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**Mathematical Methods of Operations
Research**

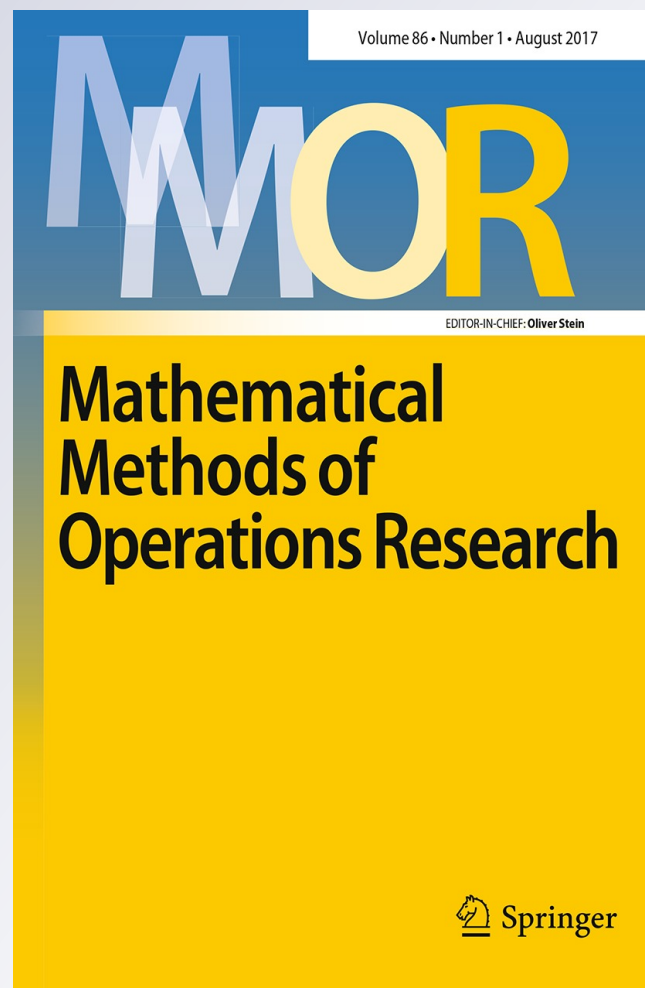
ISSN 1432-2994

Volume 86

Number 1

Math Meth Oper Res (2017) 86:29–69

DOI 10.1007/s00186-017-0582-4



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Mean–variance optimal trading problem subject to stochastic dominance constraints with second order autoregressive price dynamics

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Received: 18 September 2015 / Published online: 8 March 2017
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Abstract The efficient modeling of execution price path of an asset to be traded is an important aspect of the optimal trading problem. In this paper an execution price path based on the second order autoregressive process is proposed. The proposed price path is a generalization of the existing first order autoregressive price path in literature. Using dynamic programming method the analytical closed form solution of unconstrained optimal trading problem under the second order autoregressive process is derived. However in order to incorporate non-negativity constraints in the problem formulation, the optimal static trading problems under second order autoregressive price process are formulated. For a risk neutral investor, the optimal static trading problem of minimizing expected execution cost subject to non-negativity constraints is formulated as a quadratic programming problem. Whereas, for a risk averse investor the variance of execution cost is considered as a measure for the timing risk, and the mean–variance problem is formulated. Moreover, the optimal static trading problem subject to stochastic dominance constraints with mean–variance static trading strategy as the reference strategy is studied. Using Static approximation method the algorithm to solve proposed optimal static trading problems is presented. With numerical illustrations conducted on simulated data and the real market data, the significance of second order autoregressive price path, and the optimal static trading problems is presented.

Keywords Optimal trading · Execution cost · Stochastic dominance · Autoregressive behavior · Quadratic programming problem · Risk averse

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1 Introduction

1.1 Literature survey

The optimal trading problem is a problem of fundamental importance among market practitioners, algorithmic traders, academicians, etc. It is not efficient to trade all the required number of shares of an asset at once as doing so moves asset's price considerably if the trade size is large. Apart from trade order, other factors like arrival of some private information about the asset, changing market condition also contribute to change in prices. The price of the asset in absence of any trade or any other factor is called no-impact price. The immediate execution demand of a trade results in the temporary changes in price, and thus called as temporary price impact. Whereas the permanent changes in price due to supply-demand imbalance and other factors mentioned above is termed as permanent price impact. Thus the execution price dynamics of an asset consists of no-impact price path and the price impact factor. Based on the different models for no-impact price path and the price impact factor, many possible formulation of execution price dynamics are proposed in literature ([Bertsimas and Lo 1998](#); [Almgren and Chriss 2000](#); [Huberman and Stanzl 2005](#); [Gatheral and Schied 2011](#); [Khemchandani et al. 2013](#); [Moazeni et al. 2013](#)).

[Bertsimas and Lo \(1998\)](#) have proposed three models for execution price dynamics, referred therein as, Linear price impact model, Linear price impact with information price model, and Linear percentage temporary price impact model (LPT model). The first two models have no-impact price as arithmetic random walk whereas in LPT model the no-impact price is modeled as geometric Brownian motion. Further in the Linear price impact model the price impact is a linear function of trade size whereas in the Linear price impact with information price model it is a linear function of trade size and the information factor. The information factor captures the impact of factors like arrival of some information about the asset, changing market condition etc. on execution price. In LPT model the price impact, as a percentage of no-impact price, is modeled as a linear function of trade size and information factor. Furthermore [Almgren and Chriss \(2000\)](#) proposed the execution price dynamics with no-impact price as arithmetic random walk and linear price impact functions. [Moazeni et al. \(2013\)](#) presented the jump diffusion model for the execution price dynamics to capture the uncertain price impact of other large trades.

To reduce the effect of price impact and thus minimize execution cost an investor divides his trade into smaller parts to be traded in a fixed number of time periods. A risk neutral investor is the one whose primary concern is to find that particular division of trade, known as optimal trading strategy, which gives minimum cost of execution. [Bertsimas and Lo \(1998\)](#) are the first to give mathematical optimization model for the optimal trading problem. Further [Almgren and Chriss \(2000\)](#), [Huberman and Stanzl \(2005\)](#), [Moazeni et al. \(2013\)](#), [Khemchandani and Chandra \(2014\)](#), etc. also studied the single asset optimal trading problem. [Bertsimas et al. \(1999\)](#) and [Beutner \(2007\)](#) studied the optimal trading problem of portfolio of assets. The portfolio of all risky assets is considered in [Bertsimas et al. \(1999\)](#) whereas [Beutner \(2007\)](#) considered the portfolio of risky assets with one non-risky asset.

By splitting the trade into smaller parts to trade periodically, an investor is exposed to longer trade duration. The long trading horizon causes timing risk which is the risk occurs as a result of unfavorable price movements of the asset in the duration of trade (Bertsimas and Lo 1998; Bertsimas et al. 1999). A risk averse investor is the one who, when given two trading strategies with same expected execution cost, will choose the one with lower risk. Thus for a risk averse investor, it is plausible to minimize a measure for timing risk besides minimizing expected cost of execution. In the area of portfolio optimization, the seminal work of Markowitz (1952) had considered the variance of the portfolio return as the risk measure for a risk averse investor. On similar lines, in the area of optimal trading Huberman and Stanzl (2005) and Moazeni (2011) proposed the variance of execution cost as a measure for the timing risk. The variance as a measure for the timing risk is also suggested by Bertsimas and Lo (1998). Later on Moazeni et al. (2013) studied the optimal trading problem with Conditional Value-at-Risk of execution cost as a measure for the timing risk.

The mean-risk models are extensively used in areas like portfolio optimization (Markowitz 1952; Cai et al. 2000; Konno and Yamazaki 1991) and optimal trading (Huberman and Stanzl 2005; Moazeni 2011; Moazeni et al. 2013). However, the use of stochastic dominance approach which renders same decision to all investors whose utility functions are non-decreasing and concave, is very much concentrated to the area of portfolio optimization (Dentcheva and Ruszczyński 2006; Roman et al. 2006). The comparatively less literature is available wherein stochastic dominance approach is applied to optimal trading problems (Khemchandani 2016).

In the next subsection the motivation behind the present work and its main contributions are presented.

1.2 Motivation and contribution of the paper

It is evident from literature survey that in discrete time setting often no-impact price dynamics is modeled as arithmetic random walk. That is, the price of the asset in any time period is assumed to be dependent upon its price in the just preceding time period (first order autoregressive (AR(1)) behavior). The first contribution of this paper lies in extending the ‘Linear price impact with information price model’ of Bertsimas and Lo (1998), referred henceforth as AR(1) price model, to a more general framework. We incorporate second order autoregressive behavior (dependence of state in any time period on the states in last two preceding periods) in no-impact price dynamics and the information factor of AR(1) price model. Henceforth we refer this proposed price dynamics as AR(2) price model. The motivation behind such an extension is based on the studies which infer that the many financial time series, like price series, rate of return series, exhibit higher order autoregressive behavior (Marcek 1998, 2000; Rudoy and Rohrs 2008; Rudoy 2009). Marcek (1998, 2000) presented a comparison of the forecasts of prices obtained from fuzzy neural network model and autoregressive model. Rudoy and Rohrs (2008) and Rudoy (2009) studied the portfolio optimization problem under vector autoregressive price model for the portfolio. Apart from these studies of presence of higher order autoregressive behavior, in this paper we have presented empirical verification using real market data to justify the same.

