



Transient Analysis of a Modified Differentiated Vacation Queueing System for Energy-Saving in WiMAX

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Received: 1 May 2023 / Revised: 12 June 2024 / Accepted: 24 June 2024

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Abstract

A modified differentiated vacation queueing system with a close-down period and impatient customers is investigated in this research paper. The proposed model can be applied to study the energy management mechanisms of WiMAX and tethered high altitude platform (HAP) systems. In this system, the WiMAX mobile station (MS) switches to a close-down period for a random duration at the end of the busy period. At the end of the close-down period, if the system is non-empty, the MS switches to a functional state. Otherwise, the MS takes an ordinary vacation. The MS switches to a working vacation (WV) if the system is empty at the end of the ordinary vacation period. Jobs arriving during the WV are served at a slower service rate. Jobs may become impatient in any state of the system except during the regular busy period. Explicit expressions for the transient and steady-state probabilities of this system are obtained using continued fraction and confluent hypergeometric functions. Furthermore, performance indices such as mean, variance, average abandonment rate, and system throughput are computed. Finally, a cost-profit analysis and graphical illustrations are provided.

Keywords Single server · Close-down period · Working vacation · Impatient customers · Transient probabilities · Confluent hypergeometric function · Incomplete gamma function

1 Introduction

Today, energy costs and power consumption of mobile stations (MS) are essential to several service providers and account for a significant portion of their operating expenses. Sleep mode is one of the common energy-saving features for MS power conservation. To enhance

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the battery life of MS, the IEEE 802.16e introduced three types of power-saving classes (PSCs): PSC-I, PSC-II and PSC-III Tsao and Chen (2008). Each of these PSCs has a different sleep-mode mechanism and their functions can be compared with the queueing system with vacation. This paper analyse the PSC-II using modified differentiated vacation with a close-down period and impatient customers.

The PSC-II consists of three states: functional, sleep and a listening window. The MS communicates with the BS a request to enter sleep mode at the end of the functional period. After getting the response from the BS, the MS enters the first sleep window with duration T_{V_1} . At the end of T_{V_1} , if no data packets are waiting for service, the MS switches to the listening window of duration T_L . During the listening window, if any data packets are buffered, the MS resumes operation to receive the data packets. Otherwise, MS shifts to the energy-saving T_{V_1} sleep mode. The PCS-II sleep cycle is represented by the duration $T_V = T_{V_1} + T_L$. The PSCs also allow the service provider to meet its customers' quality of service requirements while offering longer service. Another potential application of the vacation system can be observed in the tethered HAP system Renga and Meo (2022). The HAP systems function as aerial Base Stations (BSs), representing a promising and adaptable solution for providing additional capacity that can be used flexibly to facilitate terrestrial BSs sleep modes, ultimately reducing energy consumption and making the network more sustainable. The sleep period of HAP can be treated as the vacation period of the system. The system rejuvenates itself during the vacation period and promises effective power saving in that particular period.

To make the best use of idle time, the queueing system incorporates the idea of a vacation. See Tian and Zhang (2006), Doshi (1986), Choudhury (2002) and references therein for in-depth studies on vacation queueing systems. The vacation policy can be classified as single, multiple and working vacations (WV). The jobs are not served during the vacation period under the traditional vacation queueing systems. Jobs must wait in the buffer until the server returns from its vacation. Alouf et al. (2008) applied an $M/G/1$ queueing model with an inhomogeneous vacation scheme to study the PSC-I. Baek et al. (2012) analysed the functions of WiMAX using an $M/G/1$ vacation queueing model. The authors concluded that there is a trade-off between power usage and downlink data packet delay. Lee et al. (2015) investigated the energy management mechanism of IEEE 802.16m using $M/D/1$ queueing model with vacation.

During the server vacation, it is observed that loss of data packets occurs in the WiMAX due to long waiting. In order to examine the loss of work on vacation, the queueing theory introduces the idea of WV policy. In the WV policy, jobs arriving during vacation are served at a different rate. This vacation policy reduces the chances of customers getting impatient due to long waiting. Servi and Finn (2002) were the first researchers to examine WV policy in the queueing system. The authors analysed WDM optical access network using the WV policy. Later, Liu et al. (2007) investigated an $M/M/1/WV$ queueing system using a quasi-birth-death process and obtained the system's stochastic decomposition and performance measures. Wu and Takagi (2006) extended the work of Servi and Finn to an $M/G/1/WV$ queueing model. Tian et al. (2008) applied the matrix geometry method to obtain the system size probabilities and the virtual time of an $M/M/1/WV$ queueing model. Zhang and Zhou (2017) analysed the Markovian single server queueing system with m different WV. Subsequently, many researchers analysed WV with impatient customers. Customers' impatience or signal loss in the queueing system due to long waiting times, server unavailability upon customer arrival, or buffer overflows.

Many researchers analysed WV policy with impatient customers. Sudhesh et al. (2017) derived the time-dependent solution of an $M/M/1$ queueing system with multiple WV and impatient customers. Zhang et al. (2013) studied the balking strategies of an $M/M/1$ multiple

WV queueing system. Laxmi and Rajesh (2017) studied an $M/M/1$ queueing system with variant WV and impatient customers. Using an $M/M/1/N$ vacation queueing system with impatient consumers, Misra and Goswami (2015) analysed PSC-II of IEEE.16e and provided steady-state results.

In the multiple vacation policy, if no job is waiting at the end of the vacation period, the server takes another vacation of the same duration. However, in many practical situations, the first vacation's duration differs from the other vacations. Such a type of vacation is called differentiated vacation (DV). In this vacation scheme, the server takes two types of vacations: Type-I and Type-II. The duration of type-I and type-II vacations are different. At the end of the busy period, the server switches to type-I vacation for a random time. If the system is non-empty at the end of the type-I vacation period, the server switches to the regular busy period. Otherwise, the server takes the type-II vacation. This type of vacation scheme is applied in WiMAX. The notion of DV in the queueing system was first studied by Ibe and Isijola (2014). The authors applied global and local balance equations to study the steady-state behaviour of the system. Phung-Duc (2015) discussed the steady-state probabilities of the DV queueing system using the generating function method. Vijayashree and Janani (2018) investigated the time-dependent analysis of the DV queueing system. Bouchentouf et al. (2020) analysed the PSM of WiMAX using an $M/M/1$ differentiated WV. The authors derived the steady-state behaviour of the system using the recursive technique. Sampath et al. (2020) applied DV with impatient customers to study the functions of IEEE 802.16e. Sudhesh et al. (2022) studied the Denial of Service of IEEE 802.16 using $M/M/DV$ queueing system with system disaster. Further, Ramupillai et al. (2022) analysed the PSM of WiMAX using $M/M/1/DV$ queueing system with dual-threshold policy and obtained the transient probabilities.

The literature survey shows that many researchers analysed the PSM of WiMAX using various vacation queueing models. This paper proposes an efficient sleep mode mechanism for IEEE 802.16e, guaranteeing effective power saving. To analyse the proposed model mathematically, we introduce a modified differentiated vacation (MDV) scheme with a close-down period. This model consists of the following states: Idle, functional, close-down, ordinary vacation (type-I), and WV (type-II). In this proposed model, the server switches to the close-down period for a random duration at the end of the functional period. If the system is empty at the end of the close-down period, the server takes an ordinary vacation. If no data packet is waiting at the end of the ordinary vacation, the server switches to WV. The duration of these two vacations is different.

A queueing system with a close-down period helps the system's optimal design. In recent years several vacation queueing models evolved with a close-down period. The close-down period corresponds to the time for closing down the service. Further, the server requires a setup time to restart the service. Close-down strategy is a common practice in industries, banks and communication networks. Three types of close-down/setup policies exist: interruptible, insusceptible and skippable Sun et al. (2012). In the interruptible scheme, the server switches immediately to the busy state if any customer arrives during the close-down period. In the insusceptible scheme, service cannot be provided immediately to the customers arriving during a close-down period. The server requires a setup time to resume service. In the skippable policy, the service is provided only at the end of the close-down period. Takagi (1991) first studied the concept of close-down period time. Xiu et al. (2010) investigated an $M/M/1$ queueing system with a single WV and close-down policy. Huo et al. (2009) investigated the PSM of IEEE 802.16e using multiple vacation and close-down policies. Dimitriou (2016) applied an $M/G/1$ queueing system with multiple vacation and close-down policies to analyse the PSM of a communication network. In the literature, no work has been carried out to obtain the transient and steady-state solutions of PCS-II with close-down, WV and

impatient customers. In this research article, the parameters are carefully chosen to maximise the energy efficiency of the MS without violating the quality of service requirements. In this paper, we adopt a skippable close-down policy. This close-down policy avoids frequent transitions between a close-down and a functional state.

The significance of this research work is multifaceted. This paper investigates an MDV policy with a close-down period and WV to analyze the PCS-II of IEEE 802.16e. The concept of WV reduces the chances of buffer overflow in the system. The transient solution is obtained using a modified Bessel and confluent hypergeometric function. These transient results can be used to analyze the system's status at any instant t . Additionally, a cost model for the investigated system has been analyzed. To optimize operational costs, this study addresses a cost optimization problem within a predetermined cost structure, employing the particle swarm optimization (PSO) technique. PSO, a renowned evolutionary computation method developed by Kennedy and Eberhart (1995), is leveraged to tackle the cost optimization challenge. This proposed system is crucial in the context of energy management for WiMAX and HAP systems. WiMAX systems, which provide high-speed broadband access over large areas, particularly in rural and underserved regions, benefit from efficient energy management through extended battery life of user devices, reduced operational expenditures (OPEX), and minimized carbon footprint. Similarly, HAP systems, functioning as aerial base stations (ABS), offer a flexible and cost-effective solution for enhancing network capacity and facilitating terrestrial base station (TBS) sleep modes, thereby reducing energy consumption and improving network sustainability. The insights derived from this model can be applied to optimize energy usage in these systems by implementing advanced power control algorithms, dynamic resource allocation, and energy-efficient network planning, making the network more sustainable and cost-effective.

The structure of the article is as follows. Section 2 describes the proposed model. Section 3 deals with the transient probability of the system. Section 4 discusses the model's performance measures in the transient state. Section 5 presents the system's steady-state probabilities. Section 6 presents performance metrics in the steady state. A numerical illustration is presented in Section 7. A cost-profit analysis is presented in Section 8. Finally, the conclusion and future work of the present research is presented in Section 9.

2 Model Description

The model description of the system under investigation is presented in this section. The proposed model comprises four distinct states: functional (busy), close-down, ordinary vacation and working vacation

1. **Functional state:** This is the normal operating state of the communication system when it's actively processing and transmitting data packets. In this mode, the system is open for communication, accepting incoming data packets and processing them according to protocol
2. **Close-down:** During this period, the system temporarily stops accepting new data packets. It's a scheduled downtime to ensure the system's reliability and efficiency.
3. **Ordinary Vacation:** Devices enter this state to conserve power, temporarily suspending network connections and reducing responsiveness to incoming data packets. They maintain minimal functionality, leading to potential delays until they return to full operation.
4. **Working Vacation:** During this state, components of the communication system may operate at reduced capacity or efficiency. For example, routers might have fewer available

ports or decreased throughput. It's a compromise between maintaining some level of service and conducting necessary maintenance or upgrades.

Let $\{M(t), t \geq 0\}$ represents the system status at any time t and let $N(t)$ denote the number of data packets in the system at any time t .

$$M(t) = \begin{cases} 0, & \text{the MS is in functional mode} \\ 1, & \text{the MS is in close-down mode} \\ 2, & \text{the MS is in ordinary vacation} \\ 3, & \text{the MS is in WV} \end{cases}$$

Then, $X(t) = \{M(t), N(t), t \geq 0\}$ represents the CTMC with state space $S = \{(i, n), n = 0, 1, 2, \dots; i = 0, 1, 2, 3\}$.

Packet arrivals follow a Poisson process with parameter λ and the service time for each data packet is exponentially distributed with parameter μ_b . Upon successful transmission of all data packets, the system transitions from state $(0, 1)$ to state $(1, 0)$ by activating a timer θ , which is exponentially distributed. If no data packets arrive by the end of timer θ , the server switches to the ordinary vacation state $(2, 0)$ and activates a timer γ_1 . However, if data packets are waiting at the end of timer θ , the server switches to the functional state $(0, n)$, $n = 1, 2, 3, \dots$, and starts transmitting the packets. At the end of timer γ_1 , if no packets are waiting for service, the server transitions from state $(2, 0)$ to state $(3, 0)$ and activates a timer γ_2 . If, at the end of timer γ_2 , there are packets waiting for service, the server switches to state $(0, n)$, $n = 1, 2, 3, \dots$, and starts transmitting the packets. Otherwise, the server switches to the idle state $(0, 0)$ and activates a timer γ . During the working vacation state $(3, n)$, $n = 1, 2, 3, \dots$, incoming jobs are served at a rate of μ_{v_2} ($\mu_{v_2} < \mu_b$), which follows an exponential distribution. In this model, impatience (signal loss) can occur in any system state except the functional state $(0, n)$, $n = 1, 2, 3, \dots$. Data packets trigger an independent impatience timer, which follows an exponential distribution with rate ζ . The FIFO queue discipline is followed in this system.

Figure 1 depicts the state transition diagram of the proposed system.

