



Infinite-Server Systems Driven by Finite-Dimensional Hawkes and Dynamic Contagion Arrival Processes

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Abstract

This paper investigates the infinite-server systems that use finite-dimensional Hawkes and dynamic contagion processes as arrival processes. Many real-world stochastic systems demonstrate arrival patterns that exhibit clustering behavior. In these systems, the arrival of one entity can lead to an increase in the arrival of others, which can occur through self-excitation, multi-excitation, or external excitation. This interdependent behavior significantly influences the dynamics of the system. Consequently, under suitable conditions, we derive the Markov property of these systems. Additionally, we determine the joint time-dependent probability generating functions-Laplace transform for both the system size and the arrival intensities. We also present a recursive method to identify and derive both transient and stationary moments. Furthermore, we discuss and compare several cases of these arrival models to provide deeper insights into its structure and behavior.

Keywords Infinite-server system · Two-dimensional Hawkes process · Dynamic contagion process · Markov property · Infinitesimal generator · Probability generating function · Laplace transform · Moments

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1 Introduction

The majority of classical models in queueing systems rely on the assumption that arrivals and service times are independent random variables (see Brémaud (1981)). Although this assumption simplifies the mathematical modeling and analysis, it often fails to accurately

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represent real-world systems, where arrivals tend to exhibit inter-dependencies. In many practical situations, the arrival of one entity can increase the likelihood of additional arrivals, resulting in correlated or clustered arrival patterns. This independence assumption can result in inaccurate predictions of system performance, especially in environments, where interactions influence arrival behavior. For example, in seismology, an earthquake can trigger a series of aftershocks, see Ogata (1988). In criminology, Mohler et al. (2011) observed that the occurrence of one crime may increase the likelihood of follow-up incidents in nearby areas. Roy et al. (2026) studied the evolution of populations and observed how births and deaths can trigger subsequent births and deaths. Therefore, there is a growing need for models that account for self-exciting, or contagion effects, to better reflect these dependencies.

One such model that can effectively capture this phenomenon is the Hawkes process, pioneered by Hawkes (1971a, b) and Hawkes and Oakes (1974). This model is a type of self-exciting point process in which each event increases the likelihood of future events. Unlike Poisson processes, which assume that arrivals occur independently, the Hawkes processes capture the temporal clustering, which is often observed in many complex systems. This leads to various applications across different fields, including finance, DNA modeling, and neuroscience, for instance see Lee (2025); Pasricha et al. (2022); Errais et al. (2010). However, our focus here is on examining the Hawkes processes and their generalizations within queueing systems (see Legros (2021); Koops et al. (2017, 2018); Selvamuthu and Tardelli (2022), and others). By incorporating the Hawkes process as the arrival process, queueing models can more accurately represent and predict system behavior since the arrival intensity changes based on past events.

In this regard, Daw and Pender (2018) studied an infinite-server system where an unmarked Markov Hawkes process influences the arrival process. This refers to a situation where the jump sizes in the arrival intensity are deterministic. In contrast, Koops et al. (2018) examined a more general version of the problem, where the arrival process is characterized by a marked Hawkes process, which means that the jump sizes are stochastic. Additionally, Selvamuthu and Tardelli (2022) considered the arrival process to be a marked Hawkes process, while the service process to be the state-dependent marked Hawkes process. They investigated the time-dependent behavior of the infinite-server system. For more on the application of Hawkes processes in infinite-server system, refer to Daw and Pender (2018), Rizoïu et al. (2017), Gao and Zhu (2018), and references therein.

The present work is motivated by Koops et al. (2018) and Selvamuthu and Tardelli (2022). However, the key difference between our model and that described in their work lies in the type of arrival process used. They have used one-dimensional Hawkes process to model the arrival rate in an infinite-server system. In contrast, our model employ a finite-dimensional (d -dimensional) Hawkes process, specifically a two-dimensional to model the arrivals. The one-dimensional Hawkes process effectively captures self-excitation in a single event stream. However, many real-world systems consist of multiple interacting sources of arrivals. In such cases, the finite-dimensional Hawkes process becomes crucial, as it enables the arrival intensity of one process to be influenced not only by its own past events but also by the histories of other processes.

Additionally, we have also considered the arrival process which is both self-exciting and externally-exciting (referred as dynamic contagion). By incorporating an externally-exciting component, it takes into account external factors that affect the arrival process. For instance, marketing campaigns can trigger more arrivals in a newly launched showrooms, or environmental conditions that can influence the arrivals. Dynamic contagion processes were introduced by Dassios and Zhao (2011) which combines the characteristics of both self-excited and externally-excited jumps. The dynamic contagion process is the generalization

of the Hawkes process and the Cox process, which incorporate a Poisson shot-noise intensity. Moreover, for the study of infinite server system with shot-noise driven arrival intensity, see Koops et al. (2017). For further extensions of dynamic contagion models, see Selvamuthu et al. (2023) and Pandey et al. (2025). To analyze such generalized arrival structures in a tractable manner, we derive the joint time-dependent probability generating function (PGF)–Laplace transform of the underlying system.

The remaining structure of the paper are as follows: In Section 2, we explore the infinite-server systems with dynamic contagion arrival process. Specifically, in Subsection 2.1, we provide the description of the model which include dynamic contagion process and its intensity. In Subsection 2.2, we derive the joint PGF-Laplace transform of the system size and the intensity of the dynamic contagion process in infinite-server systems. In Subsection 2.3, we examine the transient moments and simplify the first order transient and stationary moments. In Section 3, we explore the infinite-server systems with two-dimensional Hawkes arrival process and derive results analogous to those previously established for the dynamic contagion arrival process. In particular, in Subsection 3.2, we discuss the Markov property and in Subsection 3.4, we discuss the transient moments and stationary moments explicitly as well as graphically. Furthermore, in Section 4, we analyze some of the general and special cases of the discussed model. Additionally, we conclude the section with a simulated comparison graph for the number of customers in the systems and a comparison graph for their expected values, evaluated under various models for arrival processes. In Section 5, we conclude the paper with some remarks and open problems.

2 Dynamic Contagion Arrival Processes

Assume that all the processes are defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$, a filtered probability space, where $\mathcal{F}_t$ is a given filtration, which satisfies the usual conditions.

In this section, we discuss the infinite-server system where the arrival process is modeled by a dynamic contagion process with independent and identically distributed (i.i.d.) sequence of marks, which represent the jump size.

2.1 Model Description

Let $A = \{A_t, t \geq 0\}$ be the dynamic contagion process (see Dassios and Zhao (2011)) which governed the arrivals, with intensity $\lambda = \{\lambda_t, t \geq 0\}$ given by

$$\lambda_t = \lambda^* + e^{-rt} (\lambda_0 - \lambda^*) + \int_0^t X_s e^{-r(t-s)} dQ_s + \int_0^t Y_s e^{-r(t-s)} dA_s, \quad (2.1)$$

where

- λ_0 is the initial intensity and λ^* represents the baseline intensity.
- $Q = \{Q_t, t \geq 0\}$ is a Poisson process with constant intensity $\rho > 0$, $\{t_i\}_{i \geq 0}$ is the sequence of its jump times, and represents the influence of external/exogenous factors on the arrival intensity.
- $\{\tau_j\}_{j \geq 0}$ is the sequence of arrival times of the dynamic contagion process A .
- $X = \{X_{t_i}\}_{i \geq 0}$ and $Y = \{Y_{\tau_j}\}_{j \geq 0}$ are two independent i.i.d. sequences of positive random variables representing the size of the jumps.
- Assume that $\mathbb{E}[X] < \infty$, and $\mathbb{E}[Y] < \infty$.

Remark 2.1 Equivalently, the intensity process λ defined in Eq. 2.1 admits the following summation representation in terms of the jump times $\{t_i\}_{i>0}$ and $\{\tau_j\}_{j>0}$.

$$\lambda_t = \lambda^* + e^{-rt} (\lambda_0 - \lambda^*) + \sum_{0 \leq t_i < t} X_{t_i} e^{-r(t-t_i)} + \sum_{0 \leq \tau_j < t} Y_{\tau_j} e^{-r(t-\tau_j)}.$$

Remark 2.2 Note that if in Eq. 2.1 $X_i = 0$, for all i , A becomes a Hawkes process with exponential exciting function, and if $Y_i = 0$, for all i , then A becomes a Cox process with Poisson shot-noise intensity.

2.2 Joint PGF-Laplace Transform

Throughout the rest of the paper, we assume that the service times follow an exponential distribution with parameter $\mu > 0$. Let $N = \{N_t, t \geq 0\}$ denote the system size or number of customers in the system. Let the σ -algebras generated by Q and A be denoted by

$$\mathcal{F}_t^Q = \sigma\{Q_s : s \leq t\}, \quad \mathcal{F}_t^A = \sigma\{A_s : s \leq t\}, \quad \text{and} \quad \mathcal{F}_t^{Q,A} = \mathcal{F}_t^Q \vee \mathcal{F}_t^A,$$

which are assumed to be sub σ -algebras of $\mathcal{F}_t$.

Note that, Eq. 2.1, the excitation function of the dynamic contagion process is exponential, consequently, the couple given by the process and its intensity satisfies the Markov property. Therefore, in this section, we explore the main results of the paper, which include the PGF-Laplace transform of the joint process (N, λ) . To establish this result, we first examine the joint distribution of the process in terms of partial differential equation (PDE) and then apply several transformations to simplify the derivation.

The joint PGF-Laplace transform (see Dassios and Zhao (2011)) of (N, λ) is uniquely defined by the transformation

$$\zeta(t, \theta, u) = \mathbb{E} \left[\theta^{N_t} e^{-u\lambda_t} \right],$$

for $t \geq 0$, $\theta \in [0, 1]$ and $u \geq 0$.

Theorem 2.1 *Let the arrival process be a dynamic contagion process and let the service times be exponentially distributed with parameter μ . Given X and Y as in Eq. 2.1, let us define $\alpha(u) = \mathbb{E}[e^{-uX}]$ and $\beta(u) = \mathbb{E}[e^{-uY}]$. Then, for $(N_t, \lambda_t, \mu_t) |_{t=0} = (0, \lambda_0, 0)$, we have*

$$\zeta(t, \theta, u) = \mathbb{E} \left[\theta^{N_t} e^{-u\lambda_t} \right] = e^{-u(t)\lambda_0} \exp \left\{ -r\lambda^* \int_0^t u(x) dx + \rho \int_0^t (\alpha(u(x)) - 1) dx \right\},$$

where $u(\cdot)$ satisfies the ordinary differential equation (ODE) given by

$$u'(t) + ru(t) + (1 + (\theta - 1)e^{-\mu t}) \beta(u(t)) - 1 = 0, \quad (2.2)$$

with boundary condition $u(0) = u$.

Remark 2.3 (Consistency with Theorem 3.1 of Dassios and Zhao (2011)). Theorem 3.1 of Dassios and Zhao (2011) establishes that, for the joint process (N, λ) with intensity λ_t as defined in Eq. 2.1, the conditional joint PGF-Laplace transform is given by

$$\mathbb{E} \left[\theta^{(N_T - N_t)} e^{-u\lambda_T} \middle| \mathcal{F}_t \right] = e^{-(c(T) - c(t))} e^{-B(t)\lambda_t}, \quad (2.3)$$

where $0 \leq \theta \leq 1$, $u \geq 0$, $\mathcal{F}_t = \sigma\{N_t, \lambda_t\}$, and $B(t)$ satisfies

$$-B'(t) + rB(t) + \theta\beta(B(t)) - 1 = 0, \quad B(T) = u, \quad (2.4)$$

while

$$c(t) = r\lambda^* \int_0^t B(s) ds + \rho \int_0^t (1 - \alpha(B(s))) ds, \quad (2.5)$$

where $\alpha(\cdot)$ and $\beta(\cdot)$ are defined in Theorem 2.1 above. Now setting the initial time equal to 0 and replacing T by t in Eq. 2.3, we obtain

$$\mathbb{E}\left[\theta^{N_t} e^{-u\lambda_t}\right] = e^{-c(t)} e^{-B(0)\lambda_0},$$

as $\mathcal{F}_0 = \sigma\{\lambda_0, N_0\}$ is a trivial sigma-field.

Now define

$$u(t) = B(T - t), \quad 0 \leq t \leq T.$$

Then Eq. 2.4 becomes

$$u'(s) + ru(s) + \theta\beta(u(s)) - 1 = 0, \quad u(0) = u. \quad (2.6)$$

Furthermore,

$$\begin{aligned} c(t) &= r\lambda^* \int_{T-t}^T u(T-s) ds + \rho \int_{T-t}^T (1 - \alpha(u(T-s))) ds \\ &= r\lambda^* \int_0^t u(x) dx + \rho \int_0^t (1 - \alpha(u(x))) dx. \end{aligned}$$

Consequently,

$$\mathbb{E}\left[\theta^{N_t} e^{-u\lambda_t}\right] = e^{-u(t)\lambda_0} \exp\left\{-r\lambda^* \int_0^t u(s) ds + \rho \int_0^t (\alpha(u(s)) - 1) ds\right\}, \quad (2.7)$$

where $u(\cdot)$ is the solution of Eq. 2.6.

It can be observe that Eq. 2.7 coincides exactly with the expression obtained in Theorem 2.1 when the parameter $\mu = 0$. Hence, our result reduces to Theorem 3.1 of Dassios and Zhao (2011) as a special case, which implies the result is consistent. We now proceed with a proof of Theorem 2.1, based on a queueing system framework.

In order to achieve the proof of Theorem 2.1, we need some preliminaries results given by Proposition 2.1, Proposition 2.2 and Proposition 2.3 below.

In Proposition 2.1, we study the joint dynamics of the queue length and the stochastic arrival intensity. To analyze the system, we consider their joint distribution. By combining the infinitesimal evolution of the infinite-server queue with the dynamics of the underlying intensity process, we derive an integro-differential equation that describes that how this joint distribution evolves over time.