Further in this paper, by Dynamic Programming method (DP method) the closed form solution of unconstrained optimal trading problem of minimizing expected execution cost under AR(2) price model is derived. However in the trading strategy of buying some shares of an asset, a negative trade in any time period has no physical meaning. Thus, it is required to include non-negativity constraints in the formulation of optimal trading problem. Although DP method allows to obtain closed form solution of unconstrained optimal trading problem, at the same time there are some computational difficulty associated with this method to solve problem subject to non-negativity constraints (Bertsimas and Lo 1998). Bertsimas et al. (1999) suggested Static Approximation method (SA method) which is based on roll-out approach, and enables one to include constraints in problem formulation.

In order to use SA method, in this paper optimal static trading problems under AR(2) price model are formulated. For a risk neutral investor, the optimal static trading problem of minimizing expected execution cost subject to non-negativity constraints is formulated as a quadratic programming problem. Henceforth, we refer this problem as the 'Mean QP problem'. For a risk averse investor the variance of execution cost is considered as a measure for the timing risk. With this risk measure the mean-risk optimization model of minimizing a trade-off of the expectation and variance of execution cost subject to non-negativity constraints is presented as a quadratic programming problem. We will refer this problem as 'Mean-variance QP problem'. Furthermore, in this paper we present the formulation of optimal static trading problem of minimizing expected execution cost subject to stochastic dominance constraints with respect to some reference strategy. Generally the reference strategy considered is equally weighted portfolio in case of portfolio optimization (Dentcheva and Ruszczyński 2006) or the naive strategy (equal division of trade in all the time periods) in case of optimal trading problems (Khemchandani 2016). However, in this paper we consider the reference strategy as the optimal trading strategy of the Mean-variance QP problem. The proposed optimal trading problem subject to stochastic dominance constraints results in a quadratically constrained quadratic programming problem (QCQP problem). The algorithm based on SA method to solve proposed optimal static trading problems—Mean QP, Mean-variance QP and QCQP, is presented. The results obtained by conducting numerical illustrations on simulated and the real market data depict the practical relevance of the proposed AR(2) price model and the optimal static trading problems. The better performance of AR(2) price model as compared to AR(1) is inferred by comparing their efficient trading frontiers corresponding to Mean-variance QP problem.

The rest of the paper is organized as follows. Section 2 provides the mathematical formulation of the general optimal trading problem and presents AR(2) price model. In Sect. 3, by DP method the analytical closed form solution of optimal trading problem under AR(2) price model is presented. Section 4 is devoted to the formulation of optimal static trading problems—Mean QP, Mean-variance QP and QCQP, under AR(2) price model. In Sect. 5, first the algorithm to solve optimal static trading problems by SA method is presented, which is followed by the numerical illustrations conducted on simulated and the real market data. Section 6 concludes the paper.

2 Single asset optimal trading: AR(2) price model

Let us consider an investor who intends to buy $\bar{S}$ shares of an asset by some fixed time point in future. To reduce the price impact and thus minimize cost of execution, the investor will divide $\bar{S}$ into smaller parts to be traded periodically before the time of execution. Let time of execution be divided into T number of time periods of equal length. Further we assume that S_t and P_t be the number of shares purchased and the execution price of a share in t th time period, respectively, where $t = 1, 2, \dots, T$.

The mathematical formulation of finding an optimal trading strategy $(S_1, S_2, \dots, S_T)$, which gives minimum expected execution cost, is as follows:

$$\begin{aligned}
 & \textbf{(P1)} \\
 & \min_{\{S_t\}_{t=1}^T} E_1 \left(\sum_{t=1}^T P_t S_t \right) \\
 & \text{subject to} \\
 & \sum_{t=1}^T S_t = \bar{S},
 \end{aligned}$$

where $\bar{S}$ is positive as trade is to purchase. Throughout this paper, notations E_t , Var_t and Cov_t respectively denote the conditional expectation, variance and covariance with respect to t th time period, assuming that the prices realized till previous time period $t - 1$, and the information factor till t th time period are known.

To complete the formulation of **(P1)**, one needs to give the dynamics of P_t . The AR(1) price model of [Bertsimas and Lo \(1998\)](#) is given as follows:

$$P_t = P_{t-1} + \theta S_t + \gamma X_t + \epsilon_t; \quad t = 1, 2, \dots, T, \quad (1)$$

$$X_t = \rho X_{t-1} + \eta_t; \quad t = 1, 2, \dots, T, \quad (2)$$

where $\theta > 0$, $\rho \in (-1, 1)$. Symbols $\epsilon = \{\epsilon_t; t = 1, 2, \dots, T\}$ are independent identically distributed (i.i.d.) normal random variables with zero mean and variance σ_ϵ^2 . Similarly, $\eta = \{\eta_t; t = 1, 2, \dots, T\}$ are i.i.d. normal random variables with zero mean and variance σ_η^2 , and are independent to ϵ .

The paper considers a more general form of the AR(1) price dynamics, (1) and (2), by incorporating the second order autoregressive behavior in no-impact price and in the information factor. The proposed execution price dynamics is referred as AR(2) price model, and is given as follows:

$$P_t = u_1 P_{t-1} + u_2 P_{t-2} + \theta S_t + \gamma X_t + \epsilon_t; \quad t = 1, 2, \dots, T, \quad (3)$$

$$X_t = \rho_1 X_{t-1} + \rho_2 X_{t-2} + \eta_t; \quad t = 1, 2, \dots, T, \quad (4)$$

where $\theta > 0$, $\rho_1, \rho_2 \in (-1, 1)$. Other symbols have same meaning as in AR(1) price model. The AR(2) price model gets reduced to AR(1) price model for particular values of parameters $u_2 = 0$ and $\rho_2 = 0$.

Bertsimas and Lo (1998) have used DP method to give the closed form solution of (P1) under the AR(1) price model. In the next section we present the analytical closed form solution of (P1) under the AR(2) price model.

3 Optimal trading under AR(2) price model: dynamic programming method

Approach based on DP method to solve (P1) uses the fact that in any time period t , the part of optimal trading strategy $(S_t, S_{t+1}, \dots, S_T)$, which corresponds to time periods t to T must be optimal for these time periods (Bertsimas and Lo 1998). Under AR(2) price model, we consider state variables for the t th time period as the prices realized in the last two trading periods, P_{t-1} and P_{t-2} , the information factor for the current and the last trading periods, X_t and X_{t-1} , and the factor W_t which is the number of shares that remained to be traded after $(t-1)$ th time period. The control variable for the t th time period is the number of shares S_t to be traded in that period. The problem (P1), to be solved by DP method, can be re-formulated as follows:

(DP Problem)

$$\min_{\{S_t\}_{t=1}^T} E_1 \left(\sum_{t=1}^T P_t S_t \right)$$

subject to

$$W_1 = \bar{S}, \quad W_t = W_{t-1} - S_{t-1} \quad \forall t = 2, 3, \dots, T, \quad W_{T+1} = 0.$$

Let $V_t(P_{t-1}, P_{t-2}, X_t, X_{t-1}, W_t) \left[= E_t \left(\sum_{k=t}^T P_k S_k \right) \quad \forall t = 1, 2, \dots, T \right]$ be the optimal cost of execution corresponding to trades into periods t to T .

In the last trading period T , optimal strategy is to execute all the remaining shares W_T to finish the trade. That is $S_T = W_T$. Thus V_T is given as follows:

$$\begin{aligned} V_T(P_{T-1}, P_{T-2}, X_T, X_{T-1}, W_T) &= \min_{S_T} E_T(P_T S_T) \\ &= \min_{S_T} E_T(u_1 P_{T-1} + u_2 P_{T-2} + \theta S_T + \gamma X_T + \epsilon_T) S_T \\ &= u_1 P_{T-1} W_T + u_2 P_{T-2} W_T + \theta W_T^2 + \gamma X_T W_T. \end{aligned} \quad (5)$$

The Bellman equation relating V_{T-t} to V_{T-t+1} , is given as follows:

$$\begin{aligned} V_{T-t}(P_{T-t-1}, P_{T-t-2}, X_{T-t}, X_{T-t-1}, W_{T-t}) &= \min_{S_{T-t}} E_{T-t}[P_{T-t} S_{T-t} + V_{T-t+1}(P_{T-t}, P_{T-t-1}, X_{T-t+1}, X_{T-t}, W_{T-t+1})]. \end{aligned} \quad (6)$$

Starting from the last trading period T , and moving backwards recursively while solving (6) under AR(2) price model, the closed form solution of DP problem is derived in Proposition 1.