Let $P_{i,n}(t) = P\{M(t) = i, N(t) = n\}$, $i = 0, 1, 2, 3$; $n = 0, 1, 2, \dots$ be the probability that the server is in state 'i' and there are 'n' number of data packets in the system at a time 't.' Then $P_{i,n}(t)$ satisfies the following forward Kolmogorov equation.

$$P'_{0,0}(t) = -(\lambda + \gamma) P_{0,0}(t) + \gamma_2 P_{3,0}(t), \quad (2.1)$$

$$P'_{0,n}(t) = -(\lambda + \mu_b) P_{0,n}(t) + \lambda P_{0,n-1}(t) + \mu_b P_{0,n+1}(t) + \gamma_1 P_{2,n}(t) + \gamma_2 P_{3,n}(t) \\ + \theta P_{1,n}(t), n = 1, 2, 3, \dots, \quad (2.2)$$

$$P'_{1,0}(t) = -(\lambda + \theta) P_{1,0}(t) + \gamma P_{0,0}(t) + \zeta P_{1,1}(t) + \mu_b P_{0,1}(t), \quad (2.3)$$

$$P'_{1,n}(t) = -(\lambda + \theta + n\zeta) P_{1,n}(t) + \lambda P_{1,n-1}(t) + (n+1)\zeta P_{1,n+1}(t), n = 1, 2, 3, \dots, \quad (2.4)$$

$$P'_{2,0}(t) = -(\lambda + \gamma_1) P_{2,0}(t) + \zeta P_{2,1}(t) + \theta P_{1,0}(t), \quad (2.5)$$

$$P'_{2,n}(t) = -(\lambda + \gamma_1 + n\zeta) P_{2,n}(t) + \lambda P_{2,n-1}(t) + (n+1)\zeta P_{2,n+1}(t), n = 1, 2, 3, \dots, \quad (2.6)$$

$$P'_{3,0}(t) = -(\lambda + \gamma_2) P_{3,0}(t) + \gamma_1 P_{2,0}(t) + (\mu_{v_2} + \zeta) P_{3,1}(t), \quad (2.7)$$

$$P'_{3,n}(t) = -(\lambda + \gamma_2 + \mu_{v_2} + n\zeta) P_{3,n}(t) + (\mu_{v_2} + (n+1)\zeta) P_{3,n+1}(t) \\ + \lambda P_{3,n-1}(t), n = 1, 2, 3, \dots \quad (2.8)$$

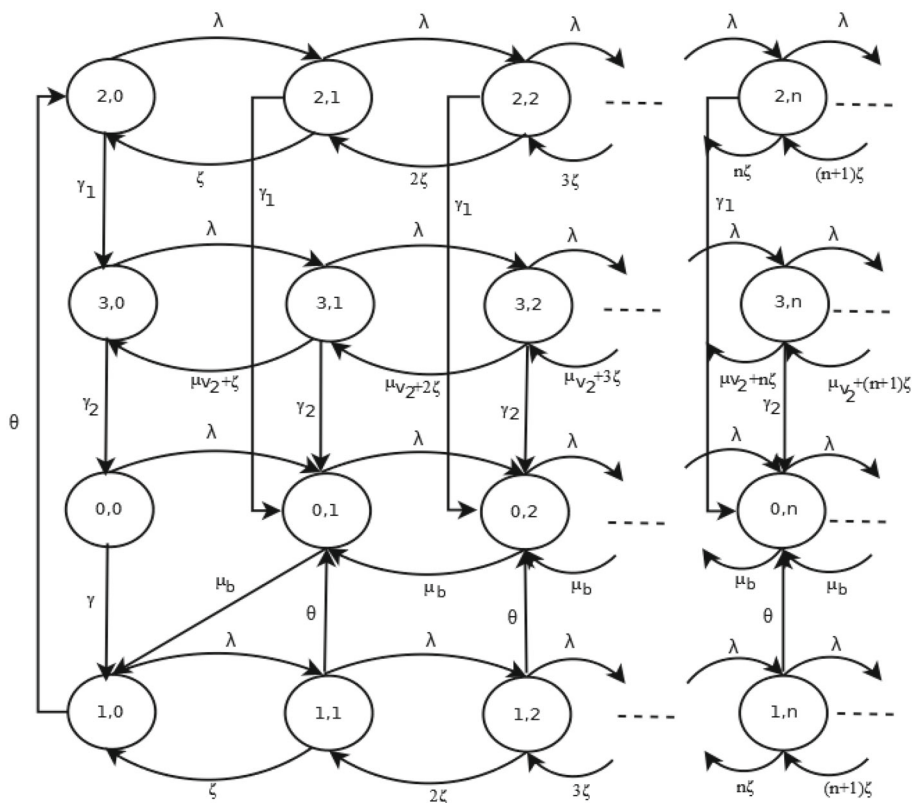


Fig. 1 State transition diagram of the proposed model

To set up initial conditions, we assume that the server is in an idle state at time $t = 0$. Therefore, $P_{0,0}(0) = 1$, $P_{i,n}(0) = 0$ for $n=1,2,3,\dots$; $i = 0, 1, 2, 3$ and $P_{j,0}(0) = 0$ for $j=1, 2, 3$.

3 Time Dependent Analysis

The probabilities of the functional state $P_{0,n}(t)$, the close-down state $P_{1,n}(t)$, vacation state $P_{2,n}(t)$, WV state $P_{3,n}(t)$ and the idle state $P_{0,0}(t)$ is presented in this section.

3.1 Evaluation of the Functional State

The functional-state probability $P_{0,n}(t)$; $n = 1, 2, 3, \dots$ is derived from Eq. (2.2) as follows.

Let

$$H(z, t) = \sum_{n=1}^{\infty} P_{0,n}(t) z^n, \quad H(z, 0) = 1.$$

Multiplying z^n on both sides of Eq. (2.2) and summing over $n = 1, 2, 3, \dots$, we get

$$\begin{aligned}
 H'(z, t) = & \left[-(\lambda + \mu_b) + \mu_b z^{-1} + \lambda z \right] H(z, t) + \gamma_1 \sum_{n=1}^{\infty} P_{2,n}(t) z^n + \gamma_2 \sum_{n=1}^{\infty} P_{3,n}(t) z^n \\
 & + \theta \sum_{n=1}^{\infty} P_{1,n}(t) z^n - \mu_b P_{0,1}(t) + \lambda z P_{0,0}(t).
 \end{aligned} \quad (3.1)$$

Solving Eq. (3.1), we obtain

$$\begin{aligned}
 H(z, t) = & \int_0^t \left[\sum_{m=1}^{\infty} z^m \left\{ \gamma_1 P_{2,m}(v) + \gamma_2 P_{3,m}(v) + \theta P_{1,m}(v) \right\} \right. \\
 & \left. + \left\{ \lambda z P_{0,0}(v) - \mu_b P_{0,1}(v) \right\} \right] \exp \left\{ (-\varsigma + \lambda z + \mu_b z^{-1})(t-v) \right\} dv \\
 & + \exp \left\{ (-\varsigma + \lambda z + \mu_b z^{-1})(t-v) \right\},
 \end{aligned} \quad (3.2)$$

where $\varsigma = \lambda + \mu_b$. Let $\varepsilon = 2\sqrt{\lambda\mu_b}$ and $\xi = \sqrt{\lambda\mu_b^{-1}}$, then

$$\exp \left[\left(\lambda z + \frac{\mu_b}{z} \right) (t-v) \right] = \sum_{n=-\infty}^{\infty} (\xi z)^n I_n(\varepsilon(t-v)), \quad (3.3)$$

where $I_m(t)$ represents the modified Bessel function of the first kind of order m . Applying (3.3) in Eq. (3.2) and equating the coefficient of z^n on both sides for $n = 1, 2, 3, \dots$

$$\begin{aligned}
 P_{0,n}(t) = & \int_0^t \left[\sum_{m=1}^{\infty} \xi^{n-m} \left\{ \gamma_1 P_{2,m}(v) + \gamma_2 P_{3,m}(v) + \theta P_{1,m}(v) \right\} I_{n-m}(\cdot) \right. \\
 & \left. + \lambda \xi^{n-1} P_{0,0}(v) I_{n-1}(\cdot) - \mu_b \xi^n P_{0,1}(v) I_n(\cdot) \right] \exp \{-\varsigma(t-v)\} dv \\
 & + \exp \{-\varsigma(t-v)\} \xi^n I_n(\cdot).
 \end{aligned} \quad (3.4)$$

Equating the coefficients of z^{-n} on both sides of Eq. (3.2) for $n = 1, 2, 3, \dots$ and using $I_{-n}(\cdot) = I_n(\cdot)$, we get

$$\begin{aligned}
 0 = & \int_0^t \left[\sum_{m=1}^{\infty} \xi^{-n-m} \left\{ \gamma_1 P_{2,m}(v) + \gamma_2 P_{3,m}(v) + \theta P_{1,m}(v) \right\} I_{n+m}(\cdot) \right. \\
 & \left. + \lambda \xi^{-n-1} P_{0,0}(v) I_{n+1}(\cdot) - \mu_b \xi^{-n} P_{0,1}(v) I_n(\cdot) \right] \exp \{-\varsigma(t-v)\} dv \\
 & + \exp \{-\varsigma(t-v)\} \xi^{-n} I_n(\cdot).
 \end{aligned} \quad (3.5)$$

After some simple manipulation, we get

$$\begin{aligned}
 P_{0,n}(t) = & \sum_{m=1}^{\infty} [\gamma_1 P_{2,m}(t) + \gamma_2 P_{3,m}(t) + \theta P_{1,m}(t)] \xi^{n-m} * [I_{n-m}(\varepsilon t) - I_{n+m}(\varepsilon t)] \\
 & \times \exp \{-\varsigma t\} + \lambda \xi^{n-1} P_{0,0}(t) * [I_{n-1}(\varepsilon t) - I_{n+1}(\varepsilon t)] \exp \{-\varsigma t\}.
 \end{aligned} \quad (3.6)$$

Thus $P_{0,n}(t)$ expressed in-terms of $P_{i,n}(t)$; $i = 0, 1, 2, 3$.

3.2 Evaluation of $P_{1,n}(t)$, $P_{2,n}(t)$ and $P_{3,n}(t)$

The time-dependent probabilities of the close-down state, ordinary vacation state and the WV state are discussed in this section. Let $\hat{P}_{i,n}(s)$ denote Laplace transform of $P_{i,n}(t)$ for $i = 1, 2, 3$; $n = 0, 1, 2, \dots$. Taking Laplace transform on Eq. (2.8) for $n=1,2,3,\dots$, we get

$$\frac{\hat{P}_{3,n}(s)}{\hat{P}_{3,n-1}(s)} = \frac{\lambda}{(s + \lambda + \mu_{v2} + \gamma_2 + n\zeta) - (\mu_{v2} + (n+1)\zeta) \frac{\hat{P}_{3,n+1}(s)}{\hat{P}_{3,n}(s)}}. \quad (3.7)$$

Applying the identity [Lorentzen and Waadeland (1992), 4.1.5, P573], Eq. (3.7) can be expressed as

$$\frac{\hat{P}_{3,n}(s)}{\hat{P}_{3,n-1}(s)} = \frac{\lambda}{\zeta \left(\frac{s + \mu_{v2} + \gamma_2}{\zeta} + n \right)} \frac{{}_1F_1 \left(\frac{\mu_{v2}}{\zeta} + n + 1; \frac{s + \mu_{v2} + \gamma_2}{\zeta} + n + 1; -\frac{\lambda}{\zeta} \right)}{{}_1F_1 \left(\frac{\mu_{v2}}{\zeta} + n; \frac{s + \mu_{v2} + \gamma_2}{\zeta} + n; -\frac{\lambda}{\zeta} \right)}. \quad (3.8)$$

For $n=1,2,3,\dots$ Equation (3.8) becomes

$$\hat{P}_{3,n}(s) = \hat{\chi}_n(s) \hat{P}_{3,0}(s), \quad (3.9)$$

where

$$\hat{\chi}_n(s) = \left(\frac{\lambda}{\zeta} \right)^n \frac{1}{\prod_{i=1}^n \left(\frac{s + \mu_{v2} + \gamma_2}{\zeta} + i \right)} \frac{{}_1F_1 \left(\frac{\mu_{v2}}{\zeta} + n + 1; \frac{s + \mu_{v2} + \gamma_2}{\zeta} + n + 1; -\frac{\lambda}{\zeta} \right)}{{}_1F_1 \left(\frac{\mu_{v2}}{\zeta} + 1; \frac{s + \mu_{v2} + \gamma_2}{\zeta} + 1; -\frac{\lambda}{\zeta} \right)}. \quad (3.10)$$

Applying the above procedure on Eqs. (2.4) and (2.6), we get

$$\hat{P}_{1,n}(s) = \hat{\phi}_n(s) \hat{P}_{1,0}(s), \quad (3.11)$$

$$\hat{P}_{2,n}(s) = \hat{\psi}_n(s) \hat{P}_{2,0}(s), \quad (3.12)$$

where

$$\hat{\psi}_n(s) = \left(\frac{\lambda}{\zeta} \right)^n \frac{1}{\prod_{i=1}^n \left(\frac{s + \gamma_1}{\zeta} + i \right)} \frac{{}_1F_1 \left(n + 1; \frac{s + \gamma_1}{\zeta} + n + 1; -\frac{\lambda}{\zeta} \right)}{{}_1F_1 \left(1; \frac{s + \gamma_1}{\zeta} + 1; -\frac{\lambda}{\zeta} \right)}, \quad (3.13)$$

$$\hat{\phi}_n(s) = \left(\frac{\lambda}{\zeta} \right)^n \frac{1}{\prod_{i=1}^n \left(\frac{s + \theta}{\zeta} + i \right)} \frac{{}_1F_1 \left(n + 1; \frac{s + \theta}{\zeta} + n + 1; -\frac{\lambda}{\zeta} \right)}{{}_1F_1 \left(1; \frac{s + \theta}{\zeta} + 1; -\frac{\lambda}{\zeta} \right)}. \quad (3.14)$$

On inversion, Eqs. (3.11), (3.12) and (3.9) respectively, yields

$$P_{1,n}(t) = \phi_n(t) * P_{1,0}(t), \quad (3.15)$$

$$P_{2,n}(t) = \psi_n(t) * P_{2,0}(t), \quad (3.16)$$

$$P_{3,n}(t) = \chi_n(t) * P_{3,0}(t). \quad (3.17)$$

An explicit expression for $P_{1,0}(t)$, $P_{2,0}(t)$ and $P_{3,0}(t)$ is given in Eqs. (3.46), (3.47) and (3.48) respectively.