Proposition 2.1 For $t > 0$, the joint distribution of the process (N, λ) is given by the function $F(t, k, \lambda) = P(N_t = k, \lambda_t \leq \lambda)$. Assume that $F(t, k, \lambda)$ is continuously differentiable in t and absolutely continuous in λ . Then $F(t, k, \lambda)$ satisfies

$$\begin{aligned} & \frac{\partial^2 F(t, k, \lambda)}{\partial \lambda \partial t} + \frac{\partial}{\partial \lambda} \left(r(\lambda^* - \lambda) \frac{\partial F(t, k, \lambda)}{\partial \lambda} \right) \\ &= \rho \int_0^\lambda \frac{\partial F(t, k, x)}{\partial x} g_X(\lambda - x) dx + \int_0^\lambda y \frac{\partial F(t, k-1, y)}{\partial y} g_Y(\lambda - y) dy \\ & \quad - \lambda \frac{\partial F(t, k, \lambda)}{\partial \lambda} - \rho \frac{\partial F(t, k, \lambda)}{\partial \lambda} + (k+1)\mu \frac{\partial F(t, k+1, \lambda)}{\partial \lambda} - k\mu \frac{\partial F(t, k, \lambda)}{\partial \lambda} \end{aligned} \quad (2.8)$$

where g_X and g_Y are the probability density function of X and Y , respectively.

Proof Considering the infinitesimal evolution of the Markov process (N_t, λ_t) over $(t, t + \Delta t]$, we obtain

$$\begin{aligned} F(t + \Delta t, k, \lambda - r\Delta t(\lambda - \lambda^*)) &= \rho \Delta t \int_0^\lambda P(X \leq \lambda - x) dF(t, k, x) - (\rho \Delta t) F(t, k, \lambda) \\ & \quad + \Delta t \int_0^\lambda y P(Y \leq \lambda - y) dF(t, k-1, y) - \Delta t \int_0^\lambda y dF(t, k, y) \\ & \quad + (k+1)\mu \Delta t F(t, k+1, \lambda) + F(t, k, \lambda) (1 - k\mu \Delta t). \end{aligned}$$

After some simplification, we get

$$\begin{aligned} \frac{1}{\Delta t} [F(t + \Delta t, k, \lambda - r\Delta t(\lambda - \lambda^*)) - F(t, k, \lambda)] &= \rho \int_0^\lambda P(X \leq \lambda - x) dF(t, k, x) - \rho F(t, k, \lambda) \\ & \quad + \int_0^\lambda y P(Y \leq \lambda - y) dF(t, k-1, y) - \int_0^\lambda y dF(t, k, y) \\ & \quad + (k+1)\mu F(t, k+1, \lambda) - k\mu F(t, k, \lambda). \end{aligned}$$

Taking $\Delta t \downarrow 0$, we obtain

$$\begin{aligned} \frac{\partial F(t, k, \lambda)}{\partial t} - r(\lambda - \lambda^*) \frac{\partial F(t, k, \lambda)}{\partial \lambda} &= \rho \int_0^\lambda P(X \leq \lambda - x) dF(t, k, x) - \rho F(t, k, \lambda) \\ & \quad + \int_0^\lambda y P(Y \leq \lambda - y) dF(t, k-1, y) - \int_0^\lambda y dF(t, k, y) + (k+1)\mu F(t, k+1, \lambda) - k\mu F(t, k, \lambda). \end{aligned} \quad (2.9)$$

Since $F(t, k, \cdot)$ is absolutely continuous, we may write

$$dF(t, k, x) = \frac{\partial F(t, k, x)}{\partial x} dx.$$

Now assuming $P(X \leq 0) = 0$ and $P(Y \leq 0) = 0$, and differentiating Eq. 2.9 successively both sides with respect to λ , and using Leibniz integral rule, we obtain Eq. 2.8. $\square$

To further analyze the joint distribution of the queue length and the intensity process, we introduce a Laplace transform with respect to the intensity variable. This transformation converts the integro-differential equation derived in Proposition 2.1 into a more tractable form, which is indexed only by the queue length.

Proposition 2.2 *Under the transformation*

$$\xi(t, k, u) = \int_0^\infty e^{-u\lambda} \frac{\partial F(t, k, \lambda)}{\partial \lambda} d\lambda, \quad (2.10)$$

Eq. 2.8 becomes

$$\begin{aligned} \frac{\partial \xi(t, k, u)}{\partial t} + (ur - 1) \frac{\partial \xi(t, k, u)}{\partial u} + \beta(u) \frac{\partial \xi(t, k - 1, u)}{\partial u} \\ = [-ur\lambda^* + \rho(\alpha(u) - 1) - k\mu] \xi(t, k, u) + (k + 1)\mu \xi(t, k + 1, u), \end{aligned} \quad (2.11)$$

with boundary conditions $\xi(0, 0, u(0)) = e^{-u(0)\lambda_0}$.

Proof In order to transform Eq. 2.8 in terms of $\xi(t, k, u)$, given in Eq. 2.10, note that

$$\int_0^\infty e^{-u\lambda} \frac{\partial^2 F(t, k, u)}{\partial \lambda \partial t} d\lambda = \frac{\partial \xi(t, k, u)}{\partial t}. \quad (2.12)$$

By integration by parts with respect to λ , we get

$$\begin{aligned} \int_0^\infty e^{-u\lambda} \frac{\partial}{\partial \lambda} \left(r(\lambda^* - \lambda) \frac{\partial F(t, k, \lambda)}{\partial \lambda} \right) d\lambda \\ = ur \int_0^\infty e^{-u\lambda} (\lambda^* - \lambda) \frac{\partial F(t, k, \lambda)}{\partial \lambda} d\lambda = ur\lambda^* \xi(t, k, u) + ur \frac{\partial \xi(t, k, u)}{\partial u}. \end{aligned} \quad (2.13)$$

Recalling that $\alpha(u) = \mathbb{E}[e^{-uX}]$ and by a change of variables between x and λ , we get

$$\begin{aligned} \rho \int_0^\infty e^{-u\lambda} \int_0^\lambda \frac{\partial F(t, k, x)}{\partial x} g_X(\lambda - x) dx d\lambda \\ = \rho \int_0^\infty \int_x^\infty e^{-u(\lambda-x)} e^{-ux} \frac{\partial F(t, k, x)}{\partial x} g_X(\lambda - x) d\lambda dx \\ = \rho \int_0^\infty \int_0^\infty e^{-uz} e^{-ux} \frac{\partial F(t, k, x)}{\partial x} g_X(z) dz dx = \rho \int_0^\infty e^{-uz} \xi(t, k, u) g_X(z) dz = \rho \alpha(u) \xi(t, k, u). \end{aligned} \quad (2.14)$$

Furthermore, by considering $\beta(u) = \mathbb{E}[e^{-uY}]$ and, again, by a change of variables between y and λ , we get

$$\begin{aligned} \int_0^\infty e^{-u\lambda} \int_0^\lambda y \frac{\partial F(t, k - 1, y)}{\partial y} g_Y(\lambda - y) dy d\lambda \\ = \int_0^\infty \int_y^\infty e^{-u(\lambda-y)} e^{-uy} y \frac{\partial F(t, k - 1, y)}{\partial y} g_Y(\lambda - y) d\lambda dy \\ = \int_0^\infty \int_0^\infty e^{-uz} e^{-uy} y \frac{\partial F(t, k - 1, y)}{\partial y} g_Y(z) dz dy \\ = - \int_0^\infty e^{-uz} \frac{\partial}{\partial u} \xi(t, k - 1, u) g_Y(z) dz = -\beta(u) \frac{\partial \xi(t, k - 1, u)}{\partial u}. \end{aligned} \quad (2.15)$$

Moreover,

$$- \rho \int_0^\infty e^{-u\lambda} \frac{\partial F(t, k, \lambda)}{\partial \lambda} d\lambda = -\rho \xi(t, k, u), \quad (2.16)$$

$$- \int_0^\infty e^{-u\lambda} \lambda \frac{\partial F(t, k, \lambda)}{\partial \lambda} d\lambda = \frac{\partial \xi(t, k, u)}{\partial u}, \quad (2.17)$$

$$(k+1)\mu \int_0^\infty e^{-u\lambda} \frac{\partial F(t, k+1, \lambda)}{\partial \lambda} d\lambda = (k+1)\mu \xi(t, k+1, u), \quad (2.18)$$

and

$$-k\mu \int_0^\infty e^{-u\lambda} \frac{\partial F(t, k, \lambda)}{\partial \lambda} d\lambda = -k\mu \xi(t, k, u). \quad (2.19)$$

Combining all the transformations made in Eqs. 2.12-2.19, we obtain Eq. 2.11. $\square$

We now introduce a PGF with respect to the queue length. This transformation removes the discrete index and converts the system as a PDE in continuous variables. The resulting PDE is useful for deriving the explicit form of the joint PGF–Laplace transform.

Proposition 2.3 Again using the transformation $\zeta(t, \theta, u) = \sum_{k=0}^\infty \theta^k \xi(t, k, u)$ in Eq. 2.11, we obtain the PDE

$$\frac{\partial \zeta(t, \theta, u)}{\partial t} + (ur + \theta\beta(u) - 1) \frac{\partial \zeta(t, \theta, u)}{\partial u} + \mu(\theta - 1) \frac{\partial \zeta(t, \theta, u)}{\partial \theta} = (-ur\lambda^* + \rho(\alpha(u) - 1))\zeta(t, \theta, u) \quad (2.20)$$

with boundary condition $\zeta(0, \theta, u(0)) = e^{-u(0)\lambda_0}$.

Proof Here the transformation $\zeta(t, \theta, u) = \sum_{k=0}^\infty \theta^k \xi(t, k, u)$ is introduced to eliminate the discrete state variable k from Eq. 2.11. In this regards, if we multiplying both sides of Eq. 2.11 by θ^k and summing over $k \geq 0$, we express each term in the form of $\zeta(t, \theta, u)$ and its partial derivatives.

Thus, by taking into account that $F(t, -1, \lambda) = 0$, we have that

$$\begin{aligned} \beta(u) \sum_{k=0}^\infty \theta^k \frac{\partial \xi(t, k-1, u)}{\partial u} &= \beta(u) \left(\frac{\partial \xi(t, -1, u)}{\partial u} + \theta \sum_{k=1}^\infty \theta^{k-1} \frac{\partial \xi(t, k-1, u)}{\partial u} \right) \\ &= \theta \beta(u) \frac{\partial}{\partial u} \sum_{k=0}^\infty \theta^k \xi(t, k, u) = \theta \beta(u) \frac{\partial \zeta(t, \theta, u)}{\partial u}. \end{aligned}$$

With a similar procedure, we obtain that

$$\begin{aligned} \mu \sum_{k=0}^\infty (k+1) \theta^k \xi(t, k+1, u) &= \mu \frac{\partial}{\partial \theta} \left(\sum_{k=0}^\infty \theta^{k+1} \xi(t, k+1, u) \right) = \mu \frac{\partial}{\partial \theta} \sum_{k=1}^\infty \theta^k \xi(t, k, u) \\ &= \mu \frac{\partial}{\partial \theta} \left(\int \left[\sum_{k=0}^\infty k \theta^{k-1} \xi(t, k, u) \right] d\theta \right) = \mu \frac{\partial \zeta(t, \theta, u)}{\partial \theta}. \end{aligned}$$

Moreover,

$$\mu \sum_{k=0}^\infty k \theta^k \xi(t, k, u) = \mu \theta \frac{\partial \zeta(t, \theta, u)}{\partial \theta}.$$

Substituting all these in Eq. 2.11, we get the PDE given by Eq. 2.20. $\square$

We now present the proof of the Theorem 2.1 by using the PDE obtained in Proposition 2.3.

Proof of Theorem 2.1 To solve PDE given in Eq. 2.20, we apply the method of characteristics. Fix (t, θ, u) and consider the characteristic curve passing through this point. Since we want to determine the value of ζ at the point (t, θ, u) , then we construct the characteristic backward in time from (t, θ, u) to the initial condition $t = 0$.

Accordingly, we consider a parameter $\tau \in [0, t]$ and define the characteristic curves as

$$\tau \mapsto (\tau, \theta(\tau), u(\tau))$$

with boundary conditions $\theta(t) = \theta$, and $u(t) = u$. The above boundary conditions ensure that

$$\zeta(t, \theta, u) = \zeta(t, \theta(t), u(t)).$$

Along a characteristic curve, the chain rule yields that

$$\frac{d\zeta(\tau, \theta(\tau), u(\tau))}{d\tau} = \frac{\partial \zeta}{\partial \tau} + \frac{d\theta(\tau)}{d\tau} \frac{\partial \zeta}{\partial \theta} + \frac{du(\tau)}{d\tau} \frac{\partial \zeta}{\partial u}. \quad (2.21)$$

Now comparing Eqs. 2.20 and 2.21, we get

$$\frac{d\theta(\tau)}{d\tau} = \mu(\theta(\tau) - 1), \quad \theta(t) = \theta, \quad (2.22)$$

and

$$\frac{du(\tau)}{d\tau} = ru(\tau) + \theta(\tau)\beta(u(\tau)) - 1, \quad u(t) = u, \quad (2.23)$$

and substituting Eqs. 2.22 and 2.23 into 2.21, Eq. 2.20 reduces to

$$\frac{d\zeta(\tau, \theta(\tau), u(\tau))}{d\tau} = \left(-r\lambda^*u(\tau) + \rho(\alpha(u(\tau)) - 1) \right) \zeta(\tau, \theta(\tau), u(\tau)).$$

which is a first-order linear ODE. Integrating from 0 to t we get

$$\zeta(t, \theta(t), u(t)) = \zeta(0, \theta(0), u(0)) \exp \left\{ -r\lambda^* \int_0^t u(s) ds + \rho \int_0^t (\alpha(u(s)) - 1) ds \right\},$$

where $\zeta(0, \theta(0), u(0)) = e^{-u(0)\lambda_0}$. Consequently,

$$\zeta(t, \theta, u) = e^{-u(0)\lambda_0} \exp \left\{ -r\lambda^* \int_0^t u(s) ds + \rho \int_0^t (\alpha(u(s)) - 1) ds \right\}, \quad (2.24)$$

where $u(\cdot)$ is the solution of the characteristic Eq. 2.23. Solving Eq. 2.22, we have that

$$\theta(\tau) = 1 + (\theta - 1)e^{-\mu(t-\tau)}. \quad (2.25)$$

Thus, substituting $\theta(\tau)$, as deduced in Eq. 2.25, into Eq. 2.23,

$$\begin{aligned} \frac{du(\tau)}{d\tau} &= ru(\tau) + \theta(\tau)\beta(u(\tau)) - 1 \\ &= ru(\tau) + \left(1 + (\theta - 1)e^{-\mu(t-\tau)} \right) \beta(u(\tau)) - 1. \end{aligned} \quad (2.26)$$