Proposition 1 *Under the AR(2) price model, (3) and (4), the optimal trade and the optimal cost in each time period corresponding to DP problem are given by the following set of equations.*

$$S_{T-t} = \delta_{P,t} P_{T-t-1} + \delta_{PP,t} P_{T-t-2} + \delta_{W,t} W_{T-t} + \delta_{X,t} X_{T-t} + \delta_{XX,t} X_{T-t-1}, \quad \forall t = 0, 1, \dots, T-1, \quad (7)$$

$$\begin{aligned} V_{T-t} = & a_t P_{T-t-1}^2 + b_t P_{T-t-2}^2 + c_t W_{T-t}^2 + d_t X_{T-t}^2 + e_t X_{T-t-1}^2 \\ & + f_t P_{T-t-1} P_{T-t-2} + g_t P_{T-t-1} W_{T-t} \\ & + h_t P_{T-t-1} X_{T-t} + i_t P_{T-t-1} X_{T-t-1} + j_t P_{T-t-2} W_{T-t} \\ & + k_t P_{T-t-2} X_{T-t} + l_t P_{T-t-2} X_{T-t-1} \\ & + m_t W_{T-t} X_{T-t} + n_t W_{T-t} X_{T-t-1} + o_t X_{T-t} X_{T-t-1} + p_t, \\ & \forall t = 0, 1, \dots, T-1, \end{aligned} \quad (8)$$

where $\delta_{P,0} = 0$, $\delta_{PP,0} = 0$, $\delta_{W,0} = 1$, $\delta_{X,0} = 0$, $\delta_{XX,0} = 0$, $a_0 = 0$, $b_0 = 0$, $c_0 = \theta$, $d_0 = 0$, $e_0 = 0$, $f_0 = 0$, $g_0 = u_1$, $h_0 = 0$, $i_0 = 0$, $j_0 = u_2$, $k_0 = 0$, $l_0 = 0$, $m_0 = \gamma$, $n_0 = 0$, $o_0 = 0$, $p_0 = 0$, and for all $t = 1, 2, \dots, T-1$,

$$\begin{aligned} \delta_{P,t} &= \frac{g_{t-1}u_1 + j_{t-1} - u_1 - f_{t-1}\theta - 2a_{t-1}u_1\theta}{2(\theta + a_{t-1}\theta^2 + c_{t-1} - g_{t-1}\theta)} \\ \delta_{PP,t} &= \frac{g_{t-1}u_2 - 2a_{t-1}u_2\theta - u_2}{2(\theta + a_{t-1}\theta^2 + c_{t-1} - g_{t-1}\theta)} \\ \delta_{W,t} &= \frac{2c_{t-1} - g_{t-1}\theta}{2(\theta + a_{t-1}\theta^2 + c_{t-1} - g_{t-1}\theta)} \\ \delta_{X,t} &= \frac{n_{t-1} + m_{t-1}\rho_1 - i_{t-1}\theta + g_{t-1}\gamma - 2a_{t-1}\theta\gamma - \gamma - h_{t-1}\theta\rho_1}{2(\theta + a_{t-1}\theta^2 + c_{t-1} - g_{t-1}\theta)} \\ \delta_{XX,t} &= \frac{m_{t-1}\rho_2 - h_{t-1}\theta\rho_1}{2(\theta + a_{t-1}\theta^2 + c_{t-1} - g_{t-1}\theta)} \\ a_t &= -(\theta + a_{t-1}\theta^2 + c_{t-1} - g_{t-1}\theta)\delta_{P,t}^2 + a_{t-1}u_1^2 + b_{t-1} + f_{t-1}u_1 \\ b_t &= -(\theta + a_{t-1}\theta^2 + c_{t-1} - g_{t-1}\theta)\delta_{PP,t}^2 + a_{t-1}u_2^2 \\ c_t &= -(\theta + a_{t-1}\theta^2 + c_{t-1} - g_{t-1}\theta)\delta_{W,t}^2 + c_{t-1} \\ d_t &= -(\theta + a_{t-1}\theta^2 + c_{t-1} - g_{t-1}\theta)\delta_{X,t}^2 + a_{t-1}\gamma^2 + d_{t-1}\rho_1^2 + e_{t-1} \\ &\quad + h_{t-1}\gamma\rho_1 + i_{t-1}\gamma + o_{t-1}\rho_1 \\ e_t &= -(\theta + a_{t-1}\theta^2 + c_{t-1} - g_{t-1}\theta)\delta_{XX,t} + d_{t-1}\rho_2^2 \\ f_t &= -2(\theta + a_{t-1}\theta^2 + c_{t-1} - g_{t-1}\theta)\delta_{P,t}\delta_{PP,t} + 2a_{t-1}u_1u_2 + f_{t-1}u_2 \\ g_t &= -2(\theta + a_{t-1}\theta^2 + c_{t-1} - g_{t-1}\theta)\delta_{P,t}\delta_{W,t} + g_{t-1}u_1 + j_{t-1} \\ h_t &= -2(\theta + a_{t-1}\theta^2 + c_{t-1} - g_{t-1}\theta)\delta_{P,t}\delta_{X,t} + 2a_{t-1}u_1\gamma + f_{t-1}\gamma \\ &\quad + h_{t-1}u_1\rho_1 + i_{t-1}u_1 + k_{t-1}\rho_1 + l_{t-1} \\ i_t &= -2(\theta + a_{t-1}\theta^2 + c_{t-1} - g_{t-1}\theta)\delta_{P,t}\delta_{XX,t} + h_{t-1}u_1\rho_2 + k_{t-1}\rho_2 \\ j_t &= -2(\theta + a_{t-1}\theta^2 + c_{t-1} - g_{t-1}\theta)\delta_{PP,t}\delta_{W,t} + g_{t-1}u_2 \end{aligned}$$

$$\begin{aligned}
 k_t &= -2(\theta + a_{t-1}\theta^2 + c_{t-1} - g_{t-1}\theta)\delta_{PP,t}\delta_{X,t} \\
 &\quad + 2a_{t-1}u_2\gamma + h_{t-1}u_2\rho_1 + i_{t-1}u_2 \\
 l_t &= -2(\theta + a_{t-1}\theta^2 + c_{t-1} - g_{t-1}\theta)\delta_{PP,t}\delta_{XX,t} + h_{t-1}u_2\rho_2 \\
 m_t &= -2(\theta + a_{t-1}\theta^2 + c_{t-1} - g_{t-1}\theta)\delta_{W,t}\delta_{X,t} + g_{t-1}\gamma + m_{t-1}\rho_1 + n_{t-1} \\
 n_t &= -2(\theta + a_{t-1}\theta^2 + c_{t-1} - g_{t-1}\theta)\delta_{W,t}\delta_{XX,t} + m_{t-1}\rho_2 \\
 o_t &= -2(\theta + a_{t-1}\theta^2 + c_{t-1} - g_{t-1}\theta)\delta_{X,t}\delta_{XX,t} + 2d_{t-1}\rho_1\rho_2 \\
 &\quad + h_{t-1}\gamma\rho_2 + o_{t-1}\rho_2 \\
 p_t &= a_{t-1}\sigma_\epsilon^2 + d_{t-1}\sigma_\eta^2 + p_{t-1}.
 \end{aligned}$$

Proof Proof of Proposition 1 is presented in “Appendix 1” section. □

Remark 1 The optimal expected cost of execution for complete trading problem is given by $V_1(P_0, P_{-1}, X_1, X_0, \bar{S})$.

Remark 2 For particular values of parameters $u_2 = 0$, $\rho_2 = 0$, S_{T-t} and V_{T-t} corresponding to AR(2) price model (given by (7) and (8), respectively) reduces to corresponding values for AR(1) price model of [Bertsimas and Lo \(1998\)](#).

4 Formulation of optimal static trading problems under AR(2) price model

Although by DP method it is possible to find closed form solution of optimal trading problem of minimizing cost of execution, there are some computational difficulty associated with this method to present solution of the problem subject to non-negativity constraints ([Bertsimas and Lo 1998](#)). However, since a negative trade in the trading strategy of buying some shares of an asset has no physical meaning, it is important to include non-negativity constraints in the problem formulation. [Bertsimas et al. \(1999\)](#) suggested SA method which allows incorporating constraints in problem definition. In order to use SA method, the formulation of optimal static trading problems under AR(2) price model is required.

In the subsequent subsections, first the formulation of optimal static trading problem for a risk neutral investor, named as Mean QP problem, is presented. It will be followed by the formulation of the Mean–variance QP problem with variance of execution cost as a measure for the timing risk. Thereafter the QCQP problem with stochastic dominance constraint is formulated. We refer the trading strategies corresponding to optimal static trading problems—Mean QP, Mean–variance QP and QCQP by ‘static trading strategies’ (denoted by $(\hat{S}_1, \hat{S}_2, \dots, \hat{S}_T)$) to distinguish these from the trading strategies corresponding to the DP problem.

4.1 Optimal static trading problem for a risk neutral investor

A risk neutral investor’s primary concern is to minimize expected execution cost subject to constraints of non-negativity of decision variables. Consider following optimal static trading problem:

(P2)

$$\begin{aligned} \min_{\{\hat{S}_t\}_{t=1}^T} \quad & E_1 \left(\sum_{t=1}^T P_t \hat{S}_t \right) \\ \text{subject to} \quad & \sum_{t=1}^T \hat{S}_t = \bar{S}, \\ & \hat{S}_t \geq 0; \quad t = 1, 2, \dots, T. \end{aligned}$$

Next, we evaluate the execution cost $\left(\sum_{t=1}^T P_t \hat{S}_t \right)$ as a quadratic function of decision variables $(\hat{S}_1, \hat{S}_2, \dots, \hat{S}_T)$.

Consider symbols $L_t, M_t; t = 0, 1, 2, \dots, T$, given by following equations, which we require in further discussion.

$$L_0 = 1, \quad M_0 = 1, \quad (9)$$

$$L_{2t} = \sum_{k=0}^t \binom{2t-k}{k} u_1^{2t-2k} u_2^k,$$

$$M_{2t} = \sum_{k=0}^t \binom{2t-k}{k} \rho_1^{2t-2k} \rho_2^k \quad \forall t = 1, 2, \dots, \left\lceil \frac{T-1}{2} \right\rceil, \quad (10)$$

$$L_{2t-1} = \sum_{k=0}^{t-1} \binom{2t-1-k}{k} u_1^{2t-1-2k} u_2^k,$$

$$M_{2t-1} = \sum_{k=0}^{t-1} \binom{2t-1-k}{k} \rho_1^{2t-1-2k} \rho_2^k \quad \forall t = 1, 2, \dots, \left\lceil \frac{T}{2} \right\rceil \quad (11)$$

where $\lceil x \rceil$ gives the smallest integer greater than or equals to x . For positive integers a and b ($b < a$), symbol $\binom{a}{b}$ means function a choose b .