3.3 Evaluation of $\psi_n(t)$, $\chi_n(t)$ and $\phi_n(t)$

By employing confluent hypergeometric function, Eq. (3.10) is expressed as

$$\frac{{}_1F_1\left(\frac{\mu_{v_2}}{\zeta} + n + 1; \frac{s + \mu_{v_2} + \gamma_2}{\zeta} + n + 1; -\frac{\lambda}{\zeta}\right)}{\prod_{i=1}^n \left(\frac{s + \mu_{v_2} + \gamma_2}{\zeta} + i\right)} = \sum_{m=0}^{\infty} \frac{\frac{1}{m!} \prod_{i=1}^m (\mu_{v_2} + (n+i)\zeta) \left(-\frac{\lambda}{\zeta}\right)^m}{\prod_{i=1}^{n+m} (s + \mu_{v_2} + \gamma_2 + i\zeta) \left(\frac{1}{\zeta}\right)^n}.$$

Using partial fraction, we obtain

$$\begin{aligned} & \frac{{}_1F_1\left(\frac{\mu_{v_2}}{\zeta} + n + 1; \frac{s + \mu_{v_2} + \gamma_2}{\zeta} + n + 1; -\frac{\lambda}{\zeta}\right)}{\prod_{i=1}^n \left(\frac{s + \mu_{v_2} + \gamma_2}{\zeta} + i\right)} \\ &= \sum_{m=0}^{\infty} \frac{1}{m!} \prod_{i=1}^m (\mu_{v_2} + (n+i)\zeta) \left(-\frac{\lambda}{\zeta}\right)^m \frac{1}{\zeta^{m-1}} \\ & \times \sum_{i=1}^{n+m} \frac{(-1)^{i-1}}{(i-1)!(n+m-i)!(s + \mu_{v_2} + \gamma_2 + i\zeta)}. \end{aligned} \quad (3.18)$$

Further,

$${}_1F_1\left(\frac{\mu_{v_2}}{\zeta} + 1; \frac{s + \mu_{v_2} + \gamma_2}{\zeta} + 1; -\frac{\lambda}{\zeta}\right) = \sum_{m=0}^{\infty} (-\lambda)^m \hat{k}_m(s), \quad (3.19)$$

where

$$\begin{aligned} \hat{k}_m(s) &= \frac{1}{m!} \frac{1}{\zeta^{2m-1}} \prod_{j=1}^m (\mu_{v_2} + j\zeta) \sum_{i=1}^m \frac{A(i)}{(s + \mu_{v_2} + \gamma_2 + i\zeta)}, \quad m = 1, 2, 3, \dots \\ A(i) &= \frac{(-1)^{i-1}}{(i-1)!(m-i)!} \text{ and } \hat{k}_0(s) = 1. \end{aligned} \quad (3.20)$$

By applying the identity Gradshteyn et al. (2007), Eq. (3.10) becomes

$$\left[{}_1F_1\left(\frac{\mu_{v_2}}{\zeta} + 1; \frac{s + \mu_{v_2} + \gamma_2}{\zeta} + 1; -\frac{\lambda}{\zeta}\right)\right]^{-1} = \sum_{m=0}^{\infty} (\lambda)^m \hat{l}_m(s) \text{ and } \hat{l}_0(s) = 1, \quad (3.21)$$

for $m = 1, 2, 3, \dots$,

$$\hat{l}_m(s) = \begin{vmatrix} \hat{k}_1(s) & 1 & 0 & \dots & 0 & 0 \\ \hat{k}_2(s) & \hat{k}_1(s) & 1 & \dots & 0 & 0 \\ \hat{k}_3(s) & \hat{k}_2(s) & \hat{k}_1(s) & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \hat{k}_{m-1}(s) & \hat{k}_{m-2}(s) & \hat{k}_{m-3}(s) & \dots & \hat{k}_1(s) & 1 \\ \hat{k}_m(s) & \hat{k}_{m-1}(s) & \hat{k}_{m-2}(s) & \dots & \hat{k}_2(s) & \hat{k}_1(s) \end{vmatrix}$$

The above expression can be written as

$$\hat{l}_m(s) = \sum_{j=1}^m (-1)^{j-1} \hat{k}_j(s) \hat{l}_{m-j}(s). \quad (3.22)$$

Substituting (3.18) and (3.21) in Eq. (3.10), we obtain

$$\hat{\chi}_n(s) = \left(\frac{\lambda}{\zeta}\right)^n \sum_{m=0}^{\infty} (-\lambda)^m \hat{k}_{n+m}(s) \sum_{j=1}^{\infty} \hat{l}_j(s) \lambda^j. \quad (3.23)$$

On inversion, Eqs. (3.23), (3.20) and (3.22) respectively, yield

$$\chi_n(t) = \left(\frac{\lambda}{\zeta}\right)^n \sum_{m=0}^{\infty} (-\lambda)^m k_{n+m}(t) * \sum_{j=1}^{\infty} l_j(t) \lambda^j, \quad (3.24)$$

$$k_m(t) = \frac{1}{m!} \frac{1}{\zeta^{2m-1}} \prod_{j=1}^m (\mu_{v_2} + j\zeta) \sum_{i=1}^m A(i) \exp\{-(\mu_{v_2} + \gamma_2 + i\zeta)t\}, m = 1, 2, 3, \dots, \quad (3.25)$$

$$l_m(t) = \sum_{j=1}^m (-1)^{j-1} k_j(t) * l_{m-j}(t), m = 1, 2, 3, \dots, k_1(t) = l_1(t). \quad (3.26)$$

Applying the same procedure on Eq. (3.13), we obtain

$$\hat{\psi}_n(s) = \lambda^n \sum_{m=0}^{\infty} (-\lambda)^m \frac{1}{m!} \prod_{i=1}^m (n+i) \hat{f}_{n+m}(s) \sum_{j=1}^{\infty} \hat{g}_j(s) \lambda^j, \quad (3.27)$$

$$\hat{f}_m(s) = \frac{1}{\zeta^{m-1}} \sum_{i=1}^m \frac{A(i)}{(s + \gamma_1 + i\zeta)}, m = 1, 2, 3, \dots, \quad (3.28)$$

$$\hat{g}_m(s) = \sum_{j=1}^m (-1)^{j-1} \hat{f}_j(s) \hat{g}_{m-j}(s). \quad (3.29)$$

On inversion,

$$\psi_n(t) = \lambda^n \sum_{m=0}^{\infty} (-\lambda)^m \frac{1}{m!} \prod_{i=1}^m (n+i) f_{n+m}(t) * \sum_{j=1}^{\infty} g_j(t) \lambda^j, \quad (3.30)$$

$$f_m(t) = \frac{1}{\zeta^{m-1}} \sum_{i=1}^m A(i) \exp\{-(\gamma_1 + i\zeta)t\}, m = 1, 2, 3, \dots, \quad (3.31)$$

$$g_m(t) = \sum_{j=1}^m (-1)^{j-1} f_j(t) * g_{m-j}(t), m = 2, 3, 4, \dots, f_1(t) = g_1(t). \quad (3.32)$$

Applying the above procedure in Eq. (3.14), we obtain

$$\phi_n(t) = \lambda^n \sum_{m=0}^{\infty} (-\lambda)^m \frac{1}{m!} \prod_{i=1}^m (n+i) h_{n+m}(t) * \sum_{j=1}^{\infty} q_j(t) \lambda^j, \quad (3.33)$$

$$h_m(t) = \frac{1}{\zeta^{m-1}} \sum_{i=1}^m A(i) \exp\{-(\theta + i\zeta)t\}, m = 1, 2, 3, \dots, \quad (3.34)$$

$$q_m(t) = \sum_{j=1}^m (-1)^{j-1} h_j(t) * q_{m-j}(t), m = 2, 3, 4, \dots, h_1(t) = q_1(t). \quad (3.35)$$

3.4 Evaluation of $P_{0,0}(t)$, $P_{1,0}(t)$, $P_{2,0}(t)$ and $P_{3,0}(t)$

This section presents the system size probabilities $P_{1,0}(t)$, $P_{2,0}(t)$, $P_{3,0}(t)$ and $P_{0,0}(t)$.

Applying Laplace transform on Eqs. (2.1), (2.3), (2.5) and (2.7) after some manipulation, we get

$$\hat{P}_{0,0}(s) = \frac{\gamma_1 \gamma_2 \theta}{s + \lambda + \gamma} \sum_{j=0}^{\infty} \hat{a}_j(s) \sum_{k=0}^{\infty} \hat{b}_k(s) \hat{P}_{1,0}(s) + \frac{1}{s + \lambda + \gamma}, \quad (3.36)$$

$$\hat{P}_{1,0}(s) = \left(\gamma \hat{P}_{0,0}(s) + \mu_b \hat{P}_{0,1}(s) \right) \sum_{l=0}^{\infty} \hat{c}_l(s), \quad (3.37)$$

$$\hat{P}_{2,0}(s) = \theta \hat{P}_{1,0}(s) \sum_{j=0}^{\infty} \hat{a}_j(s), \quad (3.38)$$

$$\hat{P}_{3,0}(s) = \gamma_1 \theta \hat{P}_{1,0}(s) \sum_{j=0}^{\infty} \hat{a}_j(s) \sum_{k=0}^{\infty} \hat{b}_k(s). \quad (3.39)$$

Using Eq. (3.36) in Eq. (3.37), we get

$$\hat{P}_{1,0}(s) = \left(\mu_b \hat{P}_{0,1}(s) + \frac{1}{s + \lambda + \gamma} \right) \hat{G}_1(s), \quad (3.40)$$

where

$$\hat{a}_j(s) = \frac{\zeta^j (\hat{\psi}_1(s))^j}{(s + \lambda + \gamma_1)^{j+1}}, \hat{b}_k(s) = \frac{(\mu_{v_2} + \zeta)^k (\hat{\chi}_1(s))^k}{(s + \lambda + \gamma_2)^{k+1}}, \hat{c}_l(s) = \frac{\zeta^l (\hat{\phi}_1(s))^l}{(s + \lambda + \theta)^{l+1}},$$

$$\hat{G}_1(s) = \sum_{l=0}^{\infty} \hat{c}_l(s) \sum_{m=0}^{\infty} \left[\frac{\gamma \gamma_1 \gamma_2 \theta}{s + \lambda + \gamma} \sum_{j=0}^{\infty} \hat{a}_j(s) \sum_{k=0}^{\infty} \hat{b}_k(s) \sum_{l=0}^{\infty} \hat{c}_l(s) \right]^m. \quad (3.41)$$

Substituting Eq. (3.40) in Eqs. (3.36), (3.38) and (3.39), we obtain

$$\hat{P}_{0,0}(s) = \frac{1}{s + \lambda + \gamma} \left[1 + \gamma_1 \gamma_2 \theta \left(\mu_b \hat{P}_{0,1}(s) + \frac{1}{s + \lambda + \gamma} \right) \sum_{j=0}^{\infty} \hat{a}_j(s) \sum_{k=0}^{\infty} \hat{b}_k(s) \hat{G}_1(s) \right], \quad (3.42)$$

$$\hat{P}_{2,0}(s) = \theta \left(\mu_b \hat{P}_{0,1}(s) + \frac{1}{s + \lambda + \gamma} \right) \hat{G}_1(s) \sum_{j=0}^{\infty} \hat{a}_j(s), \quad (3.43)$$

$$\hat{P}_{3,0}(s) = \gamma_1 \theta \left(\mu_b \hat{P}_{0,1}(s) + \frac{1}{s + \lambda + \gamma} \right) \hat{G}_1(s) \sum_{j=0}^{\infty} \hat{a}_j(s) \sum_{k=0}^{\infty} \hat{b}_k(s). \quad (3.44)$$

On inversion, Eqs. (3.40), (3.42), (3.43), and (3.44) respectively yields

$$P_{0,0}(t) = \left[\delta(t) + \gamma_1 \gamma_2 \theta \left[\mu_b P_{0,1}(t) + \exp\{-(\lambda + \gamma)t\} \right] * \sum_{j=0}^{\infty} a_j(t) * \sum_{k=0}^{\infty} b_k(t) * G_1(t) \right] * \exp\{-(\lambda + \gamma)t\}, \quad (3.45)$$

$$P_{1,0}(t) = [\mu_b P_{0,1}(t) + \exp\{-(\lambda + \gamma)t\}] * G_1(t), \quad (3.46)$$

$$P_{2,0}(t) = \theta [\mu_b P_{0,1}(t) + \exp\{-(\lambda + \gamma)t\}] * G_1(t) * \sum_{j=0}^{\infty} a_j(t), \quad (3.47)$$

$$P_{3,0}(t) = \gamma_1 \theta [\mu_b P_{0,1}(t) + \exp\{-(\lambda + \gamma)t\}] * G_1(t) * \sum_{j=0}^{\infty} a_j(t) * \sum_{k=0}^{\infty} b_k(t), \quad (3.48)$$

where

$$\begin{aligned} a_j(t) &= \frac{\zeta^j t^j \exp\{-(\lambda + \gamma_1)t\} * (\psi_1(t))^{*j}}{j!}, \quad c_l(t) = \frac{\zeta^l t^l \exp\{-(\lambda + \theta)t\} * (\phi_1(t))^{*l}}{l!}, \\ b_k(t) &= \frac{(\mu_{v_2} + \zeta)^k t^k \exp\{-(\lambda + \gamma_2)t\} * (\chi_1(t))^{*k}}{k!}, \\ G_1(t) &= \sum_{l=0}^{\infty} c_l(t) * \sum_{m=0}^{\infty} \left[\gamma \gamma_1 \gamma_2 \theta \exp\{-(\lambda + \gamma)t\} * \sum_{j=0}^{\infty} a_j(t) * \sum_{k=0}^{\infty} b_k(t) * \sum_{l=0}^{\infty} c_l(t) \right]^{*m}. \end{aligned} \quad (3.49)$$

where ‘*’ and ‘*m’ denotes convolution and m -fold convolution respectively. $\delta(t)$ represents Dirac-delta function. The expression for $P_{0,1}(t)$ is given in Eq. (3.50).

3.5 Evaluation of $P_{0,1}(t)$

Setting $n = 1$ in Eq. (3.6) and using Eqs. (3.42), (3.43) and (3.40), we get

$$P_{0,1}(t) = \sum_{h=0}^{\infty} [G_2(t)]^{*h} * \left[\frac{1}{\mu_b} G_2(t) * \exp\{-(\lambda + \gamma)t\} + G_3(t) \right], \quad (3.50)$$

where

$$\begin{aligned} G_2(t) &= \sum_{m=1}^{\infty} \left\{ \gamma_1 \psi_m(t) * \sum_{j=0}^{\infty} a_j(t) + \gamma_1 \gamma_2 \chi_m(t) * \sum_{j=0}^{\infty} a_j(t) * \sum_{k=0}^{\infty} b_k(t) + \phi_m(t) \right\} \xi^{1-m} \\ &\quad * [I_{m-1}(\varepsilon t) - I_{m+1}(\varepsilon t)] \exp\{-(\lambda + \mu_b)t\} + \lambda \gamma_1 \gamma_2 \exp\{-(\lambda + \gamma)t\} \\ &\quad * [I_0(\varepsilon t) - I_2(\varepsilon t)] \exp\{-(\lambda + \mu_b)t\} * \sum_{j=0}^{\infty} a_j(t) * \sum_{k=0}^{\infty} b_k(t), \\ G_3(t) &= \lambda \exp\{-(\lambda + \gamma)t\} * [I_0(\varepsilon t) - I_2(\varepsilon t)] \exp\{-(\lambda + \mu_b)t\}. \end{aligned}$$

3.6 Special Cases

Case (i) If $\gamma_2 = 0$, considering the initial conditions as $P_{1,0}(0) = 1$, $P_{i,j}(0) = 0$ for $i = 0, 2, 3$; $j = 0, 1, 2, \dots$, $P_{1,j}(0) = 0$ for $j = 1, 2, 3, \dots$ and if the server goes to an idle state after serving all the data packets, then Eq. (3.49) reduces to

$$G_1(t) = \sum_{l=0}^{\infty} \frac{\zeta^l (\phi_1(t))^{*l} * t^l \exp\{-(\lambda + \theta)t\}}{l!}.$$

substituting the above equation in the results (3.46) and (3.47), and further using it in the results (3.15) and (3.16), we get

$$P_{1,n}(t) = \phi_n(t) * (\mu_b P_{0,0}(t) + \delta(t)) * \sum_{l=0}^{\infty} \frac{\zeta^l (\phi_1(t))^{*l} * t^l \exp\{-(\lambda + \theta)t\}}{l!}.$$

$$P_{2,n}(t) = \psi_n(t) * \theta (\mu_b P_{0,0}(t) + \delta(t)) * \sum_{l=0}^{\infty} \frac{\zeta^l (\phi_1(t))^{*l} * t^l \exp\{-(\lambda + \theta)t\}}{l!}$$

$$* \sum_{j=0}^{\infty} \frac{\zeta^j (\psi_1(t))^{*j} * t^j \exp\{-(\lambda + \gamma_1)t\}}{j!}.$$

The above equation agrees with the results (12) and (16) of Suranga Sampath and Liu (2020).