As in Remark 2.3, we define the time-reversed variable v as

$$v(\tau) = u(t - \tau), \quad 0 \leq \tau \leq t,$$

and, Eq. 2.26 becomes

$$\frac{dv(\tau)}{d\tau} = 1 - rv(\tau) - \left(1 + (\theta - 1)e^{-\mu\tau} \right) \beta(v(\tau)),$$

with initial condition $v(0) = u(t) = u$. By renaming v as u , we get the required result. $\square$

2.3 Transient Moments

Here we are going to discuss the joint transient moment of type $\mathbb{E}[(N_t)^{n_1}(\lambda_t)^{n_2}]$. From Eq. 2.20, we deduce that

$$\begin{aligned} \frac{d}{dt} \mathbb{E}[\theta^{N_t} e^{-u\lambda_t}] - (ur + \theta\beta(u) - 1) \mathbb{E}[\theta^{N_t} \lambda_t e^{-u\lambda_t}] + \mu(\theta - 1) \mathbb{E}[N_t \theta^{N_t-1} e^{-u\lambda_t}] \\ = [-ur\lambda^* + \rho(\alpha(u) - 1)] \mathbb{E}[\theta^{N_t} e^{-u\lambda_t}]. \end{aligned} \quad (2.27)$$

Differentiating both sides of Eq. 2.27, $(n_2 \in \mathbb{N} \cup \{0\})$ -times with respect to u and, setting $u = 0$, we get

$$\begin{aligned} \frac{d}{dt} \mathbb{E}[\theta^{N_t} (\lambda_t)^{n_2}] + n_2 r \mathbb{E}[\theta^{N_t} (\lambda_t)^{n_2}] + \mathbb{E}[\theta^{N_t} (\lambda_t)^{n_2+1}] + \rho \mathbb{E}[\theta^{N_t} (\lambda_t)^{n_2}] + \mu(\theta - 1) \mathbb{E}[N_t \theta^{N_t-1} (\lambda_t)^{n_2}] \\ = n_2 r \lambda^* \mathbb{E}[\theta^{N_t} (\lambda_t)^{n_2-1}] + \mathbb{E}[\theta^{N_t+1} \lambda_t (\lambda_t + Y)^{n_2}] + \rho \mathbb{E}[\theta^{N_t} (\lambda_t + X)^{n_2}]. \end{aligned} \quad (2.28)$$

Differentiate Eq. 2.28 $(n_1 + 1)$ -times with respect to θ and set $\theta = 1$, then

$$\begin{aligned} \frac{d}{dt} \mathbb{E}[(\overline{N}_t)^{n_1} (\lambda_t)^{n_2}] + n_2 r \mathbb{E}[(\overline{N}_t)^{n_1} (\lambda_t)^{n_2}] + \mathbb{E}[(\overline{N}_t)^{n_1} (\lambda_t)^{n_2+1}] \\ + \mu \mathbb{E}[N_t (\overline{N}_t)^{n_1} (\lambda_t)^{n_2}] + \rho \mathbb{E}[(\overline{N}_t)^{n_1} (\lambda_t)^{n_2}] \\ = n_2 r \lambda^* \mathbb{E}[(\overline{N}_t)^{n_1} (\lambda_t)^{n_2-1}] + \mu \mathbb{E}[(\overline{N}_t)^{n_1+1} (\lambda_t)^{n_2}] + \rho \mathbb{E}[(\overline{N}_t)^{n_1} (\lambda_t + X)^{n_2}] \\ + (n_1 + 1) \mathbb{E}[(\overline{N}_t)^{n_1-1} \lambda_t (\lambda_t + Y)^{n_2}] + \mathbb{E}[(\overline{N}_t)^{n_1} \lambda_t (\lambda_t + Y)^{n_2}], \end{aligned} \quad (2.29)$$

where $(\overline{N}_t)^{n_1} = N_t(N_t - 1) \cdots (N_t - n_1)$ and $(\overline{N}_t)^{-1} = 1$. Hence, by Eq. 2.29, we are able to express the n -th order moments of N_t and λ_t in terms of a system of ODEs. Consequently, the first order moments is evaluated using the system of ODEs:

$$\begin{aligned} \frac{d}{dt} \mathbb{E}[N_t] &= -\mu \mathbb{E}[N_t] + \mathbb{E}[\lambda_t] \\ \frac{d}{dt} \mathbb{E}[\lambda_t] &= -(r - \mathbb{E}[Y]) \mathbb{E}[\lambda_t] + \rho \mathbb{E}[X] + r\lambda^*. \end{aligned} \quad (2.30)$$

Remark 2.4 Note that, for the system of ODEs given in Eq. 2.30, the solution exists and can be found. Here we omit finding the solutions explicitly, rather, we draw the graph of the solutions of each differential equations. We have chosen to use fixed values of the constants involved in the differential equations for the sake of graphical illustration. These constant values are based on the stability assumptions of the dynamic contagion process, i.e., $r > \mathbb{E}[Y]$. The constants values are as follows: $\rho = 0.6$, $r = 1$, $\mathbb{E}[X] = 0.6$, $\mathbb{E}[Y] = 0.5$, $\lambda^* = 0.8$, $\lambda_0 = 0.8$, $\mu = 0.7$. The corresponding results are shown in Fig. 1.

Remark 2.5 Figure 1 shows that under the condition $r > \mathbb{E}[Y]$, the expected number of customers in the system, $\mathbb{E}[N_t]$, as well as the expected arrival intensity $\mathbb{E}[\lambda_t]$, converge to steady-state values as time t become sufficiently large.

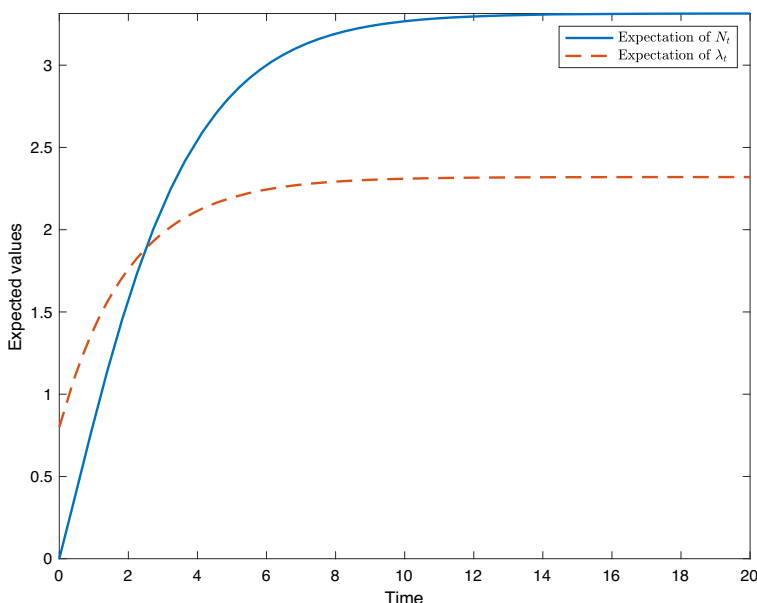


Fig. 1 Solutions to the system of ODEs with constant values given in Remark 2.4

In Proposition 2.4 below, we derive the explicit value of the stationary mean $\mathbb{E}[N_\infty]$. This result characterizes the long-run expected number of customers in the system under stability condition.

Proposition 2.4 *Let N_t denotes the number of customers in the system and let λ_t be the intensity of the dynamic contagion arrival process A . Assume that $r > \mathbb{E}[Y]$ thus, we have*

$$\mathbb{E}[\lambda_\infty] = \frac{\rho \mathbb{E}[X] + r\lambda^*}{r - \mathbb{E}[Y]} \quad \text{and} \quad \mathbb{E}[N_\infty] = \frac{\rho \mathbb{E}[X] + r\lambda^*}{\mu(r - \mathbb{E}[Y])}. \quad (2.31)$$

Proof of Theorem Under the condition, $r > \mathbb{E}[Y]$, the intensity process $\{\lambda_t, t \geq 0\}$ is stable and admits a stationary distribution with finite mean. In stationarity, the first moments of the number of customers in the system and intensity processes satisfy the balance equations

$$\begin{aligned} 0 &= -\mu \mathbb{E}[N_\infty] + \mathbb{E}[\lambda_\infty], \\ 0 &= -(r - \mathbb{E}[Y])\mathbb{E}[\lambda_\infty] + \rho \mathbb{E}[X] + r\lambda^*. \end{aligned}$$

After solving these system of equations, we obtain Eq. 2.31. $\square$

3 Two-Dimensional Hawkes Arrival Processes

This Section discusses infinite-server systems, with the arrivals driven by the two-dimensional Hawkes processes with i.i.d. sequence of marks, representing the jump sizes.

3.1 Model Description

Let $(A^{(1)}, A^{(2)}) = (A_t^{(1)}, A_t^{(2)})_{t \geq 0}$ be two-dimensional Hawkes processes that govern the arrivals, and their conditional intensity functions $(\lambda^{(1)}, \lambda^{(2)}) = (\lambda_t^{(1)}, \lambda_t^{(2)})_{t \geq 0}$, for $i = 1, 2$, which are given by

$$\lambda_t^{(i)} = \lambda^{*(i)} + e^{-r_i t} \left(\lambda_0^{(i)} - \lambda^{*(i)} \right) + \int_0^t X_s^{(i)} e^{-r_i(t-s)} dA_s^{(1)} + \int_0^t Y_s^{(i)} e^{-r_i(t-s)} dA_s^{(2)}, \quad (3.1)$$

where

- $\lambda_0^{(i)}$ is the initial intensity.
- $\lambda^{*(i)}$ represents the baseline intensity.
- r_i is the mean-reverting rate.
- $X^{(n)} = \{X_{t_i}^{(n)}\}_{i>0}$, and $Y^{(n)} = \{Y_{\tau_j}^{(n)}\}_{j>0}$, for $n = 1, 2$, are i.i.d. sequences of positive random variables, representing the size of the jumps of $A^{(1)}$ and $A^{(2)}$, respectively, and are assumed to be independent to each other. We assume that $\mathbb{E}[X^{(n)}] < \infty$, and $\mathbb{E}[Y^{(n)}] < \infty$ for $n = 1, 2$.
- $\{t_i\}_{i>0}$ and $\{\tau_j\}_{j>0}$ are the sequences of the jump times of $A^{(1)}$ and $A^{(2)}$, respectively.

Remark 3.1 Let assume that the service times follow an exponential distribution with parameter $\mu > 0$. Let N_t denote the number of customers in the system at time $t > 0$.

3.2 Markov Property

Under the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$, the σ -algebras generated by the processes $A^{(1)}$ and $A^{(2)}$ are denoted by

$$\mathcal{F}_t^{A^{(1)}} = \sigma\{A_s^{(1)} : s \leq t\}, \quad \mathcal{F}_t^{A^{(2)}} = \sigma\{A_s^{(2)} : s \leq t\} \quad \text{and} \quad \mathcal{F}_t^{A^{(1)}, A^{(2)}} = \mathcal{F}_t^{A^{(1)}} \vee \mathcal{F}_t^{A^{(2)}},$$

which are assumed to be sub σ -algebras of $\mathcal{F}_t$.

Note that, when the excitation function of a Hawkes process is exponential, then the process jointly with its intensity, exhibits the Markov property, see Roy et al. (2026). Specifically, this means that the future evolution of the system depends only on its present state, not on its past history. In the context of the two-dimensional Hawkes process, denoted as $(A^{(1)}, A^{(2)})$, with intensity $(\lambda^{(1)}, \lambda^{(2)})$, and where each component of the intensity, say $\lambda^{(1)}$ and $\lambda^{(2)}$, is governed by an exponential excitation function, the joint process, i.e., $(A^{(1)}, A^{(2)}, \lambda^{(1)}, \lambda^{(2)})$ satisfy the Markov property.

Now, we derive the Dynkin's formula as it is a direct consequence of the Markov property. Dynkin's formula provides a key relationship between the generator of the process and the expected future evolution of certain functions of the joint process, which play a crucial role in further analysis.

Proposition 3.1 Let the process $(A^{(1)}, \lambda^{(1)}, A^{(2)}, \lambda^{(2)})$ be defined on f , a function such that $f : \mathbb{N} \times \mathbb{R}_+ \times \mathbb{N} \times \mathbb{R}_+ \rightarrow \mathbb{R}$, with continuous partial derivatives with respect to $\lambda^{(1)}$ and $\lambda^{(2)}$. Setting

$$\begin{aligned} f_{1,2} &= f\left(A^{(1)}, \lambda^{(1)}, A^{(2)}, \lambda^{(2)}\right), & f_{1,2}(t) &= f\left(A_t^{(1)}, \lambda_t^{(1)}, A_t^{(2)}, \lambda_t^{(2)}\right), \\ \Delta_1 &= f\left(A^{(1)} + 1, \lambda^{(1)} + x_1, A^{(2)}, \lambda^{(2)} + x_2\right) - f\left(A^{(1)}, \lambda^{(1)}, A^{(2)}, \lambda^{(2)}\right), & \text{and} \\ \Delta_2 &= f\left(A^{(1)}, \lambda^{(1)} + y_1, A^{(2)} + 1, \lambda^{(2)} + y_2\right) - f\left(A^{(1)}, \lambda^{(1)}, A^{(2)}, \lambda^{(2)}\right), \end{aligned}$$

the infinitesimal generator $\mathcal{A}$ of $(A^{(1)}, \lambda^{(1)}, A^{(2)}, \lambda^{(2)})$ defined on f is given by

$$\begin{aligned} \mathcal{A}f_{1,2} &:= r_1\left(\lambda^{*(1)} - \lambda^{(1)}\right) \frac{\partial f_{1,2}}{\partial \lambda^{(1)}} + r_2\left(\lambda^{*(2)} - \lambda^{(2)}\right) \frac{\partial f_{1,2}}{\partial \lambda^{(2)}} \\ &\quad + \lambda^{(1)} \int_0^\infty \int_0^\infty \Delta_1 dP\left(X^{(1)} \leq x_1\right) dP\left(X^{(2)} \leq x_2\right) \\ &\quad + \lambda^{(2)} \int_0^\infty \int_0^\infty \Delta_2 dP\left(Y^{(1)} \leq y_1\right) dP\left(Y^{(2)} \leq y_2\right). \end{aligned}$$