Proposition 2 Under the AR(2) price model, (3) and (4), the execution cost function is given as follows:

$$\sum_{t=1}^T P_t \hat{S}_t = A' \hat{S} + \hat{S}' Q \hat{S} + E' \hat{S} + F' \hat{S} \quad (12)$$

where

$$A = P_0 \begin{pmatrix} L_1 \\ L_2 \\ L_3 \\ \vdots \\ \vdots \\ L_T \end{pmatrix} + u_2 P_{-1} \begin{pmatrix} L_0 \\ L_1 \\ L_2 \\ \vdots \\ \vdots \\ L_{N-1} \end{pmatrix}$$

$$\begin{aligned}
 & + \gamma X_1 \begin{pmatrix} L_0 M_0 \\ L_0 M_1 + L_1 M_0 \\ L_0 M_2 + L_1 M_1 + L_2 M_0 \\ \vdots \\ \vdots \\ \vdots \\ L_0 M_{T-1} + L_1 M_{T-2} + \dots + L_{T-1} M_0 \end{pmatrix} \\
 & + \gamma \rho_2 X_0 \begin{pmatrix} 0 \\ L_0 M_0 \\ L_0 M_1 + L_1 M_0 \\ \vdots \\ \vdots \\ \vdots \\ L_0 M_{T-2} + L_1 M_{T-3} + \dots + L_{T-2} M_0 \end{pmatrix}, \\
 E &= \begin{pmatrix} L_0 \epsilon_1 \\ L_0 \epsilon_2 + L_1 \epsilon_1 \\ L_0 \epsilon_3 + L_1 \epsilon_2 + L_2 \epsilon_1 \\ \vdots \\ \vdots \\ \vdots \\ L_0 \epsilon_T + L_1 \epsilon_{T-1} + \dots + L_{T-1} \epsilon_1 \end{pmatrix}, \\
 F &= \gamma \begin{pmatrix} 0 \\ (L_0 M_0) \eta_2 \\ (L_0 M_0) \eta_3 + \left(\sum_{l=0}^1 L_l M_{1-l} \right) \eta_2 \\ \vdots \\ \vdots \\ \vdots \\ (L_0 M_0) \eta_T + \left(\sum_{l=0}^1 L_l M_{1-l} \right) \eta_{T-1} + \dots + \left(\sum_{l=0}^{T-2} L_l M_{T-2-l} \right) \eta_2 \end{pmatrix} \\
 \hat{S} &= \begin{pmatrix} \hat{S}_1 \\ \hat{S}_2 \\ \hat{S}_3 \\ \vdots \\ \vdots \\ \vdots \\ \hat{S}_T \end{pmatrix} \text{ and}
 \end{aligned}$$

$$Q = \theta \begin{pmatrix} L_0 & \frac{1}{2}L_1 & \frac{1}{2}L_2 & \dots & \frac{1}{2}L_{T-2} & \frac{1}{2}L_{T-1} \\ \frac{1}{2}L_1 & L_0 & \frac{1}{2}L_1 & \dots & \frac{1}{2}L_{T-3} & \frac{1}{2}L_{T-2} \\ \frac{1}{2}L_2 & \frac{1}{2}L_1 & L_0 & \dots & \frac{1}{2}L_{T-4} & \frac{1}{2}L_{T-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{2}L_{T-1} & \frac{1}{2}L_{T-2} & \frac{1}{2}L_{T-3} & \dots & \frac{1}{2}L_1 & L_0 \end{pmatrix}.$$

Before giving the proof of Proposition 2, we present following three lemmas which are required to establish its proof. The proves of Lemmas 1, 2 and 3 are respectively presented in “Appendices 2–4” section.

Lemma 1 Consider L_t , M_t ; $t = 0, 1, \dots, T$ given by Eqs. (9), (10) and (11). Then for all $t = 1, 2, \dots, T - 1$, we have

$$u_1 L_t + u_2 L_{t-1} = L_{t+1}, \quad (13)$$

$$\rho_1 M_t + \rho_2 M_{t-1} = M_{t+1}. \quad (14)$$

Lemma 2 Under the dynamics of information factor X_t , given by Eq. (4), we have

$$X_t = X_1(M_{t-1}) + \rho_2 X_0(M_{t-2}) + \sum_{k=0}^{t-2} M_k \eta_{t-k} \quad \forall t = 2, 3, \dots, T. \quad (15)$$

Lemma 3 Under the AR(2) price model, (3) and (4), we have

$$P_1 = P_0(L_1) + u_2 P_{-1}(L_0) + \gamma X_1 + \theta \hat{S}_1 + \epsilon_1, \quad (16)$$

$$\begin{aligned} P_t &= P_0(L_t) + u_2 P_{-1}(L_{t-1}) + \gamma X_1 \left(\sum_{k=0}^{t-1} L_k M_{t-1-k} \right) + \gamma \rho_2 X_0 \left(\sum_{k=0}^{t-2} L_k M_{t-2-k} \right) \\ &\quad + \theta \left(\sum_{k=0}^{t-1} L_k \hat{S}_{t-k} \right) + \left(\sum_{k=0}^{t-1} L_k \epsilon_{t-k} \right) + \gamma \left\{ \sum_{k=0}^{t-2} \left(\sum_{l=0}^k L_l M_{k-l} \right) \eta_{t-k} \right\} \\ &\quad \forall t = 2, 3, \dots, T. \end{aligned} \quad (17)$$

Next we present the Proof of Proposition 2.

Proof of Proposition 2 The execution cost is

$$\begin{aligned} \sum_{t=1}^T P_t \hat{S}_t &= \sum_{t=1}^T P_t \hat{S}_t \\ &\quad \text{(By Eqs. (16) and (17) of Lemma 3)} \\ &= \left\{ P_0(L_1) + u_2 P_{-1}(L_0) + \gamma X_1 + \theta \hat{S}_1 + \epsilon_1 \right\} \hat{S}_1 \end{aligned}$$

$$\begin{aligned}
 & + \sum_{t=2}^T \left[P_0(L_t) + u_2 P_{-1}(L_{t-1}) + \gamma X_1 \left(\sum_{k=0}^{t-1} L_k M_{t-1-k} \right) \right. \\
 & + \gamma \rho_2 X_0 \left(\sum_{k=0}^{t-2} L_k M_{t-2-k} \right) + \theta \left(\sum_{k=0}^{t-1} L_k \hat{S}_{t-k} \right) + \left(\sum_{k=0}^{t-1} L_k \epsilon_{t-k} \right) \\
 & \left. + \gamma \left\{ \sum_{k=0}^{t-2} \left(\sum_{l=0}^k L_l M_{k-l} \right) \eta_{t-k} \right\} \right] \hat{S}_t \\
 & = P_0 \sum_{t=1}^T L_t \hat{S}_t + u_2 P_{-1} \sum_{t=1}^T L_{t-1} \hat{S}_t + \gamma X_1 \sum_{t=1}^T \left(\sum_{k=0}^{t-1} L_k M_{t-1-k} \right) \hat{S}_t \\
 & + \gamma \rho_2 X_0 \sum_{t=2}^T \left(\sum_{k=0}^{t-2} L_k M_{t-2-k} \right) \hat{S}_t + \theta \sum_{t=1}^T \left(\sum_{k=0}^{t-1} L_k \hat{S}_{t-k} \right) \hat{S}_t \\
 & + \sum_{t=1}^T \left(\sum_{k=0}^{t-1} L_k \epsilon_{t-k} \right) \hat{S}_t + \gamma \sum_{t=2}^T \left\{ \sum_{k=0}^{t-2} \left(\sum_{l=0}^k L_l M_{k-l} \right) \eta_{t-k} \right\} \hat{S}_t \\
 & = \mathbf{A}' \hat{\mathbf{S}} + \hat{\mathbf{S}}' \mathbf{Q} \hat{\mathbf{S}} + \mathbf{E}' \hat{\mathbf{S}} + \mathbf{F}' \hat{\mathbf{S}}.
 \end{aligned}$$

□

In the execution cost formula, given by Eq. (12), the matrices $\mathbf{E}$ and $\mathbf{F}$ are random matrices due to presence of vectors ϵ and η , respectively. Next we evaluate expectation of execution cost.