Case (ii) If $\gamma_2 = 0$, $\theta = 0$ and $\mu_{v_2} = 0$, then Eq. (3.5) reduces to

$$P_{0,n}(t) = \gamma_1 \int_0^t \sum_{m=1}^{\infty} P_{2,m}(w) \exp\{-\zeta(t-w)\} \xi^{n-m} [I_{n-m}(\cdot) - I_{n+m}(\cdot)] dw. \quad (3.51)$$

The above equation agrees with the result (4.5) of Ammar (2015).

4 Performance Indices

In this section, the time dependent mean, variance, average rate of abandonment and system throughput are discussed.

4.1 Mean

Let $E[X(t)]$ denote the expected system size at time t . For $t > 0$, we have

$$E[X(t)] = \sum_{n=1}^{\infty} n \left\{ \sum_{i=0}^3 P_{i,n}(t) \right\}.$$

Using Eqs. (2.1)–(2.8), we get

$$\frac{d}{dt} E[X(t)] = \lambda - \sum_{n=1}^{\infty} \left[\mu_b P_{0,n}(t) - n\zeta \left\{ \sum_{i=1}^3 P_{i,n}(t) \right\} - \mu_{v_2} P_{3,n}(t) \right].$$

On integration,

$$E[X(t)] = \lambda t - \sum_{n=1}^{\infty} \int_0^t \left[\mu_b P_{0,n}(w) - n\zeta \left\{ \sum_{i=1}^3 P_{i,n}(w) \right\} - \mu_{v_2} P_{3,n}(w) \right] dw.$$

4.2 Variance

Let $V[X(t)]$ denote the variance system size at time t . For $t > 0$,

$$V[X(t)] = E[X^2(t)] - (E[X(t)])^2,$$

where

$$E[X^2(t)] = \sum_{n=1}^{\infty} n^2 \left\{ \sum_{i=0}^3 P_{i,n}(t) \right\}.$$

Using Eqs. (2.1)–(2.8), we obtain

$$\begin{aligned} \frac{d}{dt} E[X^2(t)] &= \lambda - \sum_{n=1}^{\infty} (2n-1) \left[\mu_b P_{0,n}(t) - n\zeta \left\{ \sum_{i=1}^3 P_{i,n}(t) \right\} - \mu_{v_2} P_{3,n}(t) \right] \\ &\quad + 2\lambda E[X(t)] \end{aligned}$$

On integration, we get

$$\begin{aligned} E[X^2(t)] &= \lambda t - \sum_{n=1}^{\infty} (2n-1) \int_0^t \left[\mu_b P_{0,n}(w) - n\zeta \left\{ \sum_{i=1}^3 P_{i,n}(w) \right\} - \mu_{v_2} P_{3,n}(w) \right] dw \\ &\quad + 2\lambda \int_0^t E[X(w)] dw. \end{aligned}$$

4.3 Average Abandonment Rate

Let $R(t)$ denote the average abandonment rate due to impatience. Then,

$$R(t) = \zeta \left[\sum_{n=0}^{\infty} n P_{2,n}(t) + \sum_{n=0}^{\infty} n P_{3,n}(t) + \sum_{n=0}^{\infty} n P_{1,n}(t) \right].$$

4.4 System Throughput

Let $S(t)$ be the system's throughput at time t , then

$$S(t) = \mu_b \sum_{n=1}^{\infty} P_{0,n}(t) + \mu_{v_2} \sum_{n=1}^{\infty} P_{3,n}(t),$$

where the expression for $P_{0,n}(t)$ and $P_{3,n}(t)$ is given in Eqs. (3.6) and (3.17) respectively.

5 Steady-State Probabilities

This section presents the steady-state probabilities of the investigated system. Applying $\lim_{s \rightarrow 0} s \hat{P}_{i,n} = \pi_{i,n}$; $i = 0, 1, 2, 3$, on Eqs. (2.1)–(2.8), we obtain

$$(\lambda + \gamma) \pi_{0,0} = \gamma \pi_{3,0}, \quad (5.1)$$

$$(\lambda + \mu_b) \pi_{0,n} = \lambda \pi_{0,n-1} + \mu_b \pi_{0,n+1} + \gamma \pi_{2,n} + \gamma \pi_{3,n} + \theta \pi_{1,n}, \quad n = 1, 2, 3, \dots, \quad (5.2)$$

$$(\lambda + \theta) \pi_{1,0} = \gamma \pi_{0,0} + \zeta \pi_{1,1} + \mu_b \pi_{0,1}, \quad (5.3)$$

$$(\lambda + \theta + n\zeta) \pi_{1,n} = \lambda \pi_{1,n-1} + (n+1) \zeta \pi_{1,n+1}, \quad n = 1, 2, 3, \dots, \quad (5.4)$$

$$(\lambda + \gamma) \pi_{2,0} = \zeta \pi_{2,1} + \theta \pi_{1,0}, \quad (5.5)$$

$$(\lambda + \gamma_1 + n\zeta) \pi_{2,n} = \lambda \pi_{2,n-1} + (n+1) \zeta \pi_{2,n+1}, n = 1, 2, 3, \dots, \quad (5.6)$$

$$(\lambda + \gamma_2) \pi_{3,0} = \gamma_1 \pi_{2,0} + (\mu_{v_2} + \zeta) \pi_{3,1}, \quad (5.7)$$

$$(\lambda + \gamma_2 + \mu_{v_2} + n\zeta) \pi_{3,n} = \lambda \pi_{3,n-1} + (\mu_{v_2} + (n+1) \zeta) \pi_{3,n+1}, n = 1, 2, 3, \dots \quad (5.8)$$

We define a generating function

$$G_j(z) = \sum_{n=0}^{\infty} \pi_{j,n} z^n, j = 1, 2, 3.$$

Let

$$G'_j(z) = \sum_{n=1}^{\infty} n \pi_{j,n} z^{n-1}, j = 1, 2, 3.$$

Multiplying Eqs. (5.3)–(5.8) by suitable powers of z^n and summing over $n = 0, 1, 2, \dots$, we obtain

$$-G'_1(z) \zeta (1-z) + (\lambda (1-z) + \theta) G_1(z) = \gamma \pi_{0,0} + \mu_b \pi_{0,1}, \quad (5.9)$$

$$-G'_2(z) \zeta (1-z) + G_2(z) \{\lambda (1-z) + \gamma_1\} = \theta \pi_{1,0}, \quad (5.10)$$

$$\begin{aligned} -G'_3(z) \zeta z (1-z) + G_3(z) \{\lambda z (1-z) - \mu_{v_2} (1-z) + z \gamma_2\} \\ = -\mu_{v_2} \pi_{3,0} (1-z) + \gamma_1 z \pi_{2,0}. \end{aligned} \quad (5.11)$$

5.1 Evaluation of $G_1(z)$, $G_2(z)$ and $G_3(z)$

This section presents the solution of the differential Eqs. (5.9), (5.10) and (5.11). Equations (5.9)–(5.11) can be expressed as

$$G'_1(z) - \frac{1}{\zeta} \left(\lambda + \frac{\theta}{1-z} \right) G_1(z) = -\frac{1}{\zeta} \left(\frac{\gamma \pi_{0,0} + \mu_b \pi_{0,1}}{1-z} \right), \quad (5.12)$$

$$G'_2(z) - G_2(z) \frac{1}{\zeta} \left(\lambda + \frac{\gamma_1}{1-z} \right) = -\frac{\theta \pi_{1,0}}{\zeta (1-z)}, \quad (5.13)$$

$$G'_3(z) - G_3(z) \frac{1}{\zeta} \left(\lambda - \frac{\mu_{v_2}}{z} + \frac{\gamma_2}{1-z} \right) = \frac{1}{\zeta} \left(\frac{\mu_{v_2}}{z} \pi_{3,0} - \frac{\gamma_1}{1-z} \pi_{2,0} \right). \quad (5.14)$$

Multiplying both sides of Eqs. (5.12), (5.13) and (5.14) by $(1-z)^{\frac{\theta}{\zeta}} e^{-\frac{\lambda z}{\zeta}}$, $(1-z)^{\frac{\gamma_1}{\zeta}} e^{-\frac{\lambda z}{\zeta}}$ and $e^{-\frac{\lambda z}{\zeta}} z^{\frac{\mu_{v_2}}{\zeta}} (1-z)^{\frac{\gamma_2}{\zeta}}$ respectively, and integrating from 0 to z , we get

$$G_1(z) = \frac{e^{\frac{\lambda z}{\zeta}}}{(1-z)^{\frac{\theta}{\zeta}}} \left[G_1(0) - \left\{ \frac{\gamma \pi_{0,0} + \mu_b \pi_{0,1}}{\zeta} \right\} I_2(z) \right], \quad (5.15)$$

$$G_2(z) = \frac{e^{\frac{\lambda z}{\zeta}}}{(1-z)^{\frac{\gamma_1}{\zeta}}} \left[G_2(0) - \frac{\theta \pi_{1,0}}{\zeta} I_1(z) \right], \quad (5.16)$$

$$G_3(z) = \frac{e^{\frac{\lambda z}{\zeta}}}{z^{\frac{\mu_{v_2}}{\zeta}} (1-z)^{\frac{\gamma_2}{\zeta}}} \left[G_3(0) + \frac{1}{\zeta} \left\{ \mu_{v_2} \pi_{2,0} I_3(z) - \gamma_1 \pi_{v_1,0} I_4(z) \right\} \right]. \quad (5.17)$$

where

$$I_1(z) = \int_0^z (1-x)^{\frac{\gamma_1}{\zeta}-1} e^{-\frac{\lambda x}{\zeta}} dx, \quad I_2(z) = \int_0^z (1-x)^{\frac{\theta}{\zeta}-1} e^{-\frac{\lambda x}{\zeta}} dx,$$

$$I_3(z) = \int_0^z e^{-\frac{\lambda x}{\zeta}} x^{\frac{\mu v_2}{\zeta}-1} (1-x)^{\frac{\gamma_2}{\zeta}} dx \quad \text{and} \quad I_4(z) = \int_0^z e^{-\frac{\lambda x}{\zeta}} x^{\frac{\mu v_2}{\zeta}} (1-x)^{\frac{\gamma_2}{\zeta}-1} dx.$$

Setting $z = 1$ in Eq. (5.16),

$$\pi_{2,0} = \frac{\theta I_1(1)}{\zeta} \pi_{1,0}. \quad (5.18)$$

Applying Eq. (5.18) in Eq. (5.16), we obtain

$$G_2(z) = \frac{e^{\frac{\lambda z}{\zeta}}}{(1-z)^{\frac{\gamma_1}{\zeta}}} \left\{ 1 - \frac{I_1(z)}{I_1(1)} \right\} \pi_{2,0}. \quad (5.19)$$

Using the same procedure in Eqs. (5.15) and (5.17), we get

$$\pi_{2,0} = \left[\frac{\xi + \mu v_2 I_3(1)}{\gamma_1 I_4(1)} \right] \pi_{3,0}, \quad (5.20)$$

$$G_3(z) = \frac{e^{\frac{\lambda z}{\zeta}}}{z^{\frac{\mu v_2}{\zeta}} (1-z)^{\frac{\gamma_2}{\zeta}}} \left[1 + \frac{1}{\zeta} \left\{ \mu v_2 I_3(z) - \left(\frac{\xi + \mu v_2 I_3(1)}{I_4(1)} \right) I_4(z) \right\} \right] \pi_{3,0}, \quad (5.21)$$

$$\frac{\pi_{1,0}}{I_2(1)} = \frac{\gamma \pi_{0,0} + \mu_b \pi_{0,1}}{\zeta}, \quad (5.22)$$

$$G_1(z) = \frac{e^{\frac{\lambda z}{\zeta}}}{(1-z)^{\frac{\theta}{\zeta}}} \left[1 - \frac{I_2(z)}{I_2(1)} \right] \pi_{1,0}. \quad (5.23)$$

Using Eqs. (5.1), (5.18) and (5.20) in Eqs. (5.19), (5.21) and (5.23), we obtain

$$G_1(z) = \frac{e^{\frac{\lambda z}{\zeta}}}{(1-z)^{\frac{\theta}{\zeta}}} \frac{\zeta}{\theta I_1(1)} \left[1 - \frac{I_2(z)}{I_2(1)} \right] \left[\frac{\xi + \mu v_2 I_3(1)}{\gamma_1 I_4(1)} \right] \alpha_2 \pi_{0,0}, \quad (5.24)$$

$$G_2(z) = \frac{e^{\frac{\lambda z}{\zeta}}}{(1-z)^{\frac{\gamma_1}{\zeta}}} \left[1 - \frac{I_1(z)}{I_1(1)} \right] \left[\frac{\xi + \mu v_2 I_3(1)}{\gamma_1 I_4(1)} \right] \alpha_2 \pi_{0,0}, \quad (5.25)$$

$$G_3(z) = \frac{e^{\frac{\lambda z}{\zeta}}}{z^{\frac{\mu v_2}{\zeta}} (1-z)^{\frac{\gamma_2}{\zeta}}} \left[1 + \frac{1}{\zeta} \left\{ \mu v_2 I_3(z) - \left(\frac{\xi + \mu v_2 I_3(1)}{I_4(1)} \right) I_4(z) \right\} \right] \alpha_2 \pi_{0,0}, \quad (5.26)$$

where

$$\alpha_2 = \frac{\lambda + \gamma}{\gamma_2}. \quad (5.27)$$

The solution of the differential Eqs. (5.12)-(5.14) is expressed in terms of $\pi_{0,0}$.

Remark 1 In the absence of WV; i.e., $\mu v_2 = 0$ and if $\gamma_1 = \mu$, $\gamma_2 = \gamma$, $\zeta = \xi$ and $\pi_{2,0} = P_{1,k}$ then Eq. (5.17) reduce to

$$G_3(z) = \frac{e^{\frac{\lambda z}{\zeta}}}{(1-z)^{\frac{\gamma_2}{\zeta}}} \left[G_3(0) - \frac{\mu P_{1,K}}{\xi} \int_0^z e^{-\frac{\lambda x}{\zeta}} (1-x)^{\frac{\gamma}{\zeta}-1} dx \right].$$

The above expression agrees with the result (12) of Yue et al. (2014).

5.2 Evaluation of $\pi_{1,n}$, $\pi_{2,n}$ and $\pi_{3,n}$

An explicit expression for $\pi_{1,n}$, $\pi_{2,n}$ and $\pi_{3,n}$ is obtained in this section using incomplete Gamma function and confluent hypergeometric function.