If for any $0 \leq t \leq T$,

$$\mathbb{E}\left(\int_0^t \left| \mathcal{A}f_{1,2}(u) - r_1\left(\lambda^{*(1)} - \lambda_u^{(1)}\right) \frac{\partial f_{1,2}(u)}{\partial \lambda^{(1)}} - r_2\left(\lambda^{*(2)} - \lambda_u^{(2)}\right) \frac{\partial f_{1,2}(u)}{\partial \lambda^{(2)}} \right| du\right) < \infty,$$

then, for any $0 \leq t \leq T$, the following equation holds

$$\mathbb{E}\left(f_{1,2}(T) | \mathcal{F}_t^{A^{(1)}, A^{(2)}}\right) := f_{1,2}(t) + \mathbb{E}\left(\int_t^T \mathcal{A}f_{1,2}(u) du | \mathcal{F}_t^{A^{(1)}, A^{(2)}}\right). \quad (3.2)$$

Proof of Theorem For any $t > 0$,

$$\begin{aligned} f_{1,2}(t) - f_{1,2}(0) &:= \int_0^t r_1\left(\lambda^{*(1)} - \lambda_u^{(1)}\right) \frac{\partial f_{1,2}(u)}{\partial \lambda^{(1)}} du + \int_0^t r_2\left(\lambda^{*(2)} - \lambda_u^{(2)}\right) \frac{\partial f_{1,2}(u)}{\partial \lambda^{(2)}} du \\ &\quad + \sum_{0 < u \leq t} [f_{1,2}(u) - f_{1,2}(u-)], \end{aligned} \quad (3.3)$$

where the summation defined in Eq. 3.3 is such that

$$\begin{aligned} &\sum_{0 < u \leq t} [f_{1,2}(u) - f_{1,2}(u-)] \\ &:= \int_0^t \int_0^\infty \int_0^\infty \left[f\left(A_u^{(1)} + 1, \lambda_u^{(1)} + x_1, A_u^{(2)}, \lambda_u^{(2)} + x_2\right) - f_{1,2}(u) \right] dP(X^{(1)} \leq x_1) dP(X^{(2)} \leq x_2) dA_u^{(1)} \\ &\quad + \int_0^t \int_0^\infty \int_0^\infty \left[f\left(A_u^{(1)}, \lambda_u^{(1)} + y_1, A_u^{(2)} + 1, \lambda_u^{(2)} + y_2\right) - f_{1,2}(u) \right] dP(Y^{(1)} \leq y_1) dP(Y^{(2)} \leq y_2) dA_u^{(2)}. \end{aligned}$$

Note that, by the Doob's Decomposition, we have that

$$A_t^{(1)} = m_t^{(1)} + \int_0^t \lambda_u^{(1)} du \quad \text{and} \quad A_t^{(2)} = m_t^{(2)} + \int_0^t \lambda_u^{(2)} du$$

where $m_t^{(1)}$ and $m_t^{(2)}$ are $\mathcal{F}_t^{A^{(1)}, A^{(2)}}$ -martingale. Consequently, the integrability condition on the predictable integrand guarantees that

$$\begin{aligned} f_{1,2}(t) - f_{1,2}(0) - \int_0^t \mathcal{A}f_{1,2}(u) du \\ = \int_0^t \int_0^\infty \int_0^\infty \left[f\left(A_u^{(1)} + 1, \lambda_u^{(1)} + x_1, A_u^{(2)}, \lambda_u^{(2)} + x_2\right) - f_{1,2}(u) \right] dP(X^{(1)} \leq x_1) dP(X^{(2)} \leq x_2) dm_u^{(1)} \\ + \int_0^t \int_0^\infty \int_0^\infty \left[f\left(A_u^{(1)}, \lambda_u^{(1)} + y_1, A_u^{(2)} + 1, \lambda_u^{(2)} + y_2\right) - f_{1,2}(u) \right] dP(Y^{(1)} \leq y_1) dP(Y^{(2)} \leq y_2) dm_u^{(2)} \end{aligned}$$

is $\mathcal{F}_t^{A^{(1)}, A^{(2)}}$ -martingale. Therefore, the claim given by Eq. 3.2 (Dynkin's formula) is achieved. $\square$

3.3 Joint PGF-Laplace Transform

This subsection presents the joint PGF-Laplace transform of the joint process $(N, \lambda^{(1)}, \lambda^{(2)})$. To derive these results, we examine the joint distribution of the process in terms of PDEs, taking into account the Markov property of the joint process.

The joint PGF-Laplace transform of $(N, \lambda^{(1)}, \lambda^{(2)})$ is uniquely defined by

$$\zeta(t, \theta, u_1, u_2) = \mathbb{E} \left[\theta^{N_t} e^{-u_1 \lambda_t^{(1)}} e^{-u_2 \lambda_t^{(2)}} \right],$$

where $t \geq 0, \theta \in [0, 1], u_1, u_2 \geq 0$.

Theorem 3.1 *Let the arrival process be two-dimensional Hawkes process and let the service times follow an exponential distribution with parameter μ . Assuming that the initial value is $(N_t, \lambda_t^{(1)}, \lambda_t^{(2)}) \Big|_{t=0} = (0, \lambda_0^{(1)}, \lambda_0^{(2)})$, we have that*

$$\zeta(t, \theta, u_1, u_2) = e^{-u_1(t)\lambda_0^{(1)}} e^{-u_2(t)\lambda_0^{(2)}} \cdot \exp \left\{ -r_1 \lambda^{*(1)} \int_0^t u_1(x) dx - r_2 \lambda^{*(2)} \int_0^t u_2(x) dx \right\},$$

where $(u_1(\cdot), u_2(\cdot))$ solves the system of ODEs given by

$$\begin{aligned} u_1'(t) + r_1 u_1(t) + [1 + (\theta - 1)e^{-\mu t}] \alpha(u_1(t)) \gamma(u_2(t)) - 1 &= 0, \\ u_2'(t) + r_2 u_2(t) + [1 + (\theta - 1)e^{-\mu t}] \beta(u_1(t)) \delta(u_2(t)) - 1 &= 0, \end{aligned} \quad (3.4)$$

with boundary conditions $u_1(0) = u_1$ and $u_2(0) = u_2$, and where

$$\begin{aligned} \alpha(u_1) &= \mathbb{E} \left[e^{-u_1 X^{(1)}} \right], & \beta(u_1) &= \mathbb{E} \left[e^{-u_1 Y^{(1)}} \right], \\ \gamma(u_2) &= \mathbb{E} \left[e^{-u_2 X^{(2)}} \right], & \delta(u_2) &= \mathbb{E} \left[e^{-u_2 Y^{(2)}} \right]. \end{aligned}$$

In order to achieve the proof of Theorem 3.1, we prove some preliminaries results in Proposition 3.2, Proposition 3.3 and Proposition 3.4.

Proposition 3.2 *For $t > 0$, the joint distribution of $(N, \lambda^{(1)}, \lambda^{(2)})$ is given by*

$$F(t, k, \lambda^{(1)}, \lambda^{(2)}) = P \left(N_t = k, \lambda_t^{(1)} \leq \lambda^{(1)}, \lambda_t^{(2)} \leq \lambda^{(2)} \right).$$

Assuming that F is continuously differentiable in t , and absolutely continuous in $(\lambda^{(1)}, \lambda^{(2)})$, then $F(t, k, \lambda^{(1)}, \lambda^{(2)})$ satisfies the equation given by

$$\begin{aligned} & \frac{\partial^3 F(t, k, \lambda^{(1)}, \lambda^{(2)})}{\partial \lambda^{(2)} \partial \lambda^{(1)} \partial t} + \sum_{i=1,2} \frac{\partial}{\partial \lambda^{(i)}} \left[r_i \left(\lambda^{*(i)} - \lambda^{(i)} \right) \frac{\partial^2 F(t, k, \lambda^{(1)}, \lambda^{(2)})}{\partial \lambda^{(2-i+1)} \partial \lambda^{(i)}} \right] \\ &= \int_0^{\lambda^{(2)}} \int_0^{\lambda^{(1)}} x_1 \frac{\partial^2 F(t, k-1, x_1, x_2)}{\partial x_2 \partial x_1} g_{X^{(1)}}(\lambda^{(1)} - x_1) g_{X^{(2)}}(\lambda^{(2)} - x_2) dx_1 dx_2 \\ &+ \int_0^{\lambda^{(2)}} \int_0^{\lambda^{(1)}} y_2 \frac{\partial^2 F(t, k-1, y_1, y_2)}{\partial y_2 \partial y_1} g_{Y^{(1)}}(\lambda^{(1)} - y_1) g_{Y^{(2)}}(\lambda^{(2)} - y_2) dy_1 dy_2 \\ &- (\lambda^{(1)} + \lambda^{(2)}) \frac{\partial^2 F(t, k, \lambda^{(1)}, \lambda^{(2)})}{\partial \lambda^{(2)} \partial \lambda^{(1)}} + (k+1)\mu \frac{\partial^2 F(t, k+1, \lambda^{(1)}, \lambda^{(2)})}{\partial \lambda^{(2)} \partial \lambda^{(1)}} - k\mu \frac{\partial^2 F(t, k, \lambda^{(1)}, \lambda^{(2)})}{\partial \lambda^{(2)} \partial \lambda^{(1)}}, \end{aligned} \quad (3.5)$$

where, for $i = 1, 2$, $g_{X^{(i)}}$ and $g_{Y^{(i)}}$ are the pdfs of $X^{(i)}$ and $Y^{(i)}$, respectively.

Proof We consider the infinitesimal evolution of the Markov process $(N_t, \lambda_t^{(1)}, \lambda_t^{(2)})$ over $(t, t + \Delta t]$. Conditioning on the possible events in this interval, we get

$$\begin{aligned} & F\left(t + \Delta t, k, \lambda^{(1)} - r_1 \Delta t \left(\lambda^{(1)} - \lambda^{*(1)} \right), \lambda^{(2)} - r_2 \Delta t \left(\lambda^{(2)} - \lambda^{*(2)} \right)\right) \\ &= \int_0^{\lambda^{(2)}} \int_0^{\lambda^{(1)}} x_1 \Delta t P(X^{(1)} \leq \lambda^{(1)} - x_1) P(X^{(2)} \leq \lambda^{(2)} - x_2) d^2 F(t, k-1, x_1, x_2) \\ &+ \int_0^{\lambda^{(2)}} \int_0^{\lambda^{(1)}} y_2 \Delta t P(Y^{(1)} \leq \lambda^{(1)} - y_1) P(Y^{(2)} \leq \lambda^{(2)} - y_2) d^2 F(t, k-1, y_1, y_2) \\ &- \int_0^{\lambda^{(2)}} \int_0^{\lambda^{(1)}} x_1 \Delta t d^2 F(t, k, x_1, x_2) - \int_0^{\lambda^{(2)}} \int_0^{\lambda^{(1)}} y_2 \Delta t d^2 F(t, k, y_1, y_2) \\ &+ (k+1)\mu \Delta t F(t, k+1, \lambda^{(1)}, \lambda^{(2)}) + F(t, k, \lambda^{(1)}, \lambda^{(2)})(1 - k\mu \Delta t). \end{aligned}$$

After some simplification, we get that

$$\begin{aligned} & \frac{1}{\Delta t} \left[F\left(t + \Delta t, k, \lambda^{(1)} - r_1 \Delta t \left(\lambda^{(1)} - \lambda^{*(1)} \right), \lambda^{(2)} - r_2 \Delta t \left(\lambda^{(2)} - \lambda^{*(2)} \right)\right) - F(t, k, \lambda^{(1)}, \lambda^{(2)}) \right] \\ &= \int_0^{\lambda^{(2)}} \int_0^{\lambda^{(1)}} x_1 P(X^{(1)} \leq \lambda^{(1)} - x_1) P(X^{(2)} \leq \lambda^{(2)} - x_2) d^2 F(t, k-1, x_1, x_2) \\ &+ \int_0^{\lambda^{(2)}} \int_0^{\lambda^{(1)}} y_2 P(Y^{(1)} \leq \lambda^{(1)} - y_1) P(Y^{(2)} \leq \lambda^{(2)} - y_2) d^2 F(t, k-1, y_1, y_2) \\ &- \int_0^{\lambda^{(2)}} \int_0^{\lambda^{(1)}} x_1 d^2 F(t, k, x_1, x_2) - \int_0^{\lambda^{(2)}} \int_0^{\lambda^{(1)}} y_2 d^2 F(t, k, y_1, y_2) \\ &+ (k+1)\mu F(t, k+1, \lambda^{(1)}, \lambda^{(2)}) - k\mu F(t, k, \lambda^{(1)}, \lambda^{(2)}). \end{aligned}$$

Taking $\Delta t \downarrow 0$, we have

$$\begin{aligned} & \frac{\partial F(t, k, \lambda^{(1)}, \lambda^{(2)})}{\partial t} - r_1(\lambda^{(1)} - \lambda^{*(1)}) \frac{\partial F(t, k, \lambda^{(1)}, \lambda^{(2)})}{\partial \lambda^{(1)}} - r_2(\lambda^{(2)} - \lambda^{*(2)}) \frac{\partial F(t, k, \lambda^{(1)}, \lambda^{(2)})}{\partial \lambda^{(2)}} \\ &= \int_0^{\lambda^{(2)}} \int_0^{\lambda^{(1)}} x_1 P(X^{(1)} \leq \lambda^{(1)} - x_1) P(X^{(2)} \leq \lambda^{(2)} - x_2) d^2 F(t, k - 1, x_1, x_2) \\ &+ \int_0^{\lambda^{(2)}} \int_0^{\lambda^{(1)}} y_2 P(Y^{(1)} \leq \lambda^{(1)} - y_1) P(Y^{(2)} \leq \lambda^{(2)} - y_2) d^2 F(t, k - 1, y_1, y_2) \\ &- \int_0^{\lambda^{(2)}} \int_0^{\lambda^{(1)}} x_1 d^2 F(t, k, x_1, x_2) - \int_0^{\lambda^{(2)}} \int_0^{\lambda^{(1)}} y_2 d^2 F(t, k, y_1, y_2) \\ &+ (k + 1)\mu F(t, k + 1, \lambda^{(1)}, \lambda^{(2)}) - k\mu F(t, k, \lambda^{(1)}, \lambda^{(2)}). \end{aligned}$$