$$\begin{aligned}
 E_1 \left(\sum_{t=1}^T P_t \hat{S}_t \right) & = E_1 \left(\mathbf{A}' \hat{\mathbf{S}} + \hat{\mathbf{S}}' \mathbf{Q} \hat{\mathbf{S}} + \mathbf{E}' \hat{\mathbf{S}} + \mathbf{F}' \hat{\mathbf{S}} \right) \\
 & = \mathbf{A}' \hat{\mathbf{S}} + \hat{\mathbf{S}}' \mathbf{Q} \hat{\mathbf{S}} + E_1 \left(\mathbf{E}' \hat{\mathbf{S}} + \mathbf{F}' \hat{\mathbf{S}} \right) \\
 & = \mathbf{A}' \hat{\mathbf{S}} + \hat{\mathbf{S}}' \mathbf{Q} \hat{\mathbf{S}} + E_1 \sum_{t=1}^T \left[\sum_{k=0}^{t-1} L_k \epsilon_{t-k} \right] \hat{S}_t \\
 & \quad + \gamma E_1 \sum_{t=2}^T \left[\sum_{k=0}^{t-2} \left(\sum_{l=0}^k L_l M_{k-l} \right) \eta_{t-k} \right] \hat{S}_t \\
 & = \mathbf{A}' \hat{\mathbf{S}} + \hat{\mathbf{S}}' \mathbf{Q} \hat{\mathbf{S}} + \sum_{t=1}^T \left[\sum_{k=0}^{t-1} L_k E_1(\epsilon_{t-k}) \right] \hat{S}_t \\
 & \quad + \gamma \sum_{t=2}^T \sum_{k=0}^{t-2} \left(\sum_{l=0}^k L_l M_{k-l} \right) E_1(\eta_{t-k}) \hat{S}_t \\
 & \quad (\text{By } E_1(\epsilon_t) = 0 \forall t = 1, 2, \dots, T; \quad E_1(\eta_t) = 0 \\
 & \quad \forall t = 2, 3, \dots, T)
 \end{aligned}$$

$$= \mathbf{A}' \hat{\mathbf{S}} + \hat{\mathbf{S}}' \mathbf{Q} \hat{\mathbf{S}}. \quad (18)$$

By Eq. (18), the expected execution cost is a quadratic function in variables $(\hat{S}_1, \hat{S}_2, \dots, \hat{S}_T)$. Hence the problem (P2) can be re-formulated as the following quadratic programming problem:

(Mean QP problem)

$$\begin{aligned} & \min_{\hat{\mathbf{S}}} \mathbf{A}' \hat{\mathbf{S}} + \hat{\mathbf{S}}' \mathbf{Q} \hat{\mathbf{S}} \\ & \text{subject to} \\ & \quad \mathbf{e}' \hat{\mathbf{S}} = \bar{S}, \\ & \quad \hat{\mathbf{S}} \geq \mathbf{0}, \end{aligned}$$

where symbols $\mathbf{e}$ and $\mathbf{0}$ represent the unit and zero vectors of appropriate dimension, respectively.

4.2 Optimal static trading problem with variance as a measure for the timing risk

For a risk averse investor, the variance of execution cost is considered as a measure for the timing risk. Thus we tend to minimize variance of cost function $\left[\text{Var}_1 \left(\sum_{t=1}^T P_t \hat{S}_t \right) \right]$ also, apart from minimizing expected cost of execution $\left[E_1 \left(\sum_{t=1}^T P_t \hat{S}_t \right) \right]$. In the problem formulation, the Lagrange multiplier λ is used to convert the bi-objective optimization problem of minimizing expectation and variance of execution cost into single objective problem. Consider following problem:

$$\begin{aligned} & \text{(P3)} \\ & \min_{\{\hat{S}_t\}_{t=1}^T} E_1 \left(\sum_{t=1}^T P_t \hat{S}_t \right) + \lambda \text{Var}_1 \left(\sum_{t=1}^T P_t \hat{S}_t \right) \\ & \text{subject to} \\ & \quad \sum_{t=1}^T \hat{S}_t = \bar{S}, \\ & \quad \hat{S}_t \geq 0; \quad t = 1, 2, \dots, T. \end{aligned}$$

The symbol λ can be interpreted as the risk-aversion parameter. Higher value of parameter λ corresponds to high risk aversion. For a risk neutral investor λ is zero.

The variance of execution cost is given as the following quadratic function in variables $(\hat{S}_1, \hat{S}_2, \dots, \hat{S}_T)$.

$$\text{Var}_1 \left(\sum_{t=1}^T P_t \hat{S}_t \right) = \hat{\mathbf{S}}' \Sigma \hat{\mathbf{S}}, \quad (19)$$

where Σ is $T \times T$ variance–covariance matrix of the execution cost function with

$$\begin{aligned}\sum_{tt} &= \text{Var}_1(P_t) \quad \forall t = 1, 2, \dots, T, \\ \sum_{tk} &= \text{Cov}_1(P_t, P_k) \quad \forall t, k = 1, 2, \dots, T; t \neq k.\end{aligned}$$

Next, we present computation of $\sum$ matrix recursively.

Under the dynamics of information factor X_t , given by Eq. (4), we have

$$\text{Var}_1(X_2) = \text{Var}_1(\rho_1 X_1 + \rho_2 X_0 + \eta_2) = \text{Var}_1(\eta_2) = \sigma_\eta^2, \quad (20)$$

$$\begin{aligned}\text{Var}_1(X_3) &= \text{Var}_1(\rho_1 X_2 + \rho_2 X_1 + \eta_3) \\ &= \rho_1^2 \text{Var}_1(X_2) + \text{Var}_1(\eta_2) = \rho_1^2 \sigma_\eta^2 + \sigma_\eta^2.\end{aligned} \quad (21)$$

Here we make a note that for the first time period P_0 , P_{-1} , X_1 and X_0 are known quantities.

For $t = 4, 5, \dots, T$, we have

$$\begin{aligned}\text{Var}_1(X_t) &= \text{Var}_1(\rho_1 X_{t-1} + \rho_2 X_{t-2} + \eta_t) \\ &= \rho_1^2 \text{Var}_1(X_{t-1}) + \rho_2^2 \text{Var}_1(X_{t-2}) + \sigma_\eta^2 \\ &\quad + 2\rho_1 \rho_2 \text{Cov}_1(X_{t-1}, X_{t-2}),\end{aligned} \quad (22)$$

where $\text{Cov}_1(X_{t-1}, X_{t-2})$ is given by the following Lemma 4.

Lemma 4 Under the dynamics of information factor, given by Eq. (4), we have $\forall t = 3, 4, \dots, T$,

$$\begin{aligned}\text{Cov}_1(X_t, X_{t-1}) &= \rho_1 [\text{Var}_1(X_{t-1}) + \rho_2 \text{Var}_1(X_{t-2}) + \rho_2^2 \text{Var}_1(X_{t-3}) + \dots \\ &\quad + \rho_2^{t-3} \text{Var}_1(X_2)].\end{aligned} \quad (23)$$

Proof Proof of Lemma 4 is presented in “Appendix 5” section. $\square$

Furthermore, in the following Lemma 5, we evaluate covariances $\text{Cov}_1(X_t, X_{t-k})$; $k = 2, 3, \dots, t-2$; $t = 4, 5, \dots, T$, in terms of $\text{Var}(X_{t-k})$ and $\text{Cov}(X_{t-k+1}, X_{t-k})$, which are given by Eq. (22) above, and the Eq. (23) of Lemma 4, respectively.

Lemma 5 Under the dynamics of information factor, given by Eq. (4), we have for all $k = 2, 3, \dots, t-2$; $t = 4, 5, \dots, T$,

$$\text{Cov}_1(X_t, X_{t-k}) = M_{k-1} \text{Cov}_1(X_{t-k+1}, X_{t-k}) + \rho_2 M_{k-2} \text{Var}_1(X_{t-k}). \quad (24)$$

Proof Proof of Lemma 5 is presented in “Appendix 6” section. $\square$

Under the execution price dynamics P_t , given by Eq. (3), we have

$$\begin{aligned}\text{Var}_1(P_1) &= \text{Var}_1(u_1 P_0 + u_2 P_{-1} + \theta \hat{S}_1 + \gamma X_1 + \epsilon_1) \\ &= \text{Var}_1(\epsilon_1) \\ &= \sigma_\epsilon^2,\end{aligned} \quad (25)$$

$$\begin{aligned}
 \text{Var}_1(P_2) &= \text{Var}_1(u_1 P_1 + u_2 P_0 + \theta \hat{S}_2 + \gamma X_2 + \epsilon_2) \\
 &= u_1^2 \text{Var}_1(P_1) + \gamma^2 \text{Var}_1(X_2) + \text{Var}_1(\epsilon_2) \\
 &= u_1^2 \sigma_\epsilon^2 + \gamma^2 \sigma_\eta^2 + \sigma_\epsilon^2.
 \end{aligned} \tag{26}$$

For $t = 3, 4, \dots, T$,

$$\begin{aligned}
 \text{Var}_1(P_t) &= \text{Var}_1(u_1 P_{t-1} + u_2 P_{t-2} + \theta \hat{S}_t + \gamma X_t + \epsilon_t) \\
 &= u_1^2 \text{Var}_1(P_{t-1}) + u_2^2 \text{Var}_1(P_{t-2}) + \gamma^2 \text{Var}_1(X_t) + \sigma_\epsilon^2 \\
 &\quad + 2u_1 u_2 \text{Cov}_1(P_{t-1}, P_{t-2}) + 2u_1 \gamma \text{Cov}_1(P_{t-1}, X_t) \\
 &\quad + 2u_2 \gamma \text{Cov}_1(P_{t-2}, X_t),
 \end{aligned} \tag{27}$$

where $\text{Var}_1(X_t)$ is given by Eq. (22). The covariances $\text{Cov}_1(P_{t-1}, X_t)$ and $\text{Cov}_1(P_{t-2}, X_t)$ are given by Eqs. (28) and (29), and Eq. (30) of the Lemma 6. Further, the calculation of covariances $\text{Cov}_1(P_{t-1}, P_{t-2})$ are presented in Eq. (31) of Lemma 7.