From Eq. (5.19), we know that

$$I_1(z) = \int_0^z (1-x)^{\frac{\gamma_1}{\zeta}-1} e^{-\frac{\lambda x}{\zeta}} dx. \quad (5.28)$$

By applying a transformation $(x-1)\frac{\lambda}{\zeta} = t$ and using the definition of incomplete Gamma function $\gamma(\alpha, z) = \int_0^z e^{-t} t^{\alpha-1} dt$, the above equation becomes

$$I_1(z) = \frac{\exp\left(-\frac{\lambda}{\zeta}\right)}{\left(-\frac{\lambda}{\zeta}\right)^{\frac{\gamma_1}{\zeta}}} \left[\gamma\left(\frac{\gamma_1}{\zeta}, -\frac{\lambda}{\zeta}(1-z)\right) - \gamma\left(\frac{\gamma_1}{\zeta}, -\frac{\lambda}{\zeta}\right) \right]. \quad (5.29)$$

By applying the identity $\gamma(\alpha, z) = \alpha^{-1} z^\alpha {}_1F_1(\alpha, \alpha+1, -z)$. Hence

$$I_1(z) = \frac{\exp\left(-\frac{\lambda}{\zeta}\right)}{\frac{\gamma_1}{\zeta}} \left[(1-z)^{\frac{\gamma_1}{\zeta}} {}_1F_1\left(\frac{\gamma_1}{\zeta}; 1 + \frac{\gamma_1}{\zeta}; \frac{\lambda}{\zeta}(1-z)\right) - {}_1F_1\left(\frac{\gamma_1}{\zeta}; 1 + \frac{\gamma_1}{\zeta}; \frac{\lambda}{\zeta}\right) \right]. \quad (5.30)$$

By applying the identity [Abramowitz and Stegun (1970), P-505, 13.1.27], the above equation becomes

$$I_1(z) = \frac{\exp\left(-\frac{\lambda}{\zeta}z\right)}{\frac{\gamma_1}{\zeta}} \left[(1-z)^{\frac{\gamma_1}{\zeta}} \exp\left\{\frac{\lambda}{\zeta}(1-z)\right\} {}_1F_1\left(1; 1 + \frac{\gamma_1}{\zeta}; \frac{\lambda}{\zeta}(z-1)\right) - \exp\left(\frac{\lambda}{\zeta}\right) {}_1F_1\left(1; 1 + \frac{\gamma_1}{\zeta}; -\frac{\lambda}{\zeta}\right) \right]. \quad (5.31)$$

Substituting $z = 1$ in Eq. (5.31), we obtain

$$I_1(1) = -\frac{1}{\frac{\gamma_1}{\zeta}} {}_1F_1\left(1; 1 + \frac{\gamma_1}{\zeta}; -\frac{\lambda}{\zeta}\right). \quad (5.32)$$

Using Eqs. (5.31) and (5.32) in Eq. (5.19), we obtain

$$G_2(z) = \frac{{}_1F_1\left(1; 1 + \frac{\gamma_1}{\zeta}; -\frac{\lambda}{\zeta}(1-z)\right)}{{}_1F_1\left(1; 1 + \frac{\gamma_1}{\zeta}; -\frac{\lambda}{\zeta}\right)} \pi_{2,0}, \quad (5.33)$$

where

$${}_1F_1\left(1; 1 + \frac{\gamma_1}{\zeta}; -\frac{\lambda}{\zeta}(1-z)\right) = \sum_{k=0}^{\infty} (-1)^k \frac{\left(-\frac{\lambda}{\zeta}\right)^{n+k} \binom{n+k}{n} z^n}{\left(1 + \frac{\gamma_1}{\zeta}\right)_{n+k}}. \quad (5.34)$$

Applying Eq. (5.34) in (5.33) and comparing the coefficient of z^n on both sides, we get

$$\pi_{2,n} = \left(\frac{\lambda}{\zeta}\right)^n \frac{1}{\left(1 + \frac{\gamma_1}{\zeta}\right)_n} \frac{{}_1F_1\left(n+1; \frac{\gamma_1}{\zeta} + n+1; -\frac{\lambda}{\zeta}\right)}{{}_1F_1\left(1; 1 + \frac{\gamma_1}{\zeta}; -\frac{\lambda}{\zeta}\right)} \pi_{2,0}. \quad (5.35)$$

By applying the same procedure on Eqs. (5.21) and (5.23), we obtain

$$\pi_{3,n} = \left(\frac{\lambda}{\zeta}\right)^n \frac{1}{\left(\frac{\mu_{v_2} + \gamma_2}{\zeta} + 1\right)_n} \frac{{}_1F_1\left(\frac{\mu_{v_2}}{\zeta} + n+1; \frac{\mu_{v_2} + \gamma_2}{\zeta} + n+1; -\frac{\lambda}{\zeta}\right)}{{}_1F_1\left(\frac{\mu_{v_2}}{\zeta} + 1; \frac{\mu_{v_2} + \gamma_2}{\zeta} + 1; -\frac{\lambda}{\zeta}\right)} \pi_{3,0}, \quad (5.36)$$

$$\pi_{1,n} = \left(\frac{\lambda}{\zeta}\right)^n \frac{1}{\left(\frac{\theta}{\zeta} + 1\right)_n} \frac{{}_1F_1\left(n+1; \frac{\theta}{\zeta} + n+1; -\frac{\lambda}{\zeta}\right)}{{}_1F_1\left(1; \frac{\theta}{\zeta} + 1; -\frac{\lambda}{\zeta}\right)} \pi_{1,0}, \quad (5.37)$$

where $(\alpha)_n$ represents Pochhammer symbol and it is given by

$$(\alpha)_n = \begin{cases} \alpha(\alpha+1)(\alpha+2) \dots (\alpha+n-1), & n = 1, 2, 3, \dots \\ 1, & n = 0. \end{cases}$$

The probabilities $\pi_{1,0}$, $\pi_{2,0}$ and $\pi_{3,0}$ can be expressed in terms of $\pi_{0,0}$ using Eqs. (5.1), (5.18) and (5.20).

Remark 2 Equations (5.35)–(5.37) coincides with the results (3.12), (3.9) and (3.11) on applying the condition $\lim_{s \rightarrow 0} s \hat{P}_{i,n}(s) = \pi_{i,n}$, $i = 1, 2, 3$.

5.3 Evaluation of $\pi_{0,0}$

This section presents the idle state probability of the system. We define a generating function

$$G_0(z) = \sum_{n=1}^{\infty} \pi_{0,n} z^n. \quad (5.38)$$

Multiplying Eq. (5.2) by z^n and summing over $n = 1, 2, 3, \dots$, we obtain

$$G_0(z) = z \left[\frac{-\mu_b \pi_{0,1} + \lambda z \pi_{0,0} + \gamma_1 (G_2(z) - \pi_{2,0}) + \gamma_2 (G_3(z) - \pi_{3,0}) + \theta (G_1(z) - \pi_{1,0})}{(1-z)(z\lambda - \mu_b)} \right]. \quad (5.39)$$

After some algebraic manipulation, the above equation becomes

$$G_0(z) = z \left[\frac{-\lambda \pi_{0,0} (1-z) + \gamma_1 (G_2(z) - G_2(1)) + \gamma_2 (G_3(z) - G_3(1)) + \theta (G_1(z) - G_1(1))}{(1-z)(z\lambda - \mu_b)} \right]. \quad (5.40)$$

Setting $z = 1$ in Eq. (5.40) and applying L'Hopital's rule, we obtain

$$G_0(1) = \left[\frac{\lambda \pi_{0,0} + \gamma_1 G'_2(1) + \gamma_2 G'_3(1) + \theta G'_1(1)}{\mu_b - \lambda} \right]. \quad (5.41)$$

Using Eqs. (5.12)-(5.14) for $z = 1$ and using L'Hopital's rule, we get

$$G'_1(1) = \frac{\zeta^2}{\theta^2(\zeta + \theta)} \frac{\lambda}{I_1(1) I_2(1)} \left[\frac{\zeta + \mu_{v_2} I_3(1)}{\gamma_1 I_4(1)} \right] \alpha_2 \pi_{0,0}, \quad (5.42)$$

$$G'_2(1) = \frac{\lambda \zeta}{\gamma_1(\zeta + \gamma_1) I_1(1)} \left[\frac{\zeta + \mu_{v_2} I_3(1)}{\gamma_1 I_4(1)} \right] \alpha_2 \pi_{0,0}, \quad (5.43)$$

$$G'_3(1) = \frac{1}{(\zeta + \gamma_2)} \left[\mu_{v_2} + \frac{(\zeta + \mu_{v_2} I_3(1))(\lambda - \mu_{v_2})}{\gamma_2 I_4(1)} \right] \alpha_2 \pi_{0,0}. \quad (5.44)$$

Substituting Eqs. (5.42)-(5.44) in Eq. (5.41), we obtain

$$G_0(1) = \frac{1}{\mu_b - \lambda} \left[\frac{\lambda}{\alpha_2} + \left\{ \frac{\zeta^2}{\theta \gamma_1(\zeta + \theta) I_1(1) I_2(1)} + \frac{\lambda \zeta}{\gamma_1(\zeta + \gamma_1) I_1(1)} + \frac{\lambda - \mu_{v_2}}{\zeta + \gamma_2} \right\} \right. \\ \left. \times \frac{\zeta + \mu_{v_2} I_3(1)}{I_4(1)} + \frac{\mu_{v_2} \gamma_2}{\zeta + \gamma_2} \right] \alpha_2 \pi_{0,0}. \quad (5.45)$$

By the normalization condition given below, we can obtain an explicit expression for $\pi_{0,0}$.

$$\pi_{0,0} + G_0(1) + G_1(1) + G_2(1) + G_3(1) = 1 \quad (5.46)$$

From Eqs. (5.24)-(5.27), we get

$$G_1(1) = \sum_{n=0}^{\infty} \pi_{1,n} = \frac{\zeta^2}{\theta^2} \frac{1}{I_1(1) I_2(1)} \left[\frac{\zeta + \mu_{v_2} I_3(1)}{\gamma_1 I_4(1)} \right] \alpha_2 \pi_{0,0}. \quad (5.47)$$

$$G_2(1) = \sum_{n=0}^{\infty} \pi_{2,n} = \frac{\zeta}{\gamma_1 I_1(1)} \left[\frac{\zeta + \mu_{v_2} I_3(1)}{\gamma_1 I_4(1)} \right] \alpha_2 \pi_{0,0}, \quad (5.48)$$

$$G_3(1) = \sum_{n=0}^{\infty} \pi_{3,n} = \left[\frac{\zeta + \mu_{v_2} I_3(1)}{\gamma_2 I_4(1)} \right] \alpha_2 \pi_{0,0}. \quad (5.49)$$

Applying Eqs. (5.45), (5.47)-(5.49) in Eq. (5.46), we get

$$\pi_{0,0} = \left[1 + \left\{ \frac{\zeta^2}{\theta^2 \gamma_1} \frac{1}{I_1(1) I_2(1)} + \frac{\zeta}{\gamma_1^2 I_1(1)} + \frac{1}{\gamma_2} \right\} \left\{ \frac{\zeta + \mu_{v_2} I_3(1)}{I_4(1)} \right\} \alpha_2 \right. \\ \left. + \frac{1}{\mu_b - \lambda} \left\{ \frac{\lambda}{\alpha_2} + \left(\frac{\zeta^2}{\theta \gamma_1(\zeta + \theta) I_1(1) I_2(1)} + \frac{\lambda \zeta}{\gamma_1(\zeta + \gamma_1) I_1(1)} + \frac{\lambda - \mu_{v_2}}{\zeta + \gamma_2} \right) \right. \right. \\ \left. \left. \frac{\zeta + \mu_{v_2} I_3(1)}{I_4(1)} + \frac{\mu_{v_2} \gamma_2}{\zeta + \gamma_2} \right\} \alpha_2 \right]^{-1}, \quad (5.50)$$

6 Performance Measures for Steady State

This section presents the expected system size, the average rate of abandonment, the expected data packets served and proportion of data packets served.

6.1 Expected System Size

Let $E[N_s]$ be the expected number of data packets in the system and let $E[N_0]$, $E[N_1]$, $E[N_2]$ and $E[N_3]$ denote the mean number of data packets in functional state, close-down state, vacation state and WV respectively. Then

$$E[N_s] = E[N_0] + E[N_1] + E[N_2] + E[N_3],$$

where

$$E(N_j) = \lim_{z \rightarrow 1} G'_j(z), \quad j = 0, 1, 2, 3.$$

The expression for $G'_j(1)$, $j = 1, 2, 3$ is given in Eqs. (5.42)-(5.45). Differentiating Eq. (5.40) and substituting $z = 1$, we get

$$G'_0(1) = \left[\frac{\rho}{(1-\rho)} \pi_{0,0} + \frac{\gamma_1 G'_2(1) + \gamma_2 G'_3(1) + \theta G'_1(1)}{\mu_b(1-\rho)} + \frac{\gamma_1 G''_2(1) + \gamma_2 G''_3(1) + \theta G''_1(1)}{2\mu_b} \right] \times \frac{1}{(1-\rho)}. \quad (6.1)$$

The expression for $G''_j(1)$, $j = 1, 2, 3$ is obtained by differentiating Eqs. (5.9)-(5.11) twice and setting $z = 1$.

$$G''_1(1) = \frac{2\zeta^2\lambda^2}{\theta^2(2\zeta + \theta)(\zeta + \theta)I_1(1)I_2(1)} \left[\frac{\zeta + \mu_{v_2}I_3(1)}{\gamma_1 I_4(1)} \right] \alpha_2 \pi_{0,0}, \quad (6.2)$$

$$G''_2(1) = \frac{2\lambda^2\zeta}{\gamma_1(\zeta + \gamma_1)(2\zeta + \gamma_1)I_1(1)} \left[\frac{\zeta + \mu_{v_2}I_3(1)}{\gamma_1 I_4(1)} \right] \alpha_2 \pi_{0,0}, \quad (6.3)$$

$$G''_3(1) = \frac{2}{2\zeta + \gamma_2} \left[\frac{\lambda - \zeta - \mu_{v_2} - \gamma_2}{\zeta + \gamma_2} \left\{ \mu_{v_2} + \frac{(\zeta + \mu_{v_2}I_3(1))(\lambda - \mu_{v_2})}{\gamma_2 I_4(1)} \right\} + \lambda \left\{ \frac{\zeta + \mu_{v_2}I_3(1)}{\gamma_2 I_4(1)} \right\} \right] \alpha_2 \pi_{0,0}. \quad (6.4)$$

Substituting Eqs. (6.2)-(6.4) and (5.42)-(5.44) in Eq. (6.1), we get an explicit expression for $E[N_0]$.