Since $F(t, k, \cdot, \cdot)$ is absolutely continuous with respect to $\lambda^{(1)}$ and $\lambda^{(2)}$, so that

$$d^2 F(t, k, x_1, x_2) = \frac{\partial^2 F(t, k, x_1, x_2)}{\partial x_1 \partial x_2} dx_1 dx_2.$$

Now assuming, $P(X^{(1)} \leq 0) = 0$, $P(X^{(2)} \leq 0) = 0$, $P(Y^{(1)} \leq 0) = 0$ and $P(Y^{(2)} \leq 0) = 0$, differentiating, both sides with respect to $\lambda^{(1)}$ and to $\lambda^{(2)}$, and using Leibniz integral rule, we get Eq. 3.5. $\square$

Proposition 3.3 *Defining*

$$\xi(t, k, u_1, u_2) := \int_0^\infty \int_0^\infty e^{-u_1 \lambda^{(1)}} e^{-u_2 \lambda^{(2)}} \frac{\partial^2 F(t, k, \lambda^{(1)}, \lambda^{(2)})}{\partial \lambda^{(2)} \partial \lambda^{(1)}} d\lambda^{(1)} d\lambda^{(2)}, \quad (3.6)$$

and setting $\xi = \xi(t, k, u_1, u_2)$, Eq. 3.5 becomes

$$\begin{aligned} & \frac{\partial \xi}{\partial t} + (u_1 r_1 - 1) \frac{\partial \xi}{\partial u_1} + (u_2 r_2 - 1) \frac{\partial \xi}{\partial u_2} + \alpha(u_1) \gamma(u_2) \frac{\partial \xi(t, k - 1, u_1, u_2)}{\partial u_1} + \beta(u_1) \delta(u_2) \frac{\partial \xi(t, k - 1, u_1, u_2)}{\partial u_2} \\ &= (-u_1 r_1 \lambda^{*(1)} - u_2 r_2 \lambda^{*(2)}) \xi + (k + 1) \mu \xi(t, k + 1, u_1, u_2) - k \mu \xi, \end{aligned} \quad (3.7)$$

with boundary conditions $\xi(0, 0, u_1(0), u_2(0)) = e^{-u_1(0)\lambda_0^{(1)}} e^{-u_2(0)\lambda_0^{(2)}}$.

Proof In order to transform Eqs. 3.5 in term of $\xi(t, k, u, v)$, Eq. 3.6, we have

$$\int_0^\infty \int_0^\infty e^{-u_1 \lambda^{(1)}} e^{-u_2 \lambda^{(2)}} \frac{\partial^3 F(t, k, \lambda^{(1)}, \lambda^{(2)})}{\partial \lambda^{(2)} \partial \lambda^{(1)} \partial t} d\lambda^{(1)} d\lambda^{(2)} = \frac{\partial \xi}{\partial t}. \quad (3.8)$$

Using integration by parts with respect to $\lambda^{(1)}$, we get

$$\begin{aligned} & \int_0^\infty \int_0^\infty e^{-u_1 \lambda^{(1)}} e^{-u_2 \lambda^{(2)}} \frac{\partial}{\partial \lambda^{(1)}} \left(r_1(\lambda^{*(1)} - \lambda^{(1)}) \frac{\partial^2 F(t, k, \lambda^{(1)}, \lambda^{(2)})}{\partial \lambda^{(2)} \partial \lambda^{(1)}} \right) d\lambda^{(1)} d\lambda^{(2)} \\ &= u_1 r_1 \int_0^\infty \int_0^\infty e^{-u_1 \lambda^{(1)}} e^{-u_2 \lambda^{(2)}} (\lambda^{*(1)} - \lambda^{(1)}) \frac{\partial^2 F(t, k, \lambda^{(1)}, \lambda^{(2)})}{\partial \lambda^{(2)} \partial \lambda^{(1)}} d\lambda^{(1)} d\lambda^{(2)} \\ &= u_1 r_1 \lambda^{*(1)} \xi + u_1 r_1 \frac{\partial \xi}{\partial u_1}. \end{aligned} \quad (3.9)$$

In the same way, we have

$$\begin{aligned} & \int_0^\infty \int_0^\infty e^{-u_1 \lambda^{(1)}} e^{-u_2 \lambda^{(2)}} \frac{\partial}{\partial \lambda^{(2)}} \left(r_2 (\lambda^{*(2)} - \lambda^{(2)}) \frac{\partial^2 F(t, k, \lambda^{(1)}, \lambda^{(2)})}{\partial \lambda^{(1)} \partial \lambda^{(2)}} \right) d\lambda^{(1)} d\lambda^{(2)} \\ &= u_2 r_2 \lambda^{*(2)} \xi + u_2 r_2 \frac{\partial \xi}{\partial u_2}. \end{aligned} \quad (3.10)$$

Recall that $\alpha(u_1) = \mathbb{E}[e^{-u_1 X^{(1)}}]$, $\gamma(u_2) = \mathbb{E}[e^{-u_2 X^{(2)}}]$ and that $X^{(1)}, X^{(2)}$ are independent. By a change of variables between $\lambda^{(1)}$ and x_1 and simplifying, we get

$$\begin{aligned} I &= \int_0^\infty \int_0^\infty e^{-u_1 \lambda^{(1)}} e^{-u_2 \lambda^{(2)}} \\ &\quad \times \left(\int_0^{\lambda^{(2)}} \int_0^{\lambda^{(1)}} x_1 \frac{\partial^2 F(t, k-1, x_1, x_2)}{\partial x_2 \partial x_1} g_{X^{(1)}}(\lambda^{(1)} - x_1) g_{X^{(2)}}(\lambda^{(2)} - x_2) dx_1 dx_2 \right) d\lambda^{(1)} d\lambda^{(2)} \\ &= \int_0^\infty \int_0^{\lambda^{(2)}} e^{-u_2 \lambda^{(2)}} \\ &\quad \times \left(\int_0^\infty \int_0^{\lambda^{(1)}} e^{-u_1 \lambda^{(1)}} x_1 \frac{\partial^2 F(t, k-1, x_1, x_2)}{\partial x_2 \partial x_1} g_{X^{(1)}}(\lambda^{(1)} - x_1) dx_1 d\lambda^{(1)} \right) g_{X^{(2)}}(\lambda^{(2)} - x_2) dx_2 d\lambda^{(2)} \\ &= \int_0^\infty \int_0^{\lambda^{(2)}} e^{-u_2 \lambda^{(2)}} \left(\int_0^\infty \int_{x_1}^\infty e^{-u_1 (\lambda^{(1)} - x_1)} e^{-u_1 x_1} x_1 \frac{\partial^2 F(t, k-1, x_1, x_2)}{\partial x_2 \partial x_1} g_{X^{(1)}}(\lambda^{(1)} - x_1) d\lambda^{(1)} dx_1 \right) \\ &\quad \times g_{X^{(2)}}(\lambda^{(2)} - x_2) dx_2 d\lambda^{(2)}. \end{aligned}$$

To simplify last expression, by using ξ as defined in Eq. 3.6, we get that

$$\begin{aligned} I &= - \int_0^\infty \int_0^{\lambda^{(2)}} e^{-u_2 \lambda^{(2)}} \left(\int_0^\infty \int_{x_1}^\infty e^{-u_1 (\lambda^{(1)} - x_1)} \frac{\partial}{\partial u_1} \left(e^{-u_1 x_1} \frac{\partial^2 F(t, k-1, x_1, x_2)}{\partial x_2 \partial x_1} \right) \right. \\ &\quad \left. \times g_{X^{(1)}}(\lambda^{(1)} - x_1) d\lambda^{(1)} dx_1 \right) g_{X^{(2)}}(\lambda^{(2)} - x_2) dx_2 d\lambda^{(2)} \\ &= - \int_0^\infty \int_0^{\lambda^{(2)}} e^{-u_2 (\lambda^{(2)} - x_2)} \left(\int_0^\infty \int_{x_1}^\infty e^{-u_1 (\lambda^{(1)} - x_1)} \right. \\ &\quad \times \frac{\partial^3}{\partial x_2 \partial x_1 \partial u_1} \left(\int_0^\infty \int_0^\infty e^{-u_1 x_1} e^{-u_2 x_2} \frac{\partial^2 F(t, k-1, x_1, x_2)}{\partial x_2 \partial x_1} dx_1 dx_2 \right) g_{X^{(1)}}(\lambda^{(1)} - x_1) d\lambda^{(1)} dx_1 \Big) \\ &\quad \times g_{X^{(2)}}(\lambda^{(2)} - x_2) dx_2 d\lambda^{(2)} \\ &= - \int_0^\infty \int_0^{\lambda^{(2)}} e^{-u_2 (\lambda^{(2)} - x_2)} \left(\int_0^\infty \int_0^\infty e^{-u_1 \tau} \frac{\partial^3 \xi(t, k-1, u_1, u_2)}{\partial x_2 \partial x_1 \partial u_1} g_{X^{(1)}}(\tau) d\tau dx_1 \right) g_{X^{(2)}}(\lambda^{(2)} - x_2) dx_2 d\lambda^{(2)} \\ &= -\alpha(u_1) \int_0^\infty \int_0^{\lambda^{(2)}} e^{-u_2 (\lambda^{(2)} - x_2)} \frac{\partial^2}{\partial x_2 \partial u_1} \xi(t, k-1, u_1, u_2) g_{X^{(2)}}(\lambda^{(2)} - x_2) dx_2 d\lambda^{(2)}. \end{aligned}$$

Further, by applying a change of variables between $\lambda^{(2)}$ and x_2 , successively, we get

$$\begin{aligned} I &= -\alpha(u_1) \int_0^\infty \int_{x_2}^\infty e^{-u_2 (\lambda^{(2)} - x_2)} \frac{\partial^2}{\partial x_2 \partial u_1} \xi(t, k-1, u_1, u_2) g_{X^{(2)}}(\lambda^{(2)} - x_2) d\lambda^{(2)} dx_2 \\ &= -\alpha(u_1) \int_0^\infty \int_0^\infty e^{-u_2 \tau} \frac{\partial^2}{\partial x_2 \partial u_1} \xi(t, k-1, u_1, u_2) g_{X^{(2)}}(\tau) d\tau dx_2 \\ &= -\alpha(u_1) \gamma(u_2) \frac{\partial \xi(t, k-1, u_1, u_2)}{\partial u_1}. \end{aligned} \quad (3.11)$$

With an analogous procedure,

$$\begin{aligned} & \int_0^\infty \int_0^\infty e^{-u_1 \lambda^{(1)}} e^{-u_2 \lambda^{(2)}} \\ & \quad \times \left(\int_0^{\lambda^{(2)}} \int_0^{\lambda^{(1)}} y_2 \frac{\partial^2 F(t, k-1, x_1, x_2)}{\partial y_2 \partial y_1} g_{Y(1)}(\lambda^{(1)} - y_1) g_{Y(2)}(\lambda^{(2)} - y_2) dy_1 dy_2 \right) d\lambda^{(1)} d\lambda^{(2)} \\ & = -\beta(u_1) \delta(u_2) \frac{\partial \xi(t, k-1, u_1, u_2)}{\partial u_2}. \end{aligned} \quad (3.12)$$

Moreover,

$$\begin{aligned} & - \int_0^\infty \int_0^\infty e^{-u_1 \lambda^{(1)}} e^{-u_2 \lambda^{(2)}} (\lambda^{(1)} + \lambda^{(2)}) \frac{\partial^2 F(t, k, \lambda^{(1)}, \lambda^{(2)})}{\partial \lambda^{(2)} \partial \lambda^{(1)}} d\lambda^{(1)} d\lambda^{(2)} \\ & = \frac{\partial \xi(t, k, u_1, u_2)}{\partial u_1} + \frac{\partial \xi(t, k, u_1, u_2)}{\partial u_2}. \end{aligned} \quad (3.13)$$

Combining all the transformations made in Eqs. 3.8-3.13, we obtain Eq. 3.7. $\square$

Proposition 3.4 *Setting*

$$\zeta(t, \theta, u_1, u_2) = \sum_{k=0}^{\infty} \theta^k \xi(t, k, u_1, u_2),$$

and $\zeta = \zeta(t, \theta, u_1, u_2)$, in Eq. 3.7, we get

$$\begin{aligned} & \frac{\partial \zeta}{\partial t} + (u_1 r_1 + \theta \alpha(u_1) \gamma(u_2) - 1) \frac{\partial \zeta}{\partial u_1} + (u_2 r_2 + \theta \beta(u_1) \delta(u_2) - 1) \frac{\partial \zeta}{\partial u_2} + \mu(\theta - 1) \frac{\partial \zeta}{\partial \theta} \\ & = (-u_1 r_1 \lambda^{*(1)} - u_2 r_2 \lambda^{*(2)}) \zeta, \end{aligned} \quad (3.14)$$

with boundary condition $\zeta(0, \theta, u(0), v(0)) = e^{-u_1(0)\lambda_0^{(1)}} e^{-u_2(0)\lambda_0^{(2)}}$.