$$\text{Cov}_1(P_1, X_t) = 0 \quad \forall t = 1, 2, \dots, T, \tag{28}$$

$$\text{Cov}_1(P_t, X_1) = 0 \quad \forall t = 1, 2, \dots, T, \tag{29}$$

Lemma 6 Under the AR(2) price model, (3) and (4), we have for all $k = 2, 3, \dots, T$; $t = 2, 3, \dots, T$,

$$\begin{aligned}
 \text{Cov}_1(P_t, X_k) &= \gamma [L_0 \text{Cov}_1(X_t, X_k) + L_1 \text{Cov}_1(X_{t-1}, X_k) + \dots \\
 &\quad + L_{t-2} \text{Cov}_1(X_2, X_k)].
 \end{aligned} \tag{30}$$

Proof Proof of Lemma 6 is presented in “Appendix 7” section. $\square$

Lemma 7 Under the AR(2) price model, (3) and (4), we have for all $t = 2, 3, \dots, T$,

$$\begin{aligned}
 \text{Cov}_1(P_t, P_{t-1}) &= u_1 [\text{Var}_1(P_{t-1}) + u_2 \text{Var}_1(P_{t-2}) + u_2^2 \text{Var}_1(P_{t-3}) + \dots \\
 &\quad + u_2^{t-2} \text{Var}_1(P_1)] + \gamma [\text{Cov}_1(P_{t-1}, X_t) + u_2 \text{Cov}_1(P_{t-2}, X_{t-1}) \\
 &\quad + \dots + u_2^{t-2} \text{Cov}_1(P_1, X_2)].
 \end{aligned} \tag{31}$$

Proof Proof of Lemma 7 is presented in “Appendix 8” section. $\square$

Furthermore, in the following Lemma 8, we evaluate covariances $\text{Cov}_1(P_t, P_{t-k})$; $k = 2, 3, \dots, t-1$, $t = 3, 4, \dots, T$ in terms of $\text{Var}_1(P_{t-k})$, $\text{Cov}_1(P_{t-k}, X_{t-r})$; $r = 0, 1, \dots, k-2$, and $\text{Cov}_1(P_{t-k+1}, P_{t-k})$ which are respectively given by Eqs. (27), (30) and (31).

Lemma 8 Under the AR(2) price model, (3) and (4), we have for all $k = 2, 3, \dots, t-1$; $t = 3, 4, \dots, T$,

$$\text{Cov}_1(P_t, P_{t-k}) = L_{k-1} \text{Cov}_1(P_{t-k+1}, P_{t-k}) + u_2 L_{k-2} \text{Var}_1(P_{t-k})$$

$$\begin{aligned}
 & +\gamma[L_0Cov_1(P_{t-k}, X_t) + L_1Cov_1(P_{t-k}, X_{t-1}) + \dots \\
 & +L_{k-2}Cov_1(P_{t-k}, X_{t-k+2})]
 \end{aligned}
 \tag{32}$$

Proof Proof of Lemma 8 is presented in “Appendix 9” section. $\square$

The diagonal entries $\sum_{tt} \left[= Var_1(P_t) \right]; t = 1, 2, \dots, T$, of the variance-covariance matrix $\sum$, are recursively given by Eqs. (25), (26) and (27). Whereas, the off-diagonal entries $\sum_{tk} \left[= Cov_1(P_t, P_k) \right]; t, k = 1, 2, \dots, T; t \neq k$, are given by Eqs. (31) and (32).

By quadratic formulation of expected execution cost and the variance of execution cost, given by Eqs. (18) and (19) respectively, the problem (P3) is converted into following quadratic programming problem.

(Mean-variance QP problem)

$$\begin{aligned}
 & \min_{\hat{S}} A' \hat{S} + \hat{S}' Q \hat{S} + \lambda \hat{S}' \sum \hat{S} \\
 & \text{subject to} \\
 & \quad e' \hat{S} = \bar{S}, \\
 & \quad \hat{S} \geq 0.
 \end{aligned}$$

4.3 Optimal static trading problem with stochastic dominance constraints

In this section, first the basic definition of second order stochastic dominance (SSD) in context of any general distribution is given. This will be followed by stating some relations which are equivalent to SSD. Thereafter, the optimal static trading problem subject to SSD-constraints is presented.

Let X and Y be two random variables with F_X and F_Y as their respective cumulative distribution functions. Then X is said to dominate Y in SSD sense, denoted by $X \succeq_{SSD} Y$ iff

$$\int_{-\infty}^a F_X(t) dt \leq \int_{-\infty}^a F_Y(t) dt \quad \forall a \in \mathbb{R}.$$

Whitmore and Findlay (1978) have proved the following three equivalent statements in context of SSD.

1. $\int_{-\infty}^a F_X(t) dt \leq \int_{-\infty}^a F_Y(t) dt \quad \forall a \in \mathbb{R}$.
2. $E([a - X]^+) \leq E([a - Y]^+) \quad \forall a \in \mathbb{R}$ where $x^+ = \max\{x, 0\}$.
3. $E[u(X)] \geq E[u(Y)]$, for any non-decreasing concave function u .

Consider the special case where the two random variables X and Y are discrete with M equally probable outcomes. Let $(\alpha_1, \alpha_2, \dots, \alpha_M)$ and $(\beta_1, \beta_2, \dots, \beta_M)$ are the vectors of values taken by X and Y , respectively, each with probability $\frac{1}{M}$. Without loss of generality, we can assume

$$\alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_M \quad \text{and} \quad \beta_1 \leq \beta_2 \leq \dots \leq \beta_M.
 \tag{33}$$

By Roman et al. (2006), we have

$$X \succeq_{SSD} Y \quad \text{iff} \quad \frac{1}{i} \sum_{j=1}^i \alpha_j \geq \frac{1}{i} \sum_{j=1}^i \beta_j, \quad \forall i = 1, 2, \dots, M, \quad (34)$$

with at least one strict inequality.

We name the values $\frac{1}{i} \sum_{j=1}^i \alpha_j$; $i = 1, 2, \dots, M$, corresponding to random variable X as the ‘Left tail average realizations’ of X .

In the optimal trading problem an investor, who wants to buy some fixed number of shares of an asset, tends to minimize the total cost of execution. Thus, in context of stochastic dominance the investor would prefer the trading strategy for which the execution cost function is being dominated by the execution cost function of some reference strategy. In this paper we have taken reference strategy as the optimal static trading strategy of the Mean-variance QP problem for some fixed λ , say $\hat{S}^*$. Let $C_{\hat{S}} \left(\sum_{t=1}^T P_t \hat{S}_t \right)$ and $C_{\hat{S}^*} \left(\sum_{t=1}^T P_t \hat{S}_t^* \right)$ be the execution cost corresponding to strategies $\hat{S}$ and $\hat{S}^*$, respectively. We require constraint that $C_{\hat{S}}$ should be dominated by $C_{\hat{S}^*}$ in SSD sense. That is

$$C_{\hat{S}^*} \succeq_{SSD} C_{\hat{S}}. \quad (35)$$

The optimal trading problem of minimizing expected execution cost subject to SSD-constraint (35) and the non-negativity constraints is as follows:

$$\begin{aligned} & \text{(P4)} \\ & \min_{\hat{S}} A' \hat{S} + \hat{S}' Q \hat{S} \\ & \text{subject to} \\ & e' \hat{S} = \bar{S}, \\ & C_{\hat{S}^*} \succeq_{SSD} C_{\hat{S}}, \\ & \hat{S} \geq \mathbf{0}. \end{aligned}$$

Next, we make an attempt to further simplify constraint $C_{\hat{S}^*} \succeq_{SSD} C_{\hat{S}}$ of problem (P4).

Consider M number of simulations of random processes ϵ and η of Eqs. (3) and (4), respectively. That is consider M realizations of ϵ and η as $(\epsilon^i$; $i = 1, 2, \dots, M)$ and $(\eta^i$; $i = 1, 2, \dots, M)$, respectively. We have $\epsilon^i = \{\epsilon_1^i, \epsilon_2^i, \dots, \epsilon_T^i\}$ and $\eta^i = \{\eta_1^i, \eta_2^i, \dots, \eta_T^i\}$. Let E^i and F^i be the i th realization of vectors E and F respectively, present in the execution cost given by Eq. (12), which respectively correspond to i th simulation ϵ^i and η^i .