6.2 Average Abandonment Rate

Let R_I denote the average abandonment rate due to impatience. Then,

$$R_I = \zeta \left[\sum_{n=0}^{\infty} n \pi_{1,n} + \sum_{n=0}^{\infty} n \pi_{2,n} + \sum_{n=0}^{\infty} n \pi_{3,n} \right].$$

Using Eqs. (5.42)-(5.44), we get

$$R_I = \left[\frac{\zeta + \mu_{v_2}I_3(1)}{I_4(1)} \left\{ \frac{\zeta^2}{\theta^2\gamma_1(\zeta + \theta)I_1(1)I_2(1)} + \frac{\lambda}{\gamma_1^2(\zeta + \gamma_1)I_1(1)} + \frac{\lambda\zeta}{\gamma_2(\zeta + \gamma_2)} + \frac{\lambda - \mu_{v_2}}{\gamma_2(\zeta + \gamma_2)} \right\} + \frac{\mu_{v_2}}{\zeta + \gamma_2} \right] \alpha_2 \pi_{0,0}.$$

6.3 System Throughput

Let E_S denote the expected number of data-frame served per unit of time. Then

$$E_S = \left[\mu_b G_0(1) + \mu_{v_2} (G_3(1) - \pi_{3,0}) \right].$$

Using Eqs. (5.45) and (5.49), we get

$$E_S = \frac{\mu_b}{\mu_b - \lambda} \left[\frac{\lambda}{\alpha_2} + \left\{ \frac{\zeta^2}{\theta \gamma_1 (\zeta + \theta)} \frac{\lambda}{I_1(1) I_2(1)} + \frac{\lambda \zeta}{\gamma_1 (\zeta + \gamma_1) I_1(1)} + \frac{\lambda - \mu_{v_2}}{\zeta + \gamma_2} \right\} \right. \\ \left. \frac{\zeta + \mu_{v_2} I_3(1)}{I_4(1)} + \frac{\mu_{v_2} \gamma_2}{\zeta + \gamma_2} \right] \alpha_2 \pi_{b,0} + \mu_{v_2} \left\{ \frac{\zeta + \mu_{v_2} I_3(1)}{\gamma_2 I_4(1)} - 1 \right\} \alpha_2 \pi_{0,0}.$$

$$\text{Proportion of data packets served} = P_S = \frac{E_S}{\lambda}.$$

7 Cost Analysis

This section presents cost analysis of the model considered. The total expected cost (TEC) and total expected revenue (TER) are defined as follows

$$TEC = f(\mu_b^*, \mu_{v_2}^*) = C_0 G_0(1) + C_1 G_1(1) + C_2 G_2(1) + C_3 G_3(1) \\ + C_4 \mu_b + C_5 \mu_{v_2} + C_h E(N_s) + C_L R_I. \quad (7.1)$$

where

C_0 : CPUT (Cost per unit time) when the server is busy

C_1 : CPUT when the MS is on close-down state

C_2 : CPUT when the MS is on vacation

C_3 : CPUT when the MS is on WV

C_L : CPUT when a data packet is lost

C_4 : Cost per data packet served with rate μ_b

C_5 : Cost per data packet served with rate μ_{v_2}

C_h : Carrying cost

It is important to note that the cost function seems to be a complex and nonlinear expression. This is due to the fact that there are many parameters such λ , μ_b , μ_{v_2} , ζ , γ , γ_1 , γ_2 and θ . Our aim is to determine the optimal service rates of the server during the busy period μ_b and working vacation period μ_{v_2} to minimize the TEC function. The mathematical representation of the cost minimization problem is as follows

$$f(\mu_b^*, \mu_{v_2}^*) = \min f(\mu_b, \mu_{v_2}),$$

subject to

$$\mu_b > \mu_{v_2} \text{ and } \left\{ 1 - \frac{\gamma_2 I_4(1)}{\zeta + I_3(1) \mu_{v_2}} \right\} < \frac{\lambda}{\mu_{v_2}}.$$

Let R as the revenue generated from delivering services to a customer, then total expected revenue (TER) is given by

$$TER = R \times E(N_s).$$

8 Numerical Illustrations

This section presents the graphical illustration of the system size probabilities. The performance indices of the proposed system are numerically computed using MATLAB software.

8.1 Numerical Illustrations of the Time-Dependent Probabilities

The system's performance measures in a transient state are analysed in this section. The values of the parameter are chosen as follows. $\lambda = 1$, $\mu_b = 1.25$, $\mu_{v2} = 1.1$, $\gamma = 0.8$, $\gamma_1 = 0.4$, $\gamma_2 = 0.7$, $\theta = 0.5$ and $\zeta = 0.1$.

Figure 2 illustrates the performance of the functional state probability $P_{0,n}(t)$. Since it is assumed $P_{0,0}(0) = 1$, it is clear from the graph that the curve for $P_{0,0}(t)$ starts at 1 and gradually declines as 't' rises until it reaches the steady state. The other $P_{0,0}(t)$ curves begin at zero and approach the steady state as 't' increases.

The behaviour of the close-down state $P_{1,n}(t)$, vacation $P_{2,n}(t)$, and WV state $P_{3,n}(t)$ are each illustrated in Figs. 3, 4 and 5. We observe that, as 't' increases, the probability curves of the vacation states increase to a certain extent and then the curves start decreasing. As 't' increases, many data packets may wait for service during vacation. As a result, the vacation state $P_{2,n}(t)$ curves increase as 't' increases. If any data packets are still waiting at the end of the vacation period, the MS enters the busy state. As a result, the $P_{2,n}(t)$ curve decreases. It is clear from Figs. 6 and 7 that the mean system size reduces as the rate of signal loss ζ rises.

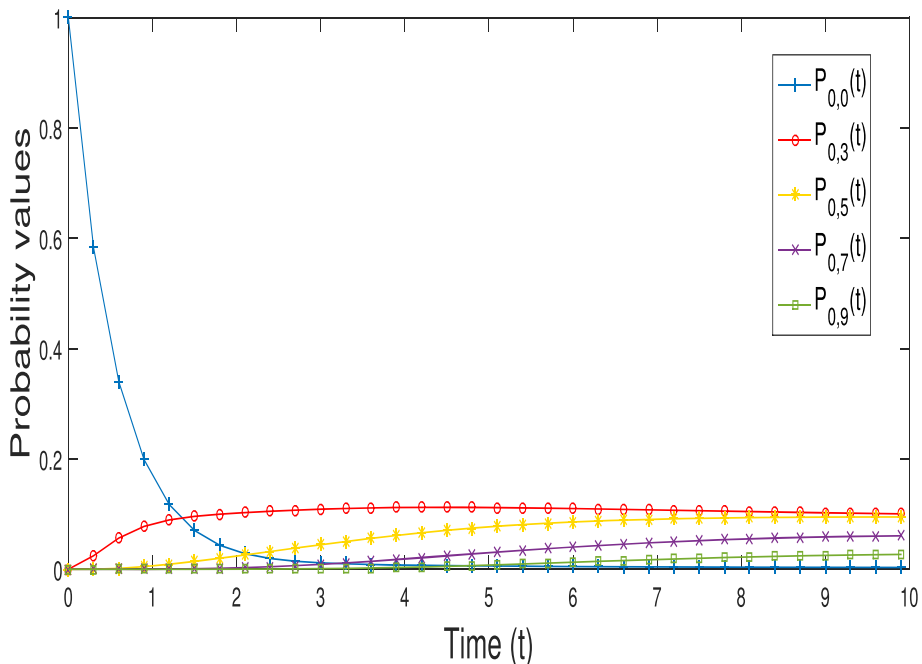


Fig. 2 Probability of data packets in the functional state $P_{0,n}(t)$ versus t

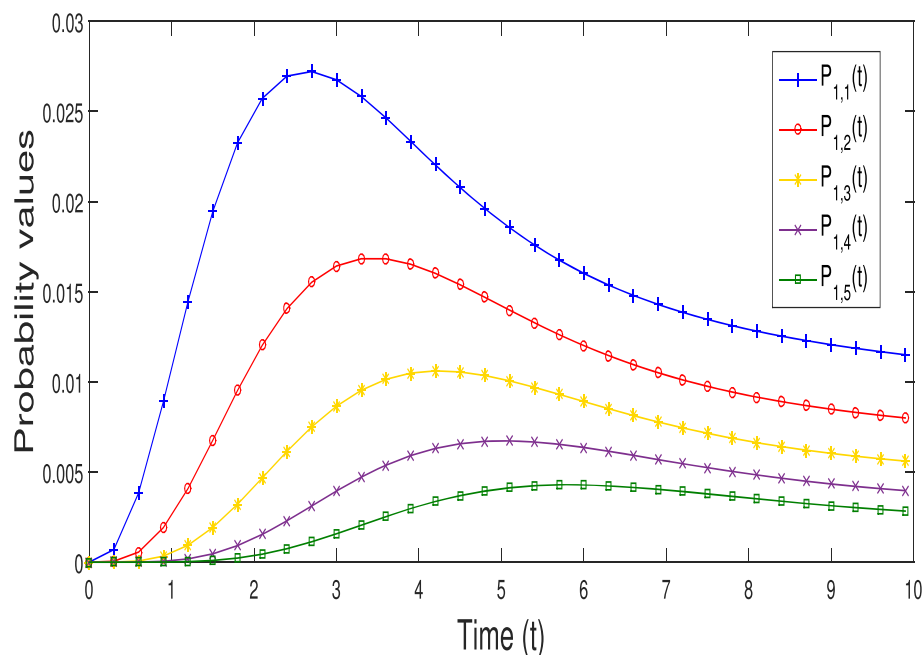


Fig. 3 Probability of data packets in the close-down state $P_{1,n}(t)$ versus t

8.2 Numerical Illustrations of Steady-State Probabilities

This section focuses on analyzing the performance measures of the system in a steady-state. The parameter values selected for the analysis are as follows: $\lambda = 1$, $\mu_b = 3$, $\mu_{v_2} = 0.9$, $\gamma = 1$, $\gamma_1 = 1.5$, $\gamma_2 = 1.7$, and $\theta = 1.8$.

Figures 8, 9, 10 and 11 present the expected number of data packets in the busy state $E(N_0)$, close-down state $E(N_1)$, vacation $E(N_2)$, and waiting for vacation $E(N_3)$, respectively. The graphs are plotted against λ , with the impatience rate ζ being varied. It is observed that as λ increases, the expected system size also increases initially, and then the curve starts to decrease. This behavior can be attributed to the phenomenon where an increase in the number of data packets (buffer overflow) leads to more customers reneging and leaving the system.

Figures 12, 13 and 14 depict the probability of having n data packets in the close-down state $\pi_{1,n}$, vacation state $\pi_{2,n}$, and waiting for vacation state $\pi_{3,n}$, respectively. The graphs are plotted against n , while varying the arrival rate λ and keeping the impatience rate ζ fixed at 0.5. As n increases, the curves of these graphs steadily deteriorate until they reach zero.

Figure 15 demonstrates the mean rate of abandonment R_I due to impatience. From the graph, it is observed that as the impatience rate ζ increases, the curves of R_I also increase. This indicates that a higher impatience rate leads to more jobs leaving the system.

Lastly, Fig. 16 displays the graph of the proportion of served data packets (P_s).

8.3 Numerical Illustrations of the Cost Model

The PSO algorithm is employed to address the numerical optimization of the cost function. The following values are chosen for the PSO algorithm: $C_0 = 500$, $C_1 = 300$, $C_2 = 350$,