Proof Successively, note that

$$\sum_{k=0}^{\infty} \theta^k \frac{\partial \xi}{\partial t} = \frac{\partial}{\partial t} \sum_{k=0}^{\infty} \theta^k \xi = \frac{\partial \zeta}{\partial t}, \quad (3.15)$$

$$(u_1 r_1 - 1) \sum_{k=0}^{\infty} \theta^k \frac{\partial \xi}{\partial u_1} = (u_1 r_1 - 1) \frac{\partial \zeta}{\partial u_1} \quad \text{and} \quad (u_2 r_2 - 1) \sum_{k=0}^{\infty} \theta^k \frac{\partial \xi}{\partial u_2} = (u_2 r_2 - 1) \frac{\partial \zeta}{\partial u_2}. \quad (3.16)$$

Taking into account that $F(t, -1, \lambda, \mu) = 0$, then

$$\begin{aligned} \sum_{k=0}^{\infty} \theta^k \frac{\partial \xi(t, k-1, u_1, u_2)}{\partial u_1} & = \frac{\partial \xi(t, -1, u_1, u_2)}{\partial u_1} + \theta \sum_{k=1}^{\infty} \theta^{k-1} \frac{\partial \xi(t, k-1, u_1, u_2)}{\partial u_1} \\ & = \theta \frac{\partial}{\partial u_1} \sum_{k=0}^{\infty} \theta^k \xi(t, k, u_1, u_2) = \theta \frac{\partial}{\partial u_1} \sum_{k=0}^{\infty} \theta^k \xi = \theta \frac{\partial \zeta}{\partial u_1}. \end{aligned} \quad (3.17)$$

In the same way

$$\sum_{k=0}^{\infty} \theta^k \frac{\partial \xi(t, k-1, u_1, u_2)}{\partial u_2} = \theta \frac{\partial \zeta(t, \theta, u_1, u_2)}{\partial u_2} = \theta \frac{\partial \zeta}{\partial u_2}. \quad (3.18)$$

Analogously,

$$\begin{aligned}\sum_{k=0}^{\infty} (k+1)\theta^k \xi(t, k+1, u_1, u_2) &= \frac{\partial}{\partial \theta} \sum_{k=0}^{\infty} \theta^{k+1} \xi(t, k+1, u_1, u_2) = \frac{\partial}{\partial \theta} \sum_{k=1}^{\infty} \theta^k \xi(t, k, u_1, u_2) \\ &= \frac{\partial}{\partial \theta} \sum_{k=1}^{\infty} \theta^k \xi = \frac{\partial}{\partial \theta} \int \left(\sum_{k=0}^{\infty} k \theta^{k-1} \xi \right) d\theta = \frac{\partial \zeta}{\partial \theta}.\end{aligned}\quad (3.19)$$

Moreover,

$$\sum_{k=0}^{\infty} k \theta^k \xi = \theta \frac{\partial \zeta}{\partial \theta}.\quad (3.20)$$

Substituting the results obtained in Eqs. 3.15–3.20 in Eq. 3.7, we obtain the PDE given by Eq. 3.14. $\square$

Proof of Theorem 3.1 As in the proof of Theorem 2.1, we solve the PDE given in Eq. 3.14, using the method of characteristics. Fix (t, θ, u_1, u_2) and consider the characteristic curve passing through this point.

Let $\tau \in [0, t]$ and define the characteristic curves as $\tau \mapsto (\tau, \theta(\tau), u_1(\tau), u_2(\tau))$, with boundary conditions $\theta(t) = \theta$, $u_1(t) = u_1$, and $u_2(t) = u_2$, which ensure that

$$\zeta(t, \theta, u_1, u_2) = \zeta(\tau, \theta(\tau), u_1(\tau), u_2(\tau)).$$

By the chain rule, we get

$$\frac{d\zeta(\tau, \theta(\tau), u_1(\tau), u_2(\tau))}{d\tau} = \frac{\partial \zeta}{\partial \tau} + \frac{d\theta(\tau)}{d\tau} \frac{\partial \zeta}{\partial \theta} + \frac{du_1(\tau)}{d\tau} \frac{\partial \zeta}{\partial u_1} + \frac{du_2(\tau)}{d\tau} \frac{\partial \zeta}{\partial u_2},\quad (3.21)$$

and comparing Eqs. 3.14 and 3.21,

$$\frac{d\zeta(\tau, \theta(\tau), u_1(\tau), u_2(\tau))}{d\tau} = (-u_1(\tau)r_1\lambda^{*(1)} - u_2(\tau)r_2\lambda^{*(2)})\zeta(\tau, \theta(\tau), u_1(\tau), u_2(\tau)).\quad (3.22)$$

Integrating from 0 to t , the solution of Eq. 3.22 is given by

$$\zeta(t, \theta(t), u(t), v(t)) = e^{-u_1(0)\lambda_0^{(1)}} e^{-u_2(0)\lambda_0^{(2)}} \exp\left\{-r_1\lambda^{*(1)} \int_0^t u_1(x)dx - r_2\lambda^{*(2)} \int_0^t u_2(x)dx\right\}.$$

Similarly, comparing Eqs. 3.14 and 3.21, we get $\frac{d\theta(\tau)}{d\tau} = \mu(\theta(\tau) - 1)$, which solution is given by $\theta(\tau) = 1 + ce^{\mu\tau}$. Recalling that $\theta(t) = \theta$, which implies that $c = (\theta - 1)e^{-\mu t}$, we get

$$\theta(\tau) = 1 + (\theta - 1)e^{-\mu(t-\tau)}.\quad (3.23)$$

Now comparing Eqs. 3.14 and 3.21 and substituting $\theta(\tau)$ obtained in Eq. 3.23, we get

$$\begin{aligned}\frac{du_1(\tau)}{d\tau} &= r_1u_1(\tau) + \theta(\tau)\alpha(u_1(\tau))\gamma(u_2(\tau)) - 1 \\ &= r_1u_1(\tau) + \left[1 + (\theta - 1)e^{-\mu(t-\tau)}\right]\alpha(u_1(\tau))\gamma(u_2(\tau)) - 1\end{aligned}\quad (3.24)$$

$$\begin{aligned}\frac{du_2(\tau)}{d\tau} &= r_2u_2(\tau) + \theta(\tau)\beta(u_1(\tau))\delta(u_2(\tau)) - 1 \\ &= r_2u_2(\tau) + \left[1 + (\theta - 1)e^{-\mu(t-\tau)}\right]\beta(u_1(\tau))\delta(u_2(\tau)) - 1.\end{aligned}\quad (3.25)$$

We introduce the time-reversed variables $v_i, i = 1, 2$, defined by

$$v_i(\tau) = u_i(t - \tau), \quad 0 \leq \tau \leq t.$$

Rewriting the ODEs defined in Eqs. 3.24 and 3.25 in terms of v_1 and v_2 , respectively, and then relabelling v_i as u_i , we obtain the system of ODEs defined in Eq. 3.4, with initial conditions $u_1(0) = u_1$ and $u_2(0) = u_2$. $\square$

3.4 Transient Moments

Under some suitable assumptions, we are going to discuss the expression of the joint transient moment of the type $\mathbb{E} \left[(N_t)^{n_1} \left(\lambda_t^{(1)} \right)^{n_2} \left(\lambda_t^{(2)} \right)^{n_3} \right]$.

Setting $A = \theta^{N_t} e^{-u_1 \lambda_t^{(1)}} e^{-u_2 \lambda_t^{(2)}}$ and noting that $\mathbb{E} \left[\theta^{N_t} e^{-u_1 \lambda_t^{(1)}} e^{-u_2 \lambda_t^{(2)}} \right]$ is a function of t, θ, u_1 , and u_2 , by Eq. 3.14, we have that

$$\begin{aligned} \frac{d}{dt} \mathbb{E}[A] - (u_1 r_1 + \theta \alpha(u_1) \gamma(u_2) - 1) \mathbb{E} \left[\lambda_t^{(1)} A \right] - (u_2 r_2 + \theta \beta(u_1) \delta(u_2) - 1) \mathbb{E} \left[\lambda_t^{(2)} A \right] \\ + \mu(\theta - 1) \mathbb{E} \left[N_t \theta^{N_t-1} e^{-u_1 \lambda_t^{(1)}} e^{-u_2 \lambda_t^{(2)}} \right] = \left(-u_1 r_1 \lambda^{*(1)} - u_2 r_2 \lambda^{*(2)} \right) \mathbb{E}[A]. \end{aligned} \quad (3.26)$$

Now differentiating both sides of the Eq. 3.26, ($n_2 \in \mathbb{N} \cup \{0\}$) times with respect to u_1 , after that taking $u_1 = 0$, we obtain

$$\begin{aligned} \frac{d}{dt} \mathbb{E} \left[\theta^{N_t} \left(\lambda_t^{(1)} \right)^{n_2} e^{-u_2 \lambda_t^{(2)}} \right] + n_2 r_1 \mathbb{E} \left[\theta^{N_t} \left(\lambda_t^{(1)} \right)^{n_2} e^{-u_2 \lambda_t^{(2)}} \right] \\ + u_2 r_2 \lambda^{*(2)} \mathbb{E} \left[\theta^{N_t} \left(\lambda_t^{(1)} \right)^{n_2} e^{-u_2 \lambda_t^{(2)}} \right] + \mu(\theta - 1) \mathbb{E} \left[N_t \theta^{N_t-1} \left(\lambda_t^{(1)} \right)^{n_2} e^{-u_2 \lambda_t^{(2)}} \right] \\ = -\mathbb{E} \left[\theta^{N_t} \left(\lambda_t^{(1)} \right)^{n_2+1} e^{-u_2 \lambda_t^{(2)}} \right] + (u_2 r_2 - 1) \mathbb{E} \left[\theta^{N_t} \left(\lambda_t^{(1)} \right)^{n_2} e^{-u_2 \lambda_t^{(2)}} \lambda_t^{(2)} \right] \\ + n_2 r_1 \lambda^{*(1)} \mathbb{E} \left[\theta^{N_t} \left(\lambda_t^{(1)} \right)^{n_2-1} e^{-u_2 \lambda_t^{(2)}} \right] \\ + \theta \gamma(u_2) \sum_{j=0}^{n_2} \binom{n_2}{j} \mathbb{E} \left[\theta^{N_t} \left(\lambda_t^{(1)} \right)^{j+1} e^{-u_2 \lambda_t^{(2)}} \right] \mathbb{E} \left[\left(X^{(1)} \right)^{n_2-j} \right] \\ + \theta \delta(u_2) \sum_{j=0}^{n_2} \binom{n_2}{j} \mathbb{E} \left[\theta^{N_t} \left(\lambda_t^{(1)} \right)^j e^{-u_2 \lambda_t^{(2)}} \lambda_t^{(2)} \right] \mathbb{E} \left[\left(Y^{(1)} \right)^{n_2-j} \right]. \end{aligned} \quad (3.27)$$

In the same way differentiating both sides of Eq. 3.27, ($n_3 \in \mathbb{N} \cup \{0\}$) times with respect to u_2 and after that taking $u_2 = 0$, we get

$$\begin{aligned} \frac{d}{dt} \mathbb{E} \left[\theta^{N_t} \left(\lambda_t^{(1)} \right)^{n_2} \left(\lambda_t^{(2)} \right)^{n_3} \right] + (n_2 r_1 + n_3 r_2) \mathbb{E} \left[\theta^{N_t} \left(\lambda_t^{(1)} \right)^{n_2} \left(\lambda_t^{(2)} \right)^{n_3} \right] \\ + \mu(\theta - 1) \mathbb{E} \left[N_t \theta^{N_t-1} \left(\lambda_t^{(1)} \right)^{n_2} \left(\lambda_t^{(2)} \right)^{n_3} \right] \\ = n_2 r_1 \lambda^{*(1)} \mathbb{E} \left[\theta^{N_t} \left(\lambda_t^{(1)} \right)^{n_2-1} \left(\lambda_t^{(2)} \right)^{n_3} \right] + n_3 r_2 \lambda^{*(2)} \mathbb{E} \left[\theta^{N_t} \left(\lambda_t^{(1)} \right)^{n_2} \left(\lambda_t^{(2)} \right)^{n_3-1} \right] \\ + \theta \sum_{j=0}^{n_2} \sum_{k=0}^{n_3} \binom{n_2}{j} \binom{n_3}{k} \mathbb{E} \left[\theta^{N_t} \left(\lambda_t^{(1)} \right)^{j+1} \left(\lambda_t^{(2)} \right)^k \right] \mathbb{E} \left[\left(X^{(1)} \right)^{n_2-j} \right] \mathbb{E} \left[\left(X^{(2)} \right)^{n_3-k} \right] \end{aligned}$$

$$\begin{aligned}
& + \theta \sum_{j=0}^{n_2} \sum_{k=0}^{n_3} \binom{n_2}{j} \binom{n_3}{k} \mathbb{E} \left[\theta^{N_t} (\lambda_t^{(1)})^j (\lambda_t^{(2)})^{k+1} \right] \mathbb{E} \left[(Y^{(1)})^{n_2-j} \right] \mathbb{E} \left[(Y^{(2)})^{n_3-k} \right] \\
& - \mathbb{E} \left[\theta^{N_t} (\lambda_t^{(1)})^{n_2+1} (\lambda_t^{(2)})^{n_3} \right] - \mathbb{E} \left[\theta^{N_t} (\lambda_t^{(1)})^{n_2} (\lambda_t^{(2)})^{n_3+1} \right]. \quad (3.28)
\end{aligned}$$