Thus for M number of simulations of ϵ and η , by Eq. (12) we have corresponding M number of simulations of $C_{\hat{S}}$ and $C_{\hat{S}^*}$ as follows:

$$C_{\hat{S}}^i = A' \hat{S} + \hat{S}' Q \hat{S} + E^{i'} \hat{S} + F^{i'} \hat{S}, \quad i = 1, 2, \dots, M, \quad (36)$$

$$C_{\hat{S}^*}^i = A' \hat{S}^* + \hat{S}^{*'} Q \hat{S}^* + E^{i'} \hat{S}^* + F^{i'} \hat{S}^*, \quad i = 1, 2, \dots, M. \quad (37)$$

By the equivalent definition of SSD, we have

$$C_{\hat{S}}^* \succeq_{SSD} C_{\hat{S}} \quad \text{iff} \quad E_1 \left(\left[a - C_{\hat{S}}^* \right]^+ \right) \leq E_1 \left(\left[a - C_{\hat{S}} \right]^+ \right) \quad \forall a \in \mathbb{R}. \quad (38)$$

Further by [Dentcheva and Ruszczyński \(2003\)](#), if $\{C_{\hat{S}}^i; i = 1, 2, \dots, M\}$ is the set of realizations of $C_{\hat{S}}$, we have following equivalence.

$$\begin{aligned} E_1 \left(\left[a - C_{\hat{S}}^* \right]^+ \right) \leq E_1 \left(\left[a - C_{\hat{S}} \right]^+ \right) \quad \forall a \in \mathbb{R} \quad \text{iff} \quad E_1 \left(\left[C_{\hat{S}}^i - C_{\hat{S}}^* \right]^+ \right) \\ \leq E_1 \left(\left[C_{\hat{S}}^i - C_{\hat{S}} \right]^+ \right) \quad \forall i = 1, 2, \dots, M. \end{aligned} \quad (39)$$

Consider following approximates for the expectations:

$$E_1 \left(\left[C_{\hat{S}}^i - C_{\hat{S}}^* \right]^+ \right) \approx \frac{1}{M} \sum_{j=1}^M \left[C_{\hat{S}}^i - C_{\hat{S}}^j \right]^+ \quad (40)$$

$$E_1 \left(\left[C_{\hat{S}}^i - C_{\hat{S}} \right]^+ \right) \approx \frac{1}{M} \sum_{j=1}^M \left[C_{\hat{S}}^i - C_{\hat{S}}^j \right]^+. \quad (41)$$

From Eqs. (38)–(41), we have

$$C_{\hat{S}}^* \succeq_{SSD} C_{\hat{S}} \quad \text{iff} \quad \frac{1}{M} \sum_{j=1}^M \left[C_{\hat{S}}^i - C_{\hat{S}}^j \right]^+ \leq \frac{1}{M} \sum_{j=1}^M \left[C_{\hat{S}}^i - C_{\hat{S}}^j \right]^+ \quad \forall i = 1, 2, \dots, M. \quad (42)$$

To further simplify the constraint $C_{\hat{S}}^* \succeq_{SSD} C_{\hat{S}}$, let us consider $\forall i, j = 1, 2, \dots, M$

$$w_{ij} = \left[C_{\hat{S}}^i - C_{\hat{S}}^j \right]^+ \quad \text{and} \quad v_{ij} = \left[C_{\hat{S}}^i - C_{\hat{S}}^j \right]^+.$$

By Eq. (42), we get

$$C_{\hat{S}}^* \succeq_{SSD} C_{\hat{S}} \quad \text{iff} \quad \frac{1}{M} \sum_{j=1}^M w_{ij} \leq \frac{1}{M} \sum_{j=1}^M v_{ij}. \quad (43)$$

Further, by definition $x^+ = \max\{x, 0\}$, we get

$$w_{ij} = \max \left\{ C_{\hat{S}}^i - C_{\hat{S}}^j, 0 \right\} \quad \text{and} \quad v_{ij} = \max \left\{ C_{\hat{S}}^i - C_{\hat{S}}^j, 0 \right\}.$$

Therefore,

$$u_{ij} \geq 0, \quad u_{ij} \geq C_{\hat{S}}^i - C_{\hat{S}}^j, \quad v_{ij} \geq 0, \quad v_{ij} \geq C_{\hat{S}}^i - C_{\hat{S}}^j \quad \forall i, j = 1, 2, \dots, M. \quad (44)$$

From Eq. (36), we get

$$C_{\hat{S}}^i - C_{\hat{S}^*}^j = A' \hat{S} + \hat{S}' Q \hat{S} + E^{i'} \hat{S} + F^{i'} \hat{S} - C_{\hat{S}^*}^j \quad \forall i, j = 1, 2, \dots, M, \quad (45)$$

$$C_{\hat{S}}^i - C_{\hat{S}}^j = (E^i - E^j)' \hat{S} + (F^i - F^j)' \hat{S} \quad \forall i, j = 1, 2, \dots, M. \quad (46)$$

In view of (43)–(46), the problem (P4) can be re-written as follows:

(QCQP problem)

$$\begin{aligned} & \min_{\hat{S}} \quad A' \hat{S} + \hat{S}' Q \hat{S} \\ & \text{subject to} \\ & \quad e' \hat{S} = \bar{S}, \\ & \quad \hat{S} \geq \mathbf{0}, \\ & \quad \frac{1}{M} \sum_{j=1}^M w_{ij} \leq \frac{1}{M} \sum_{j=1}^M v_{ij}; \quad \forall i = 1, 2, \dots, M, \\ & \quad w_{ij} \geq A' \hat{S} + \hat{S}' Q \hat{S} + E^{i'} \hat{S} + F^{i'} \hat{S} - C_{\hat{S}^*}^j; \quad \forall i, j = 1, 2, \dots, M, \\ & \quad v_{ij} \geq (E^i - E^j)' \hat{S} + (F^i - F^j)' \hat{S}; \quad \forall i, j = 1, 2, \dots, M, \\ & \quad w_{ij} \geq 0; \quad \forall i, j = 1, 2, \dots, M, \\ & \quad v_{ij} \geq 0; \quad \forall i, j = 1, 2, \dots, M. \end{aligned}$$

For known strategy $\hat{S}^*$ the corresponding realization of execution cost $C_{\hat{S}^*}^j$ is known. Thus, in the QCQP problem following constraints are quadratic in decision variables $\hat{S}$.

$$w_{ij} \geq A' \hat{S} + \hat{S}' Q \hat{S} + E^{i'} \hat{S} + F^{i'} \hat{S} - C_{\hat{S}^*}^j \quad \forall i, j = 1, 2, \dots, M.$$

Also the objective function of QCQP problem is a quadratic function in $\hat{S}$. Thus, the QCQP problem is a quadratically constraint quadratic programming problem and hence the name QCQP problem. Conic optimization, semi-definite programming, etc. are some areas in which the QCQP class of problems are well studied and thus have all nice properties associated with the class. However, in context of optimal trading problem the QCQP problem is considered for the first time in the present paper.

5 Numerical illustrations

In this section, first the algorithm based on SA method to solve optimal static trading problems—Mean QP, Mean–variance QP and QCQP is presented. This is followed by the computational experiments conducted on simulated and the real market data drawn from National Stock Exchange of India (NSE India) database.

5.1 Implementation of static approximation method using roll-out heuristic approach

In Mean QP, Mean–variance QP and QCQP problems $\{\hat{S}_t, t = 1, 2, \dots, T\}$ are decision variables, whereas $\{W_t, t = 1, 2, \dots, T\}$ are internal variables which depend upon $\{\hat{S}_t, t = 1, 2, \dots, T\}$. Further, the variables $\{X_t, t = 1, 2, \dots, T\}$, in execution price dynamics P_t (Eq. 3), are external state variables which change due to their own natural dynamics given by (4). Because of the presence of external variables in the proposed problems, the optimal trading strategies can not be determined in advance of trading. One requires to revise these in each time period according to the new values taken by external variables (Bertsimas et al. 1999). Bertsimas et al. (1999) mentioned SA method which revises the trading strategy in each time period with new realization of external variables. Later on Khemchandani et al. (2013) used the SA method for solving their formulation of optimal trading problem.

We detail the algorithm to solve Mean QP, Mean–variance QP and QCQP optimal static trading problems by SA method. (The algorithm is provided in context of QCQP problem. For Mean QP and Mean–variance QP problems same steps will be followed.)

Step 1 Formulate the QCQP problem corresponding to the first time period with decision variables $\{\hat{S}_1, \hat{S}_2, \dots, \hat{S}_T\}$, initial prices P_0, P_{-1} , known external state variables X_1 and X_0 , and $\bar{S}$ as the total no. of shares to be traded at the beginning. Name the problem as QCQP(1) and solve it by standard optimization method to obtain the optimal static trading strategy as $\{\hat{S}_1^{(1)}, \hat{S}_2^{(1)}, \dots, \hat{S}_T^{(1)}\}$.

Step 2 Execute $\hat{S}_1^{(1)}$ in the first time period. Calculate price P_1 with help of $\hat{S}_1^{(1)}$ and ϵ_1 . Formulate problem QCQP(2) corresponding to the second time period with decision variables $\{\hat{S}_2, \hat{S}_3, \dots, \hat{S}_T\}$, prices P_1, P_0 , external variables X_2, X_1 and $W_1 - \hat{S}_1^{(1)}$ as the total no. of shares remained to be traded after the first time period. Name the optimal solution of QCQP(2) as $\{\hat{S}_2^{(2)}, \hat{S}_3^{(2)}, \dots, \hat{S}_T^{(2)}\}$.

Step 3 Execute $\hat{S}_2^{(2)}$ in the second time period. Calculate price P_2 with help of $\hat{S}_2^{(2)}$ and ϵ_2 . Formulate problem QCQP(3) corresponding to the third time period with decision variables $\{\hat{S}_3, \hat{S}_4, \dots, \hat{S}_T\}$, prices P_2, P_1 , updated external variables X_3, X_2 and $W_2 - \hat{S}_2^{(2)}$ as the total no. of shares remained to be traded after the second time period. Let the optimal solution of QCQP(3) be $\{\hat{S}_3^{(3)}, \hat{S}_4^{(3)}, \dots, \hat{S}_T^{(3)}\}$.

Step 4 Repeat above steps for all the remaining time periods successively till we reach the last time period where optimal static trading strategy is nothing but to trade all the remaining shares $W_T (= W_{T-1} - \hat{S}_{T-1}^{T-1} = \hat{S}_T^{(T)})$ in the last time period.