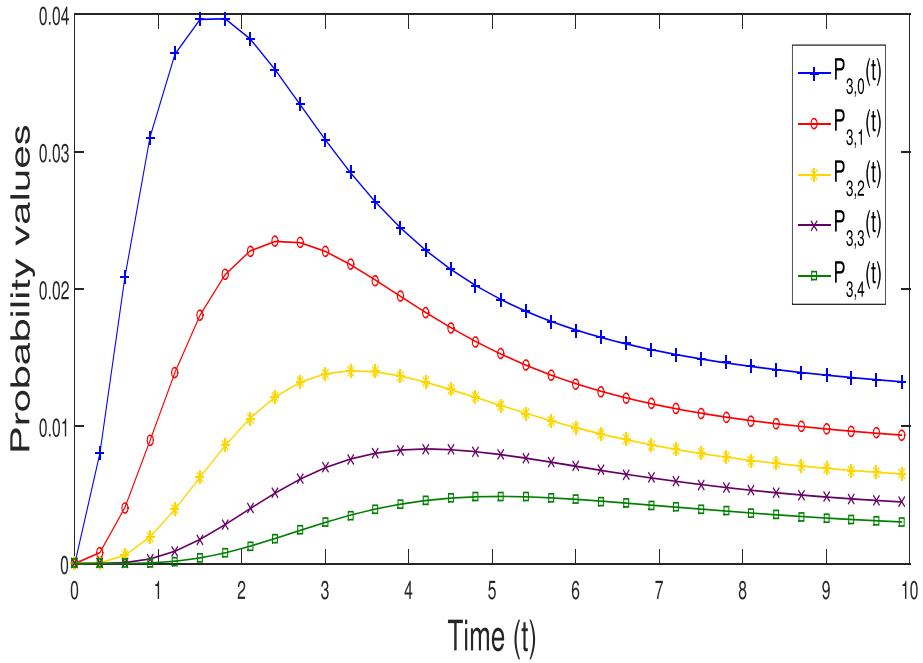


Fig. 4 Probability of data packets in the vacation state $P_{2,n}(t)$ versus t

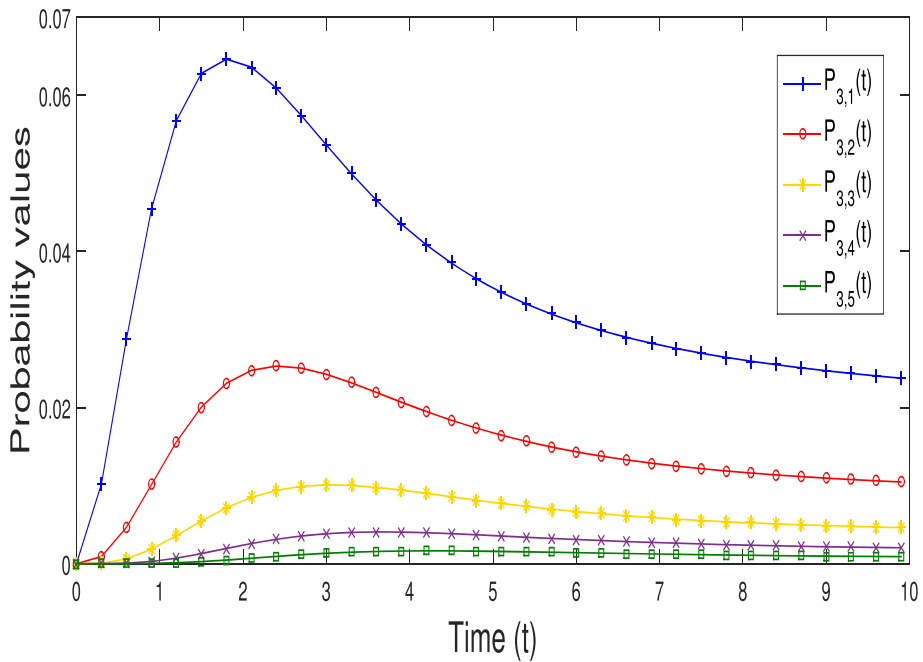


Fig. 5 Probability of data packets in the WV $P_{3,n}(t)$ versus t

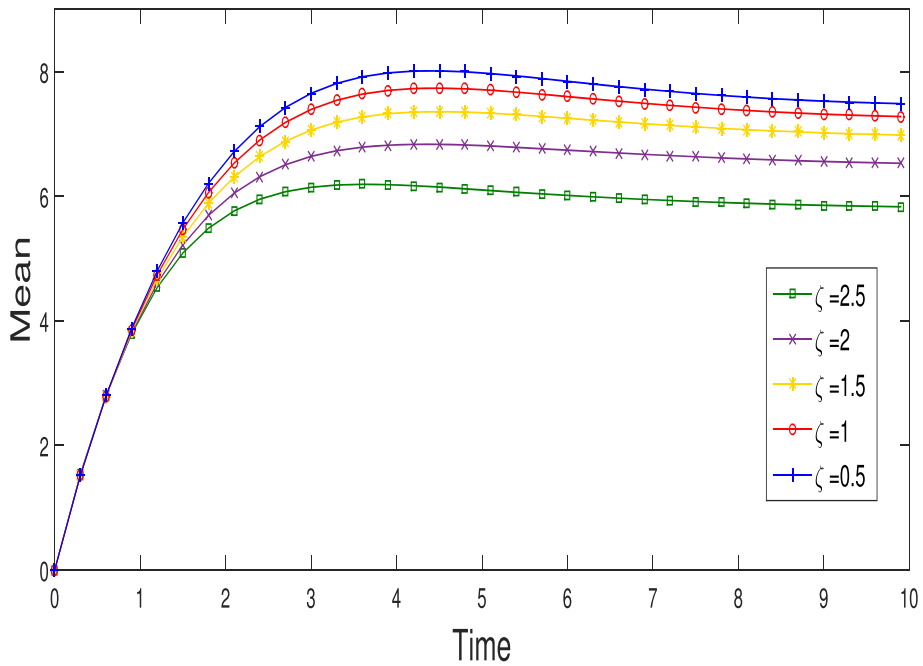


Fig. 6 Expected number of data packets in the system $E[X(t)]$ versus t

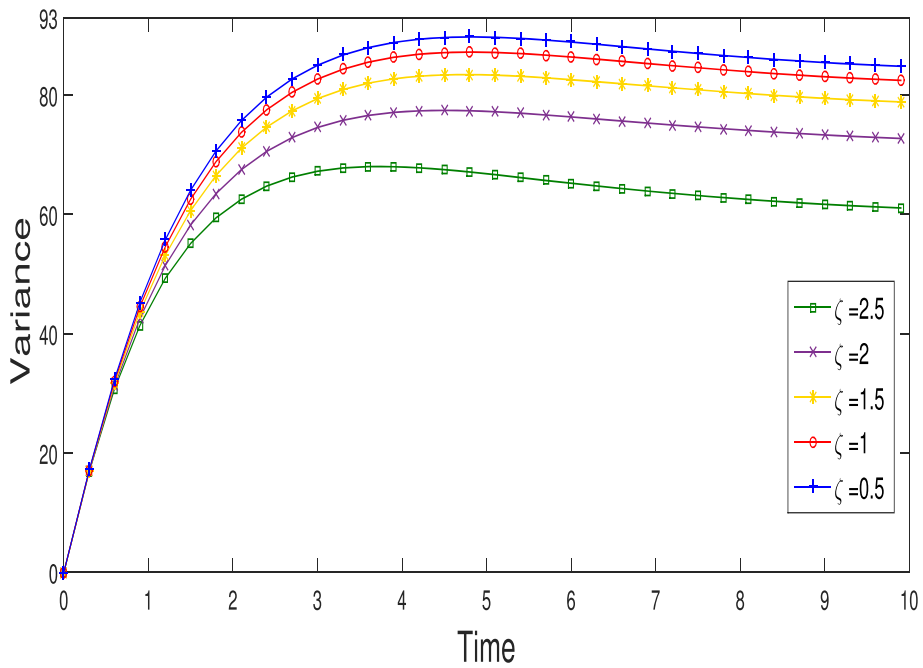


Fig. 7 Variance of the system $V(t)$ versus t

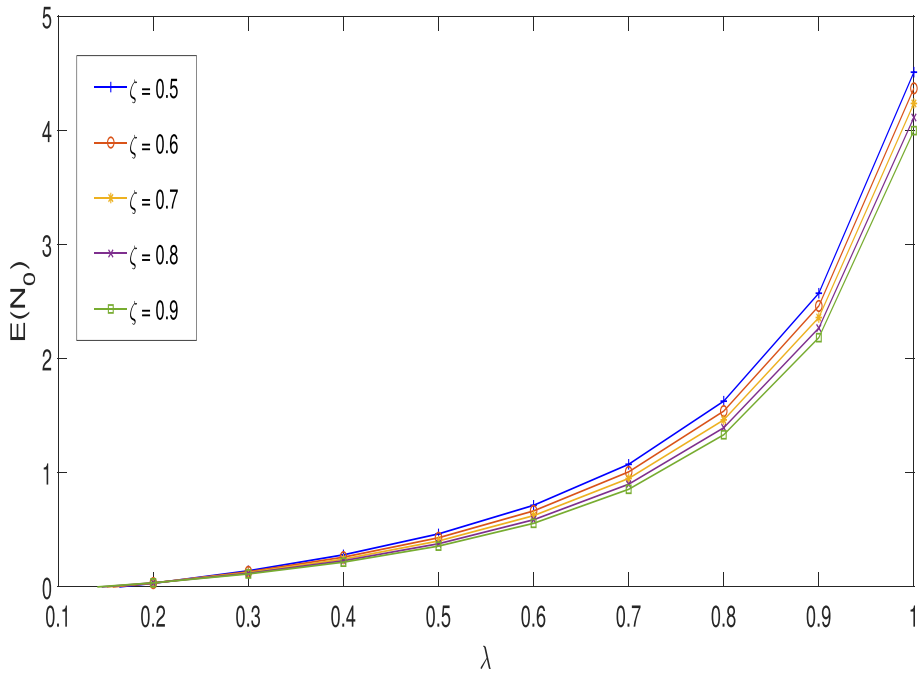


Fig. 8 Expected number of data packets in the functional state $E(N_0)$ versus λ

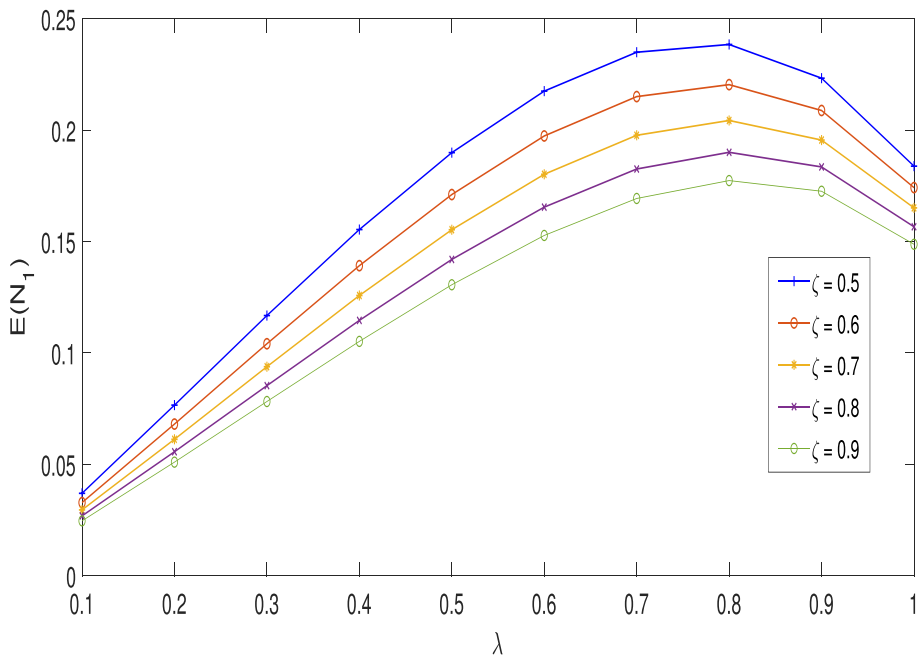


Fig. 9 Expected number of data packets in the close-down state versus λ

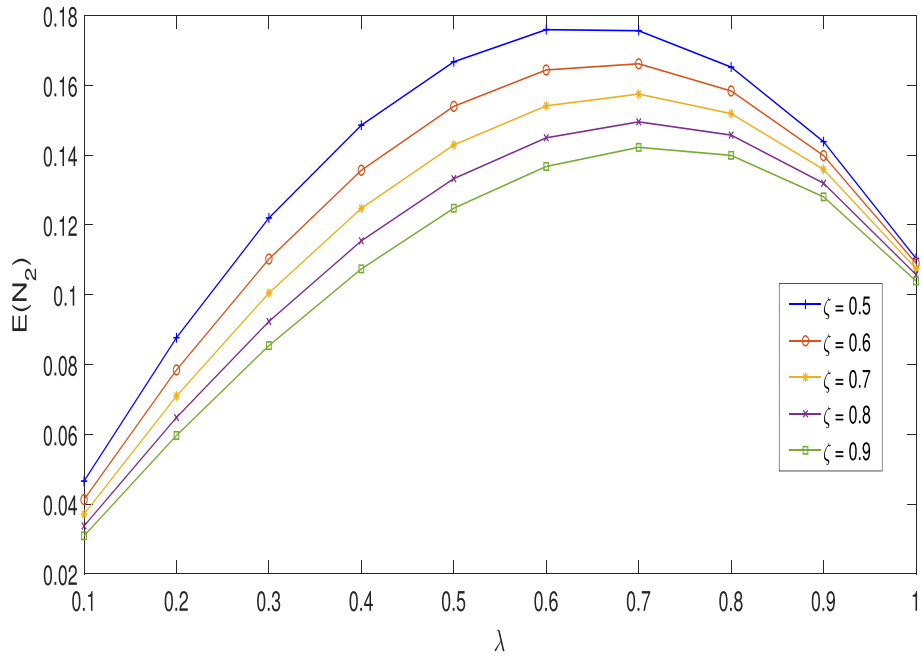


Fig. 10 Expected number of data packets in the vacation state versus λ

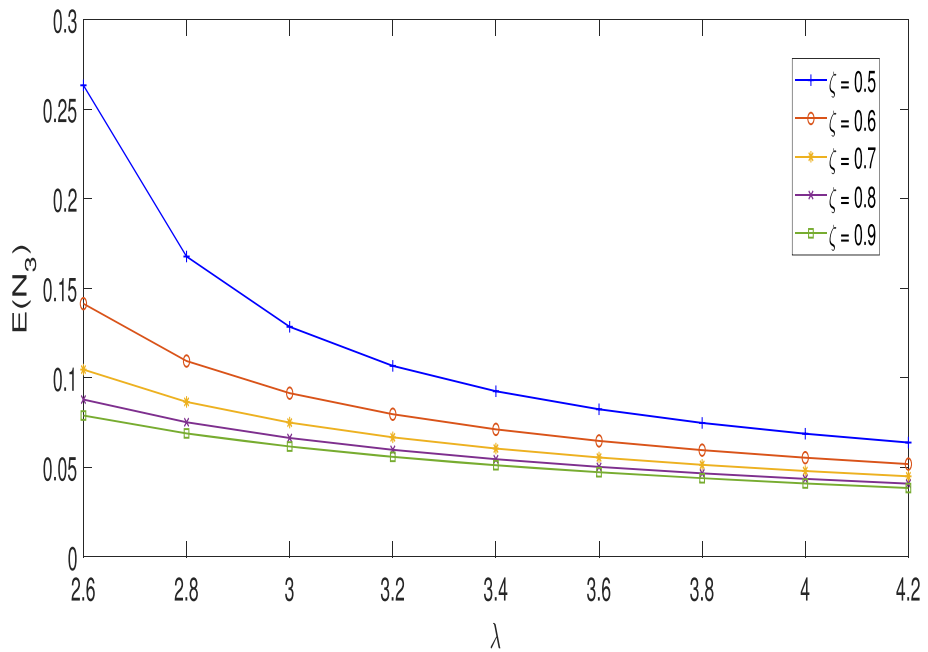


Fig. 11 Expected number of data packets in the WV state versus λ

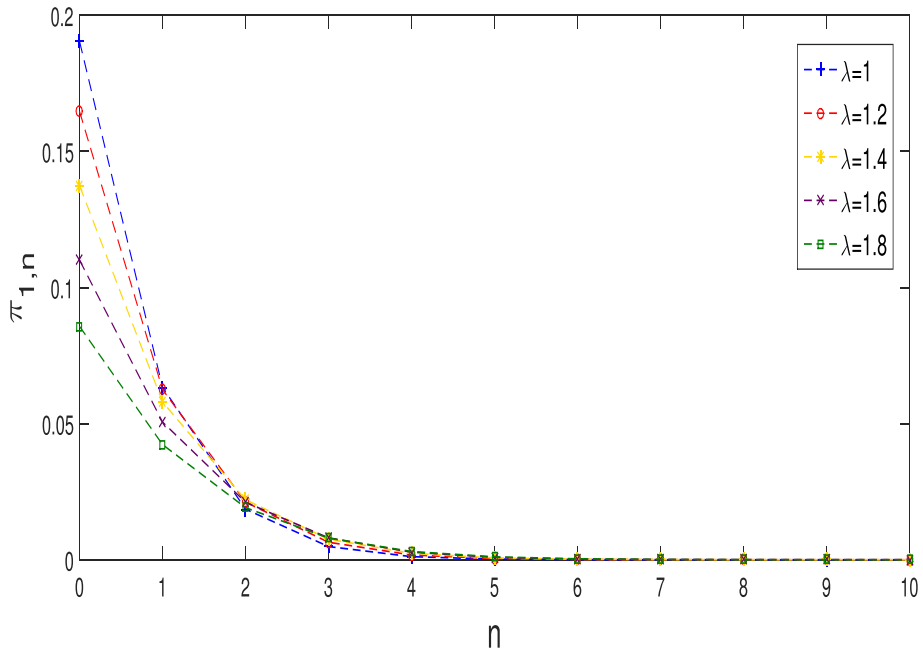


Fig. 12 Probability of n data packets in the close-down state versus n

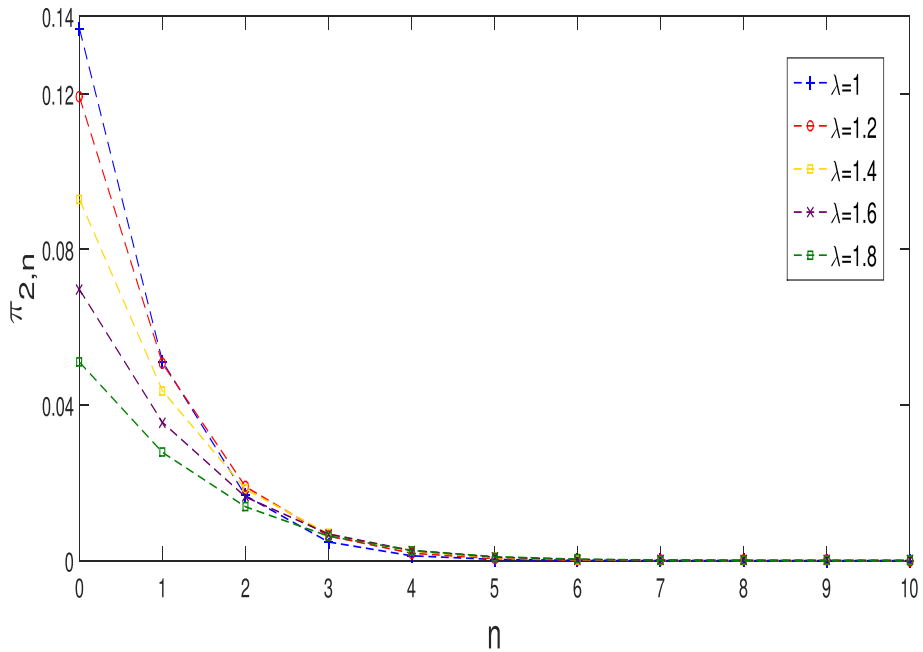


Fig. 13 Probability of n data packets in the vacation state $\pi_{2,n}$ versus n

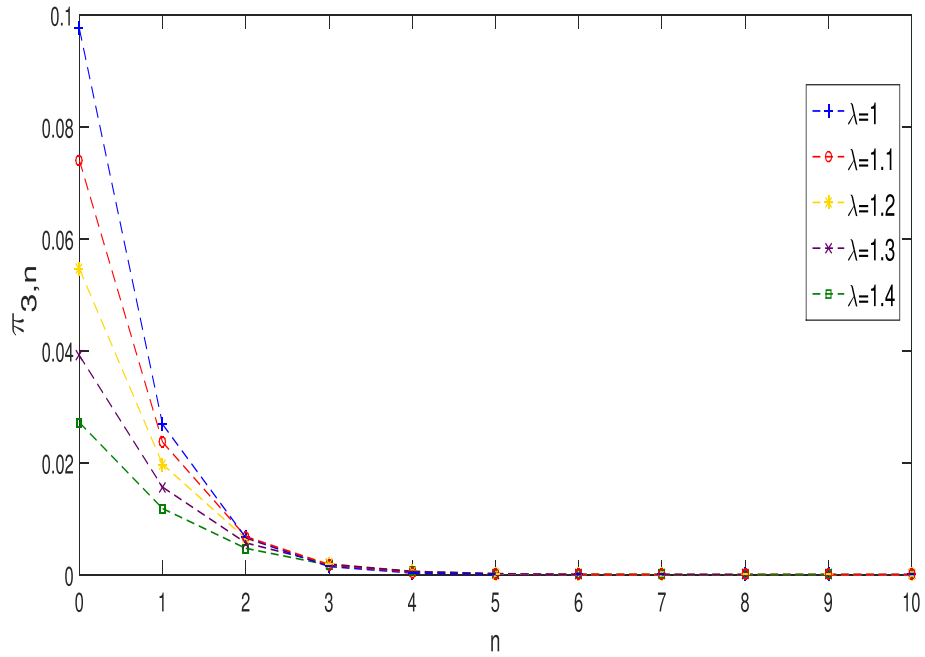


Fig. 14 Probability of n data packets in the WV state $\pi_{3,n}$ versus n

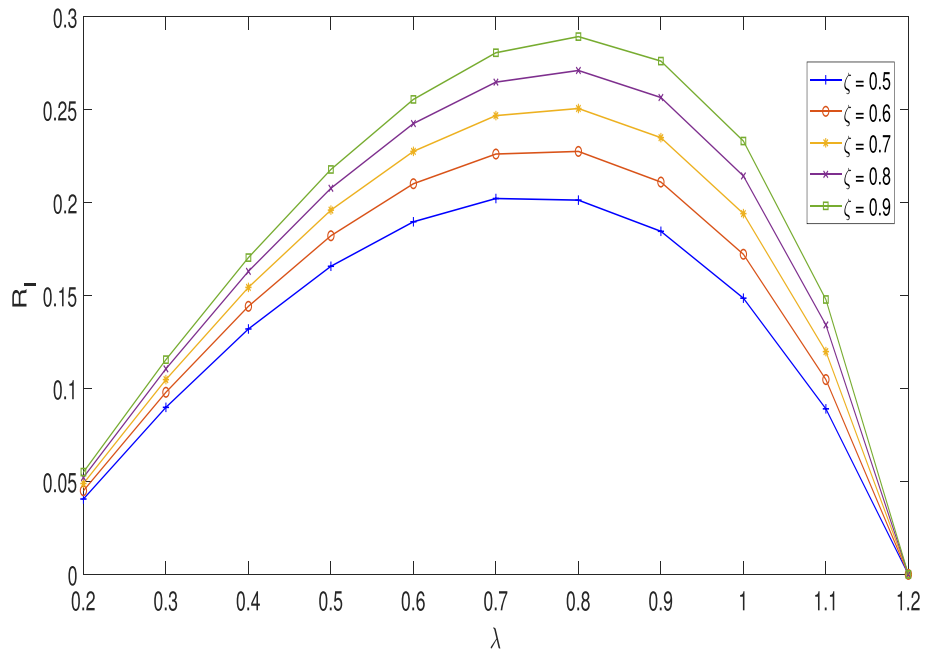


Fig. 15 Mean rate of abandonment R_I versus λ

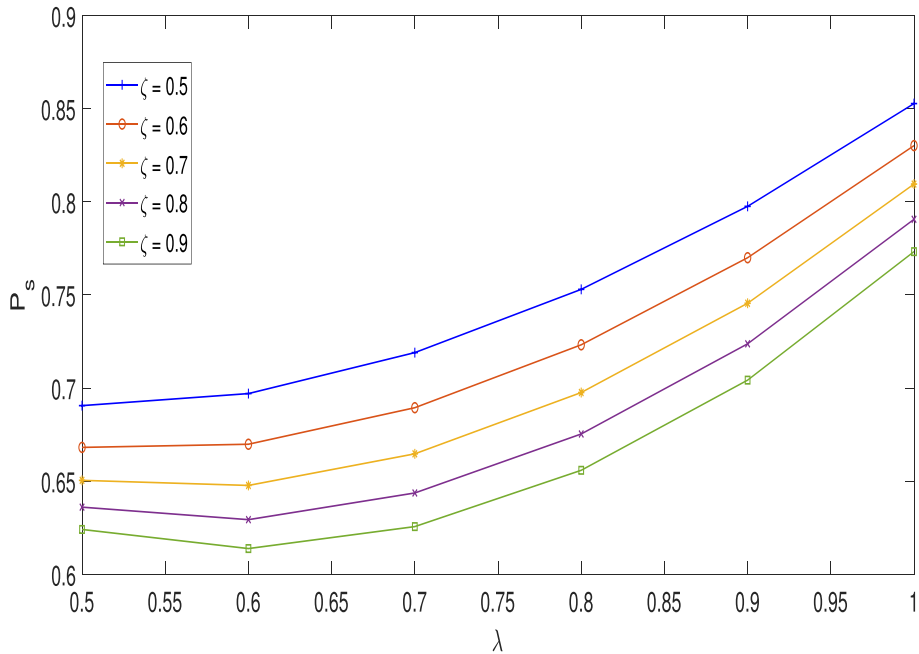


Fig. 16 Proportion of data packets served P_S versus λ

$C_3 = 300$, $C_4 = 50$, $C_5 = 40$, $C_h = 100$, $C_L = 120$, $R = 350$, number of particles = 30, number of iterations = 100, inertia weight = 0.7, cognitive constant = 1.5 and social constant = 1.5. Using PSO, the optimal values for the service rates μ_b and μ_{v_2} , which lead to the minimization of TEC, are determined for various values of λ , considering the specified cost values

Case 1: $C_0 = 500$, $C_1 = 300$, $C_2 = 350$, $C_3 = 300$, $C_4 = 50$, $C_5 = 40$, $C_h = 100$, $C_L = 120$;

Case 2: $C_0 = 300$, $C_1 = 300$, $C_2 = 350$, $C_3 = 300$, $C_4 = 50$, $C_5 = 40$, $C_h = 100$, $C_L = 120$;

Case 3: $C_0 = 500$, $C_1 = 200$, $C_2 = 350$, $C_3 = 300$, $C_4 = 50$, $C_5 = 40$, $C_h = 100$, $C_L = 120$;

Case 4: $C_0 = 500$, $C_1 = 300$, $C_2 = 250$, $C_3 = 300$, $C_4 = 50$, $C_5 = 40$, $C_h = 100$, $C_L = 120$;

Case 5: $C_0 = 500$, $C_1 = 300$, $C_2 = 350$, $C_3 = 200$, $C_4 = 50$, $C_5 = 40$, $C_h = 100$, $C_L = 120$;

Case 6: $C_0 = 500$, $C_1 = 300$, $C_2 = 350$, $C_3 = 300$, $C_4 = 40$, $C_5 = 40$, $C_h = 100$, $C_L = 120$;

Case 7: $C_0 = 500$, $C_1 = 300$, $C_2 = 350$, $C_3 = 300$, $C_4 = 50$, $C_5 = 30$, $C_h = 100$, $C_L = 120$;

Case 8: $C_0 = 500$, $C_1 = 300$, $C_2 = 350$, $C_3 = 300$, $C_4 = 50$, $C_5 = 40$, $C_h = 80$, $C_L = 120$;

Case 9: $C_0 = 500$, $C_1 = 300$, $C_2 = 350$, $C_3 = 300$, $C_4 = 50$, $C_5 = 40$, $C_h = 100$, $C_L = 100$;

Table 1 presents the optimal values of μ_b and μ_{v_2} , along with their corresponding TEC. From the table, it is observed that μ_b and TEC exhibit an increase with the rising values of λ , irrespective of the specific set of cost values. Further, it is observed from Table 1 that the optimal service rates and the resulting TEC are influenced by the arrival rate, indicating a direct relationship between the arrival rate λ and the system's performance metrics.

