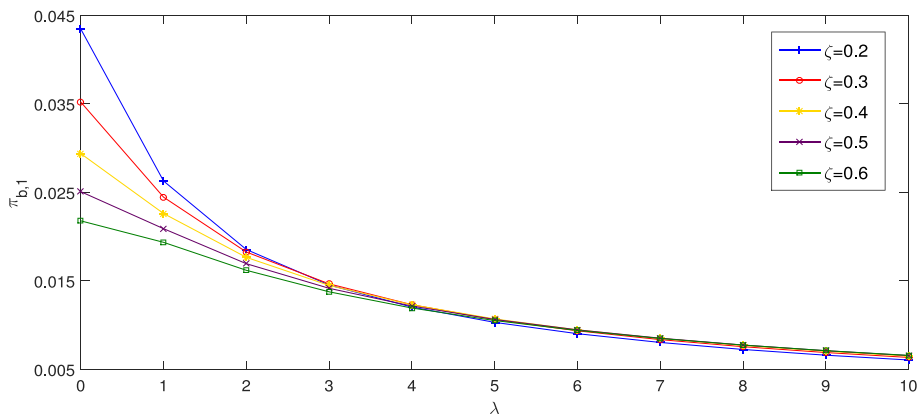


Fig. 10 Steady-state probability of $\pi_{b,1}$ against λ for varying values of γ_1

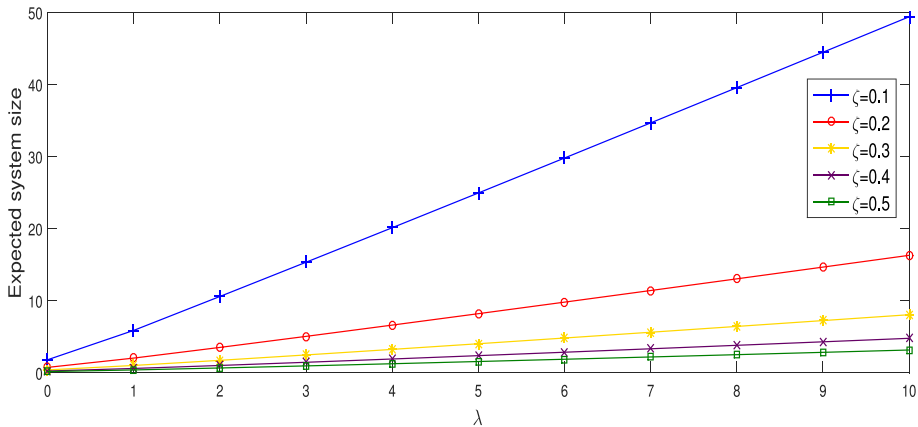


Fig. 11 Mean system size against λ varying disaster rates ζ

8 Conclusion and Future Work

A dual-stage vacation queueing system with disaster and subsequent repair is investigated in this paper. In the proposed model, the dual-stage vacation and disaster are compared with the sleep-modes and DoS attack in the WiMAX system. For this system, a closed-form solution is obtained for both the transient and steady-state probability. Further, some performance indices of the system are computed. The results obtained enable the system designers to understand the behaviour of the WiMAX system when rate of uplink/down-link data frame changes. From the results, we conclude that the increase in Dos attack in the WiMAX leads to loss of data frame in the system. This work may be extended to a dual-stage $M/M/1$ queueing model, retrial queueing systems with single and multiple servers. One can also extend this work to a non-Markovian queueing models.

Acknowledgements We thank the editor and the anonymous reviewers for their insightful comments, which helped in improving the paper. This research work was carried out during the visit of R. Sudhesh to IIT Delhi. R. Sudhesh gratefully acknowledges for the support received from the Department of Mathematics, IIT Delhi.

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