To obtain the joint moments and the moments that correspond to the system size, we differentiate Eq. 3.28, $(n_1 + 1)$ times with respect to θ and after setting $\theta = 1$, namely, we obtain

$$\begin{aligned}
& \frac{d}{dt} \mathbb{E} \left[(\overline{N}_t)^{n_1} (\lambda_t^{(1)})^{n_2} (\lambda_t^{(2)})^{n_3} \right] + (n_2 r_1 + n_3 r_2) \mathbb{E} \left[(\overline{N}_t)^{n_1} (\lambda_t^{(1)})^{n_2} (\lambda_t^{(2)})^{n_3} \right] \\
& + (n_1 + 1) \mu \mathbb{E} \left[(\overline{N}_t)^{n_1} (\lambda_t^{(1)})^{n_2} (\lambda_t^{(2)})^{n_3} \right] \\
& = n_2 r_1 \lambda^{*(1)} \mathbb{E} \left[(\overline{N}_t)^{n_1} (\lambda_t^{(1)})^{n_2-1} (\lambda_t^{(2)})^{n_3} \right] + n_3 r_2 \lambda^{*(2)} \mathbb{E} \left[(\overline{N}_t)^{n_1} (\lambda_t^{(1)})^{n_2} (\lambda_t^{(2)})^{n_3-1} \right] \\
& + (n_1 + 1) \sum_{j=0}^{n_2} \sum_{k=0}^{n_3} \binom{n_2}{j} \binom{n_3}{k} \mathbb{E} \left[(\overline{N}_t)^{n_1-1} (\lambda_t^{(1)})^{j+1} (\lambda_t^{(2)})^k \right] \mathbb{E} \left[(X^{(1)})^{n_2-j} \right] \mathbb{E} \left[(X^{(2)})^{n_3-k} \right] \\
& + \sum_{j=0}^{n_2} \sum_{k=0}^{n_3} \binom{n_2}{j} \binom{n_3}{k} \mathbb{E} \left[(\overline{N}_t)^{n_1} (\lambda_t^{(1)})^{j+1} (\lambda_t^{(2)})^k \right] \mathbb{E} \left[(X^{(1)})^{n_2-j} \right] \mathbb{E} \left[(X^{(2)})^{n_3-k} \right] \\
& + (n_1 + 1) \sum_{j=0}^{n_2} \sum_{k=0}^{n_3} \binom{n_2}{j} \binom{n_3}{k} \mathbb{E} \left[(\overline{N}_t)^{n_1-1} (\lambda_t^{(1)})^j (\lambda_t^{(2)})^{k+1} \right] \mathbb{E} \left[(Y^{(1)})^{n_2-j} \right] \mathbb{E} \left[(Y^{(2)})^{n_3-k} \right] \\
& + \sum_{j=0}^{n_2} \sum_{k=0}^{n_3} \binom{n_2}{j} \binom{n_3}{k} \mathbb{E} \left[(\overline{N}_t)^{n_1} (\lambda_t^{(1)})^j (\lambda_t^{(2)})^{k+1} \right] \mathbb{E} \left[(Y^{(1)})^{n_2-j} \right] \mathbb{E} \left[(Y^{(2)})^{n_3-k} \right] \\
& - \mathbb{E} \left[(\overline{N}_t)^{n_1} (\lambda_t^{(1)})^{n_2+1} (\lambda_t^{(2)})^{n_3} \right] - \mathbb{E} \left[(\overline{N}_t)^{n_1} (\lambda_t^{(1)})^{n_2} (\lambda_t^{(2)})^{n_3+1} \right]
\end{aligned}$$

where $(\overline{N}_t)^{n_1} = N_t(N_t - 1) \cdots (N_t - n_1)$ and $(\overline{N}_t)^{-1} = 1$. In order to find the first order moments, we have

$$\begin{aligned}
& \frac{d}{dt} \mathbb{E}[N_t] = -\mu \mathbb{E}[N_t] + \mathbb{E}[\lambda_t^{(1)}] + \mathbb{E}[\lambda_t^{(2)}] \\
& \frac{d}{dt} \mathbb{E}[\lambda_t^{(1)}] = -\left(r_1 - \mathbb{E}[X^{(1)}]\right) \mathbb{E}[\lambda_t^{(1)}] + \mathbb{E}[Y^{(1)}] \mathbb{E}[\lambda_t^{(2)}] + r_1 \lambda^{*(1)} \quad (3.29) \\
& \frac{d}{dt} \mathbb{E}[\lambda_t^{(2)}] = -\left(r_2 - \mathbb{E}[Y^{(2)}]\right) \mathbb{E}[\lambda_t^{(2)}] + \mathbb{E}[X^{(2)}] \mathbb{E}[\lambda_t^{(1)}] + r_2 \lambda^{*(2)}.
\end{aligned}$$

Remark 3.2 The graphical illustration of the solutions of each differential equations defined in Eq. 3.29 are shown in Fig. 2, with constant values

$$\begin{aligned}
r_1 &= 1, & \mathbb{E}[X^{(1)}] &= 0.5, & \mathbb{E}[Y^{(1)}] &= 0.6, \\
r_2 &= 1.2, & \mathbb{E}[X^{(2)}] &= 0.3, & \mathbb{E}[Y^{(2)}] &= 0.4, \\
\lambda^{*(1)} &= 0.8, & \lambda^{*(2)} &= 0.7, & \lambda_0^{(1)} &= 0.8, & \lambda_0^{(2)} &= 0.7 \quad \text{and} \quad \mu = 0.7.
\end{aligned}$$

Remark 3.3 Note that, similarly to the case discussed in Remark 2.5, Fig. 2 shows that the expected number of customers in the system, $\mathbb{E}[N_t]$, along with the expected arrival intensities

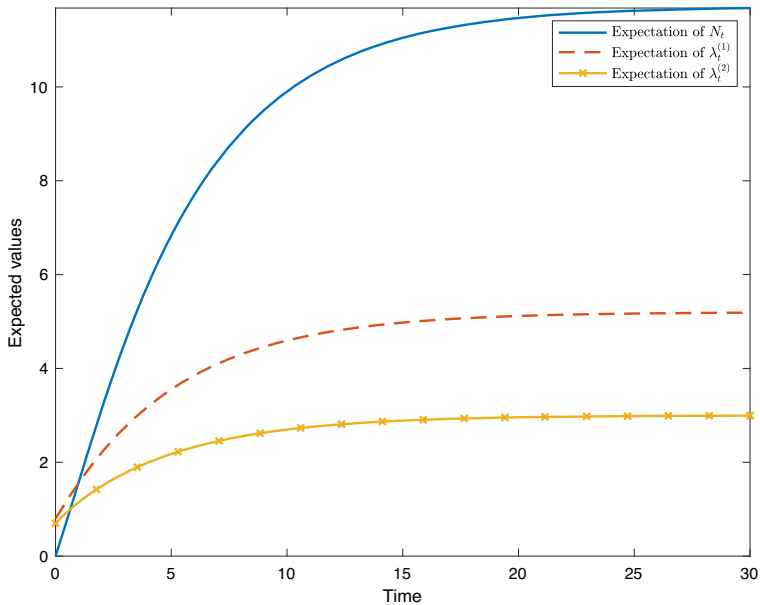


Fig. 2 Solutions to the system of differential equations with constant values given in Remark 3.2

$\mathbb{E}[\lambda_t^{(1)}]$ and $\mathbb{E}[\lambda_t^{(2)}]$, converges to finite steady-state values as t becomes sufficiently large, provided that $\rho(M) < 1$, where $\rho(M)$ denotes the spectral radius of the matrix

$$M = \begin{pmatrix} \frac{\mathbb{E}[X^{(1)}]}{r_1} & \frac{\mathbb{E}[Y^{(1)}]}{r_1} \\ \frac{\mathbb{E}[X^{(2)}]}{r_2} & \frac{\mathbb{E}[Y^{(2)}]}{r_2} \end{pmatrix}.$$

Under this stability condition, the intensity process $\{\lambda_t^{(i)}, t \geq 0\}$, $i = 1, 2$, admits a stationary distribution. Consequently, the associated infinite-server queueing system reaches a steady state.

Proposition 3.5 Let N_t denotes the number of customers in the system and let $\lambda_t^{(1)}$ and $\lambda_t^{(2)}$ be the intensities of two-dimensional Hawkes arrival process $(A^{(1)}, A^{(2)})$. Assume that $r_1 > \mathbb{E}[X^{(1)}]$, $r_2 > \mathbb{E}[Y^{(2)}]$ thus, we get

$$\mathbb{E}[\lambda_\infty^{(1)}] = \frac{r_1 \lambda^{*(1)} \mathbb{E}(Y^{(2)}) - r_2 \lambda^{*(2)} \mathbb{E}(Y^{(1)}) - r_1 r_2 \lambda^{*(1)}}{r_2 \mathbb{E}(X^{(1)}) + r_1 \mathbb{E}(Y^{(2)}) - r_1 r_2 - \mathbb{E}(X^{(1)}) \mathbb{E}(Y^{(2)}) + \mathbb{E}(X^{(2)}) \mathbb{E}(Y^{(1)})}, \quad (3.30)$$

$$\mathbb{E}[\lambda_\infty^{(2)}] = \frac{r_2 \lambda^{*(2)} \mathbb{E}(X^{(1)}) - r_1 \lambda^{*(1)} \mathbb{E}(X^{(2)}) - r_1 r_2 \lambda^{*(2)}}{r_2 \mathbb{E}(X^{(1)}) + r_1 \mathbb{E}(Y^{(2)}) - r_1 r_2 - \mathbb{E}(X^{(1)}) \mathbb{E}(Y^{(2)}) + \mathbb{E}(X^{(2)}) \mathbb{E}(Y^{(1)})}, \quad (3.31)$$

and

$$\begin{aligned} \mathbb{E}[N_\infty] = & \frac{1}{\mu} \left(r_1 \lambda^{*(1)} \mathbb{E}[Y^{(2)}] - r_2 \lambda^{*(2)} \mathbb{E}[Y^{(1)}] - r_1 r_2 \lambda^{*(1)} \right. \\ & \left. + r_2 \lambda^{*(2)} \mathbb{E}[X^{(1)}] - r_1 \lambda^{*(1)} \mathbb{E}[X^{(2)}] - r_1 r_2 \lambda^{*(2)} \right) \\ & \cdot \left(r_2 \mathbb{E}[X^{(1)}] + r_1 \mathbb{E}[Y^{(2)}] - r_1 r_2 - \mathbb{E}[X^{(1)}] \mathbb{E}[Y^{(2)}] + \mathbb{E}[X^{(2)}] \mathbb{E}[Y^{(1)}] \right)^{-1}. \end{aligned} \quad (3.32)$$

Proof Given the system of differential equations defined in Eq. 3.29, from Remark 3.3, we are able to conclude that the stationary solution of the differential equation exists and can be evaluate with the following system of equations

$$\begin{aligned} 0 &= -\mu \mathbb{E}[N_\infty] + \mathbb{E}[\lambda_\infty^{(1)}] + \mathbb{E}[\lambda_\infty^{(2)}] \\ 0 &= -\left(r_1 - \mathbb{E}[X^{(1)}]\right) \mathbb{E}[\lambda_\infty^{(1)}] + \mathbb{E}[Y^{(1)}] \mathbb{E}[\lambda_\infty^{(2)}] + r_1 \lambda^{*(1)} \\ 0 &= -\left(r_2 - \mathbb{E}[Y^{(2)}]\right) \mathbb{E}[\lambda_\infty^{(2)}] + \mathbb{E}[X^{(2)}] \mathbb{E}[\lambda_\infty^{(1)}] + r_2 \lambda^{*(2)}. \end{aligned}$$

After solving these system of equations, we obtain Eqs. 3.30-3.32. $\square$

4 General and Special Cases

In this section, specifically, in Subsection 4.1, we examine the Markov property and the joint PGF-Laplace transform for the d -dimensional Hawkes arrival process. In Subsection 4.2, we explore the one-dimensional Hawkes arrival process. In Subsection 4.3, we discuss the arrival process as a Poisson process and, subsequently, we provide a graphical comparison of all the described arrival processes.

4.1 d -Dimensional Hawkes Arrival Process

This Section discusses infinite-server systems, with the arrivals driven by the d -dimensional Hawkes processes with i.i.d. sequence of marks, representing the jump sizes.

4.1.1 Model Description and Markov Property

Let $(A^{(1)}, \dots, A^{(d)}) = (A_t^{(1)}, \dots, A_t^{(d)})_{t \geq 0}$ be a d -dimensional Hawkes process with $(\lambda^{(1)}, \dots, \lambda^{(d)}) = (\lambda_t^{(1)}, \dots, \lambda_t^{(d)})_{t \geq 0}$, the conditional intensity function governing the arrivals, such that

$$\lambda_t^{(i)} = \lambda^{*(i)} + e^{-r_i t} (\lambda_0^{(i)} - \lambda^{*(i)}) + \int_0^t X_s^{(i1)} e^{-r_i(t-s)} dA_s^{(1)} \dots + \int_0^t X_s^{(id)} e^{-r_i(t-s)} dA_s^{(d)},$$

for $i = 1, \dots, d, d \geq 1$, and where

- $\lambda_0^{(i)}$ are the initial intensities.
- $\lambda^{*(i)}$ represent the baseline intensities.
- $r_1, \dots, r_d$ are the mean-reverting rates.

- $X^{(ki)} = \{X_{t_j^{(i)}}^{(ki)}\}_{j>0}$, for $i, k = 1, 2, \dots, d$, are the i.i.d. sequence of positive random variables representing the size of the jumps of $A^{(i)}$, and are assumed to be independent of each other. We assume that $\mathbb{E}[X^{(ki)}] < \infty$, for $i, k = 1, 2, \dots, d$.
- $\{t_j^{(i)}\}_{j>0}$ are the sequences of the jump times of $A^{(i)}$.

Let a d -dimensional process and its intensity, $(A_t^{(1)}, \lambda_t^{(1)}, \dots, A_t^{(d)}, \lambda_t^{(d)})$, be defined on a function f such that $f: \mathbb{N} \times \mathbb{R}_+ \times \dots \times \mathbb{N} \times \mathbb{R}_+ \rightarrow \mathbb{R}$, with continuous partial derivatives with respect to $\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(d)}$. Setting

$$f_{1,\dots,d} = f(A^{(1)}, \lambda^{(1)}, \dots, A^{(d)}, \lambda^{(d)}), \quad f_{1,\dots,d}(t) = f(A_t^{(1)}, \lambda_t^{(1)}, \dots, A_t^{(d)}, \lambda_t^{(d)}),$$

and, for $i = 1, \dots, d$,

$$\Delta_{id} = f(A^{(1)} + \delta_{1i}, \lambda^{(1)} + x_{1i}, \dots, A^{(d)} + \delta_{di}, \lambda^{(d)} + x_{di}) - f(A^{(1)}, \lambda^{(1)}, \dots, A^{(d)}, \lambda^{(d)}),$$

the infinitesimal generator $\mathcal{A}$ of $(A^{(1)}, \lambda^{(1)}, \dots, A^{(d)}, \lambda^{(d)})$ defined on f is given by

$$\begin{aligned} \mathcal{A}f_{1,\dots,d} &:= r_1 \left(\lambda^{*(1)} - \lambda^{(1)} \right) \frac{\partial f_{1,\dots,d}}{\partial \lambda^{(1)}} + \dots + r_d \left(\lambda^{*(d)} - \lambda^{(d)} \right) \frac{\partial f_{1,\dots,d}}{\partial \lambda^{(d)}} \\ &+ \lambda^{(1)} \int_0^\infty \dots \int_0^\infty \Delta_{1d} dP(X^{(11)} \leq x_{11}) \dots dP(X^{(d1)} \leq x_{d1}) \\ &\dots \\ &+ \lambda^{(d)} \int_0^\infty \dots \int_0^\infty \Delta_{dd} dP(X^{(1d)} \leq x_{1d}) \dots dP(X^{(dd)} \leq x_{dd}). \end{aligned}$$

For any $0 \leq t \leq T$, we have that

$$\mathbb{E} \left(\int_0^t \left| \mathcal{A}f_{1,\dots,d}(u) - r_1 \left(\lambda^{*(1)} - \lambda_u^{(1)} \right) \frac{\partial f_{1,\dots,d}(u)}{\partial \lambda^{(1)}} - \dots - r_d \left(\lambda^{*(d)} - \lambda_u^{(d)} \right) \frac{\partial f_{1,\dots,d}(u)}{\partial \lambda^{(d)}} \right| du \right) < \infty.$$