Table 2 shows the impact of μ_b , μ_{v_2} , and λ on TEC and TER. It is evident that as the service rates increase, TEC and TER decrease. This happens because higher service rates allow more data packets to be serviced and leave the system, which reduces $E(N_s)$. Consequently, both

Table 1 Values of μ_b^* , $\mu_{v_2}^*$ and TEC* for different cases and λ values

Case	λ	0.6	0.8	1	1.2
1	$\mu_b^*, \mu_{v_2}^*$	1.8802, 0.1000	2.1807, 0.1000	2.4482, 0.1000	2.6976, 0.1000
	TEC*	609.6321	664.436	714.3803	760.6595
2	$\mu_b^*, \mu_{v_2}^*$	1.4638, 0.1000	1.7548, 0.1000	2.0290, 0.1000	2.2914, 0.1000
	TEC*	546.4857	591.9151	633.9575	673.4539
3	$\mu_b^*, \mu_{v_2}^*$	2.0030, 0.1000	2.3242, 0.1000	2.6070, 0.1000	2.8648, 0.1000
	TEC*	567.6516	621.5998	671.3998	717.9981
4	$\mu_b^*, \mu_{v_2}^*$	1.9548, 0.1000	2.2531, 0.1000	2.5182, 0.1000	2.7614, 0.1000
	TEC*	584.6528	642.966	695.8114	744.483
5	$\mu_b^*, \mu_{v_2}^*$	1.8990, 0.1000	2.1957, 0.1000	2.4602, 0.1000	2.7069, 0.1000
	TEC*	603.5869	660.1271	711.2019	758.2475
6	$\mu_b^*, \mu_{v_2}^*$	2.0447, 0.1000	2.3520, 0.1000	2.6232, 0.1000	2.8690, 0.1000
	TEC*	590.0512	641.8198	689.0744	732.8693
7	$\mu_b^*, \mu_{v_2}^*$	1.8805, 0.1000	2.1804, 0.1000	2.4483, 0.1000	2.6980, 0.1000
	TEC*	608.6321	663.436	713.3803	759.6595
8	$\mu_b^*, \mu_{v_2}^*$	1.8090, 0.1000	2.1049, 0.1000	2.3722, 0.1000	2.6201, 0.1000
	TEC*	583.589	632.0508	676.1882	717.0926
9	$\mu_b^*, \mu_{v_2}^*$	1.8852, 0.1000	2.1803, 0.1000	2.4566, 0.1000	2.7082, 0.1000
	TEC*	608.1314	664.436	712.1636	758.1483

Table 2 TEC and TER versus μ_b , μ_{v_2} , λ

μ_b	TEC	TER	μ_{v_2}	TEC	TER	λ	TEC	TER
1.6	768.4405	884.1158	0.2	746.0309	895.3804	0.6	585.7867	454.3625
1.8	730.6434	774.7614	0.4	752.4631	892.2193	0.8	668.5192	641.6457
2.0	710.8856	715.5731	0.6	758.8691	889.0082	1.0	768.4405	884.1158
2.2	700.4191	679.8509	0.8	765.2551	885.7568	1.2	925.5754	1321.90
2.4	695.4096	656.6590	1.0	771.6208	882.4644	1.4	1339.50	2653.80

TEC and TER decrease with an increase in the service rate. On the other hand, TEC and TER increase as the arrival rate λ rises. This is because a higher arrival rate leads to an increase in $E(N_s)$, which in turn causes TEC and TER to increase.

9 Conclusion and Future Work

The PCS-II of IEEE 802.16e is analysed using a modified differentiated vacation queueing system with the close-down period, working vacation and impatient customers. The time-dependent probabilities are obtained explicitly by applying a confluent hypergeometric function for this system. The steady-state solution is computed through the generating function technique, incomplete Gamma function and Kummer's function. Performance metrics of the system are computed. Finally, a cost model is developed to determine the optimal

service rate. The performance indices discussed in this paper have potential applications in algorithm improvement and determining the best parameters for power-saving mechanisms. This work may be extended to multiple server queueing systems with finite buffers.

Author Contributions A. Mohammed shapique: Identified the problem, designed the model and analysed the data. R. Sudhesh: Interpretation of the data and Critical revision of the article. S. Dharmaraja: Aided in interpreting the results.

Funding No funding was obtained for this study.

Availability of Data and Materials The data and materials generated during the current study are available from the corresponding author on reasonable request.

Declarations

Competing Interests The authors declare no competing interests.

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