Thus, for any $0 \leq t \leq T$, the following Dynkin's formula holds

$$\mathbb{E} \left(f_{1,\dots,d}(T) | \mathcal{F}_t^{A^{(1)}, \dots, A^{(d)}} \right) := f_{1,\dots,d}(t) + \mathbb{E} \left(\int_t^T \mathcal{A}f_{1,\dots,d}(u) du | \mathcal{F}_t^{A^{(1)}, \dots, A^{(d)}} \right).$$

4.1.2 Joint Transient Moments for d -Dimensional Hawkes Process

Theorem 4.1 *Let the arrival process be a d -dimensional Hawkes process and let the service times follow an independent exponential distribution with parameter μ . Assuming the initial value $(N_t, \lambda_t^{(1)}, \dots, \lambda_t^{(d)}) \Big|_{t=0} = (0, \lambda_0^{(1)}, \dots, \lambda_0^{(d)})$, we have*

$$\begin{aligned} \zeta(t, \theta, u_1, \dots, u_d) &= \mathbb{E} \left[\theta^{N_t} e^{-u_1 \lambda_t^{(1)}} \dots e^{-u_d \lambda_t^{(d)}} \right] \\ &= e^{-u_1(t) \lambda_0^{(1)}} \dots e^{-u_d(t) \lambda_0^{(d)}} \times \exp \left\{ -r_1 \lambda^{*(1)} \int_0^t u_1(x) dx - \dots - r_d \lambda^{*(d)} \int_0^t u_d(x) dx \right\}, \end{aligned}$$

where $(u_1(\cdot), \dots, u_d(\cdot))$ solves the system of ODEs given by

$$\begin{aligned} u'_1(t) + r_1 u_1(t) + [1 + (\theta - 1)e^{-\mu t}] \alpha_{11}(u_1(t)) \cdots \alpha_{d1}(u_d(t)) - 1 &= 0 \\ &\vdots \\ u'_d(t) + r_d u_d(t) + [1 + (\theta - 1)e^{-\mu t}] \alpha_{1d}(u_1(t)) \cdots \alpha_{dd}(u_d(t)) - 1 &= 0, \end{aligned}$$

with boundary conditions $u_1(0) = u_1, \dots, u_d(0) = u_d$, where

$$\alpha_{i1}(u_i) = \mathbb{E} \left[e^{-u_i X^{i1}} \right], \dots, \alpha_{id}(u_i) = \mathbb{E} \left[e^{-u_i X^{id}} \right], \quad \text{for } i = 1, 2, \dots, d.$$

Proof The outline of the proof is similar to that of Theorem 3.1.

In addition using the same approach as in Proposition 3.4, we can extend the PDE in Eq. 3.14 to the d -dimensional Hawkes process. In particular, if $\zeta = \zeta(t, \theta, u_1, \dots, u_d)$, then

$$\begin{aligned} \frac{\partial \zeta}{\partial t} + (u_1 r_1 + \theta \alpha_{11}(u_1(t)) \cdots \alpha_{d1}(u_d(t)) - 1) \frac{\partial \zeta}{\partial u_1} + \cdots \\ \cdots + (u_d r_d + \theta \alpha_{1d}(u_1(t)) \cdots \alpha_{dd}(u_d(t)) - 1) \frac{\partial \zeta}{\partial u_d} + \mu(\theta - 1) \frac{\partial \zeta}{\partial \theta} \\ = - \left(u_1 r_1 \lambda^{*(1)} + \cdots + u_d r_d \lambda^{*(d)} \right) \zeta, \end{aligned} \quad (4.1)$$

with boundary condition $\zeta(0, \theta, u_1(0), \dots, u_d(0)) = e^{-u_1(0)\lambda_0^{(1)}} \cdots e^{-u_d(0)\lambda_0^{(d)}}$.

Similarly, by setting $A = \theta^{N_t} e^{-u_1 \lambda_t^{(1)}} \cdots e^{-u_d \lambda_t^{(d)}}$ and noting that $\mathbb{E} \left[\theta^{N_t} e^{-u_1 \lambda_t^{(1)}} \cdots e^{-u_d \lambda_t^{(d)}} \right]$ is a function of $t, \theta, u_1, \dots, u_d$, and by Eq. 4.1, we have that

$$\begin{aligned} \frac{\partial}{\partial t} \mathbb{E} \left[\theta^{N_t} e^{-u_1 \lambda_t^{(1)}} \cdots e^{-u_d \lambda_t^{(d)}} \right] - (u_1 r_1 + \theta \alpha_{11}(u_1) \cdots \alpha_{d1}(u_d) - 1) \mathbb{E} \left[\lambda_t^{(1)} A \right] - \cdots \\ - (u_d r_d + \theta \alpha_{1d}(u_1) \cdots \alpha_{dd}(u_d) - 1) \mathbb{E} \left[\lambda_t^{(d)} A \right] + \mu(\theta - 1) \mathbb{E} \left[N_t \theta^{N_t-1} e^{-u_1 \lambda_t^{(1)}} \cdots e^{-u_d \lambda_t^{(d)}} \right] \\ = - \left(u_1 r_1 \lambda^{*(1)} + \cdots + u_d r_d \lambda^{*(d)} \right) \mathbb{E} \left[\theta^{N_t} e^{-u_1 \lambda_t^{(1)}} \cdots e^{-u_d \lambda_t^{(d)}} \right]. \end{aligned} \quad (4.2)$$

However, using the Eq. 4.2, we can able to find the expression of the joint transient moment of the type $\mathbb{E} \left[(N_t)^{n_1} \left(\lambda_t^{(1)} \right)^{n_2} \cdots \left(\lambda_t^{(d)} \right)^{n_{d+1}} \right]$ as demonstrated in Subsections 2.3 and 3.4.

4.2 One-Dimensional Hawkes Arrival Process

Let $A = \{A_t, t \geq 0\}$, the arrival process, be a one-dimensional Hawkes process with conditional intensity function $\lambda = \{\lambda_t, t \geq 0\}$, given by

$$\lambda_t = \lambda^* + e^{-rt} (\lambda_0 - \lambda^*) + \int_0^t X_s e^{-r(t-s)} dA_s. \quad (4.3)$$

Let the service times follow an independent exponential distribution with parameter μ and let N_t represent the number of customers in the system up to time t . Thus, by Theorem 3.1, we obtain

$$\zeta(t, \theta, u) = \mathbb{E} \left[\theta^{N_t} e^{-u \lambda_t} \right] = e^{-u(t)\lambda_0} \exp \left\{ -r \lambda^* \int_0^t u(x) dx \right\}, \quad (4.4)$$

where $u(\cdot)$ solves the ODE given by

$$u'(t) + ru(t) - 1 + (1 + (\theta - 1)e^{-\mu t})\alpha(u(t)) = 0, \quad (4.5)$$

with boundary condition $u(0) = u$ and $\alpha(u) = \mathbb{E}[e^{-uX}]$.

Remark 4.1 Koops et al. (2018) studied an infinite-server system under Hawkes/ M/∞ and Eq. 4.4 coincides with their result for the joint transient distribution of $\{(N_t, \lambda_t), t \geq 0\}$.

4.3 Poisson Arrival Process

In a one-dimensional Hawkes arrival case (as defined in Eq. 4.3), we consider $r = 0$ and $X_t = 0$, then $\{A_t, t \geq 0\}$ becomes a Poisson process with parameter λ_0 , and the service times follow an independent exponential distribution with parameter μ . By Eqs. 4.4 and 4.5, we deduce that

$$\zeta(t, \theta) = \mathbb{E}[\theta^{N_t}] = e^{-u(t)\lambda_0},$$

where $u'(t) = -(\theta - 1)e^{-\mu t}$ with boundary condition $u(0) = 0$, which implies that $u(t) = -\frac{1}{\mu}[(\theta - 1)(1 - e^{-\mu t})]$. Therefore,

$$\zeta(t, \theta) = \mathbb{E}[\theta^{N_t}] = \exp\left(\frac{\lambda_0}{\mu}[(\theta - 1)(1 - e^{-\mu t})]\right). \quad (4.6)$$

If we differentiate both sides of Eq. 4.6 with respect to θ and put $\theta = 1$, we get that $\mathbb{E}[N_t] = \frac{\lambda_0}{\mu}(1 - e^{-\mu t})$, which corresponds to the result of $M/M/\infty$ of the usual queueing models.

In order to compare the number of customers in the system with respect to time under different arrival processes, we present the following simulation-based graphical illustration in Fig. 3. The comparison includes four cases where arrivals: a two-dimensional Hawkes process, a dynamic contagion process, a one-dimensional Hawkes process, and a Poisson process. All processes are simulated using the same initial, baseline and constant parameters. Moreover, Fig. 4 is the graphical representation for the comparison of expected value of number of customers in the system.

By Figs. 3 and 4, we are able to observe that the number of customers in the system and its expected values in the two-dimensional Hawkes arrival takes greater values than in the other cases. However, this is true because the number of customers in the system is influenced by the more general arrival process, which can cause bursts of arrivals and lead to higher numbers of customers at times. Many real-world systems exhibit emergent behavior in which simple local interactions lead to complex global patterns. This type of behavior can be observed in communication and cloud-computing systems, where requests generated by one user activity may trigger additional requests; healthcare and emergency-response systems, where an initial event can lead to a cascade of subsequent arrivals; and financial markets, where activity in one asset or market segment may stimulate activity in others. Thus, studying infinite-server systems with such multi-exciting arrivals is essential

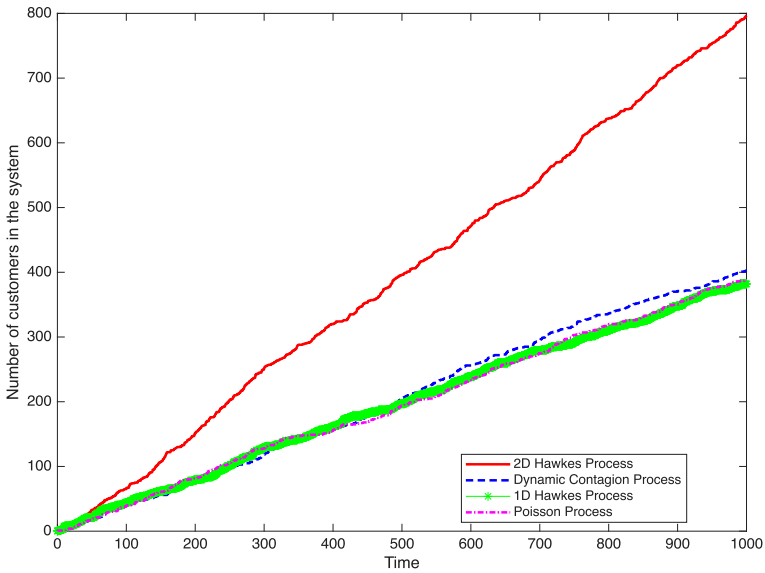


Fig. 3 Simulated sample path of the number of customers in the system across different arrival processes

for modeling and understanding complex systems in the real world. By understanding these systems, we can better predict behavior, manage risks, and improve performance in systems where demand can change rapidly and unpredictably.

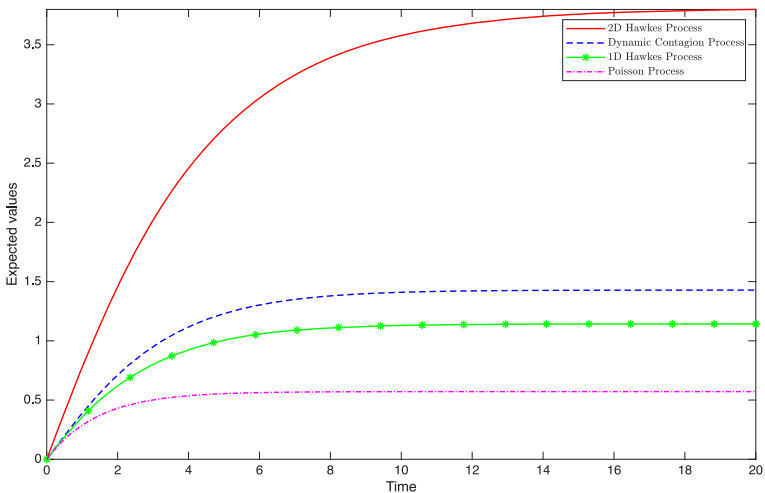


Fig. 4 Expected value of number of customers in the system across different arrival processes

5 Conclusion and Future Problems

In this paper, we examined infinite-server systems using different arrival process models. Specifically, we considered the arrival process as a finite-dimensional Hawkes process and a dynamic contagion process. It is well known that the joint process comprising the Hawkes process and its intensity function, characterized by an exponential excitation function, satisfies the Markov property. Consequently, we have discussed the Markov property of the finite-dimensional Hawkes arrival process and its intensities with exponential excitation functions. This analysis utilized some essential concepts such as the infinitesimal generator and Dynkin's formula. Subsequently, for all the arrival processes discussed, we derived the joint time-dependent PGF-Laplace transform. This discussion includes several specific instances of the underlying model. Furthermore, we employed the graphical illustrations to compare the expected number of customers in infinite-server systems across the various arrival models.

The framework developed in this work is suitable for a wide range of practical systems in which arrivals are not independent but exhibit temporal dependence and clustering effects. Such features are commonly observed in finance and insurance, for example in correlated claim arrivals, risk contagion, and clustered default events. It is also relevant to queueing systems arising in communication networks, where traffic often arrives in bursts rather than independently. Moreover, the multivariate extension allows for the study of interacting systems, where events in one component can influence the intensity of arrivals in another.

An open problem in this area is to investigate the Large Deviation Principle (LDP). Since, LDP plays an important role in the analysis of rare but significant events in queueing systems, such as extreme congestion. Understanding these rare events is essential for assessing system reliability and risk. Furthermore, other potential future work could focus on studying infinite-server systems with a higher dimensional Hawkes process while considering exciting functions other than the exponential, i.e., for general exciting functions. Additionally, service time can also be considered as non-exponential.

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Competing interests The authors declare no competing interests.

Conflicts of Interest The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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