With above Steps 1–4, we solve a sequence of T quadratically constrained quadratic programming problems QCQP(t), $t = 1, 2, \dots, T$, to obtain final optimal static trading strategy as $\{\hat{S}_1^{(1)}, \hat{S}_2^{(2)}, \dots, \hat{S}_T^{(T)}\}$. Similarly to obtain optimal static trading strategies for Mean QP and Mean–variance QP problems, we solve a sequence of T quadratic programming problems QP(t), $t = 1, 2, \dots, T$ for both the problems to obtain corresponding trading strategy. In this paper the MOSEK solver with MATLAB software is used to solve sequence of optimization problems QCQP(t) and QP(t).

Table 1 Optimal cost of execution corresponding to Mean QP problem for different combination of the parameters γ , P_0 , P_{-1} , u_1 , u_2 , ρ_1 and ρ_2

γ	P_0 (Rs.)	P_{-1} (Rs.)	Optimal cost of execution (Rs.)							
			$\rho_1 = .25$	$\rho_2 = .25$	$\rho_1 = .25$	$\rho_2 = .25$	$\rho_1 = .5$	$\rho_2 = 0$		
			$u_1 = .5$	$u_2 = .5$	$u_1 = .75$	$u_2 = .25$	$u_1 = 1$	$u_2 = 0$		
0	60	50	581602.5		600540.9		625652.6			
	55	55	567651.7		570916.9		575652.6			
	50	60	550184.6		540919.7		525652.6			
5	60	50	580979.0		599683.2		624511.2			
	55	55	566830.6		570005.8		574511.4			
	50	60	549402.8		539982.7		524511.2			
20	60	50	579113.3		597116.1		621192.6			
	55	55	564561.5		567277.2		571198.0			
	50	60	547059.8		537177.3		521194.0			
100	60	50	569099.8		583691.5		605556.9			
	55	55	552194.9		552957.3		555548.7			
	50	60	535078.6		522488.0		505548.9			

Other parameters are considered as $T = 20$, $\bar{S} = 10000$, $\sigma_\epsilon = 0.125$, $\sigma_\eta = 0.316$, $\theta = 5 \times 10^{-4}$, $X_1 = 0$ and $X_0 = 0$.

5.2 Simulated data

The numerical analysis is conducted on the simulated data. The realizations of vectors ϵ and η are obtained by using Carlosimulation method. With these realizations the Mean QP, Mean–variance QP and QCQP problems are solved by SA method detailed in Sect. 5.1.

Table 1 lists the optimal cost of execution corresponding to Mean QP problem for different combinations of the parameters of AR(2) price model. The four row-subpanels correspond to four different values of γ with each subpanel is further divided into the three rows according to criteria $P_0 > P_{-1}$, $P_0 = P_{-1}$ and $P_0 < P_{-1}$. Considering one row-subpanel for some value of γ , like $\gamma = 5$, the optimal cost of execution decreases as we move down in the column in accordance to $P_0 > P_{-1}$, $P_0 = P_{-1}$ and $P_0 < P_{-1}$. The conditions $P_0 > P_{-1}$, $P_0 = P_{-1}$ and $P_0 < P_{-1}$ respectively show bullish, stable and bearish market scenarios. These market scenarios correspond to expectation of rise in prices, no change in prices and decline in prices in future, and thus give optimal cost of execution high, medium and low, respectively.

Moreover in Table 1, for any fixed value of γ , P_0 and P_{-1} the optimal execution cost corresponding to Mean QP problem is obtained for different combinations of the values of u_1 , u_2 , ρ_1 and ρ_2 . The last column correspond to AR(1) price model as $u_2 = 0$ and $\rho_2 = 0$. Owing to bullish market behavior, the optimal execution cost increases as we move from fourth to sixth column in the case of $P_0 > P_{-1}$. That is AR(2) price model gives less cost of execution as compared to AR(1). Similarly for the case of $P_0 < P_{-1}$, the optimal execution cost decreases from fourth to sixth column.

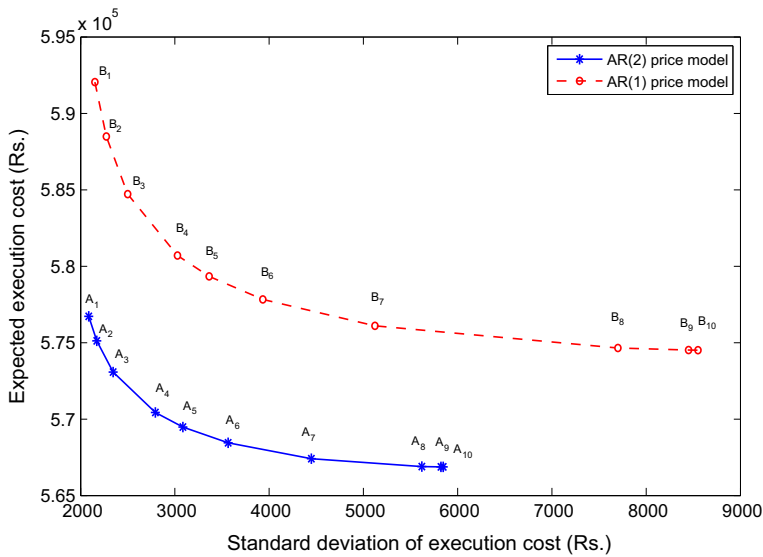


Fig. 1 The efficient trading frontiers corresponding to Mean–variance QP problem for AR(2) price model ($u_1 = .5$, $u_2 = .5$, $\rho_1 = .25$, $\rho_2 = .25$) and AR(1) price model ($u_1 = 1$, $u_2 = 0$, $\rho_1 = .5$, $\rho_2 = 0$). The parameters considered are $T = 20$, $\bar{S} = 10,000$, $\sigma_\epsilon = 0.125$, $\sigma_\eta = 0.316$, $\theta = 5 \times 10^{-4}$, $\gamma = 5$, $P_0 = \text{Rs. } 55$, $P_{-1} = \text{Rs. } 55$, $X_1 = 0$ and $X_0 = 0$. The points A_i , B_i ; $i = 1, 2, \dots, 10$ correspond respectively to decreasing values of risk aversion parameter λ as $\lambda = 10^{-2}$, 6×10^{-3} , 3×10^{-3} , 10^{-3} , 6×10^{-4} , 3×10^{-4} , 10^{-4} , 10^{-5} , 10^{-6} , 0

In case of stable market $P_0 = P_{-1}$, the optimal execution cost is less for AR(2) price model as compared to AR(1) price model.

Furthermore, we observe from Table 1 that for any fixed combination of values of P_0 , P_{-1} , u_1 and u_2 , the optimal cost of execution decreases with increase in γ . This is due to the fact that an increase in γ increases the effect of information factor in the execution price. As a result the optimal static trading strategy is more information driven and leads to low cost of execution. This result is also demonstrated in Bertsimas and Lo (1998) for the AR(1) price model.

Kissell and Glantz (2007) introduced the concept of efficient trading frontier in context of optimal trading problem. Figure 1 compares the efficient trading frontiers corresponding to Mean–variance QP problem under the AR(2) and AR(1) price models. For both the price models the Mean–variance QP problem is solved for different values of risk aversion parameter λ . From the formulation of Mean–variance QP problem it is evident that the high values of λ give optimal static trading strategies which have less variance (thus less standard deviation) and more execution cost. Hence in Fig. 1 the highest point on both the curves correspond to the largest value of $\lambda = 10^{-2}$. It is evident from Fig. 1 that for any value of λ the optimal trading strategy under the AR(2) price model has less execution cost and variance as compared to those under the AR(1) model. Thus AR(2) price model is always preferable over AR(1) model.

Table 2 gives the optimal static trading strategy $\hat{S}$ obtained by solving QCQP problem with reference strategy as $\hat{S}^*$ which is the optimal static trading strategy of

Table 2 The comparison of optimal static trading strategy $\hat{S}$ of QCQP problem with reference strategy $\hat{S}^*$ (optimal trading strategy of Mean–variance QP problem with $\lambda = 10^{-5}$)

Time period	$\hat{S}$	$\hat{S}^*$	Time period	$\hat{S}$	$\hat{S}^*$
1	465.1	553.3	12	491.3	473.7
2	465.8	529.6	13	496.8	489.2
3	465.1	504.8	14	503.2	491.5
4	468.6	520.5	15	512.3	495.9
5	470.3	503.9	16	520.9	497.6
6	472.0	476.9	17	537.7	501.3
7	474.1	481.0	18	544.0	504.1
8	476.8	492.0	19	599.4	542.6
9	479.8	493.6	20	592.5	532.8
10	482.2	461.5	Expected execution cost (Rs.)	566873.5	566892.9
11	482.2	454.2			

The parameters considered are $T = 20$, $\bar{S} = 10000$, $\sigma_\epsilon = \text{Rs. } 0.125$, $\sigma_\eta = 0.316$, $\theta = 5 \times 10^{-4}$, $\gamma = 5$, $u_1 = .5$, $u_2 = .5$, $\rho_1 = .25$, $\rho_2 = .25$, $P_0 = \text{Rs. } 55$, $P_{-1} = \text{Rs. } 55$, $X_1 = 0$ and $X_0 = 0$.

Mean–variance QP problem for $\lambda = 10^{-5}$. The optimal cost of execution corresponding to $\hat{S}$ and $\hat{S}^*$ are also listed. It is evident that the QCQP problem results in the trading strategy $\hat{S}$ which has less expected execution cost as compared to that of the Mean–variance efficient strategy $\hat{S}^*$. Moreover as shown in Fig. 2, due to equivalence relation of SSD given by (34), the strategy $\hat{S}$ has less ‘Left tail average realization of execution cost’ $\left(\frac{1}{i} \sum_{j=1}^i C_{\hat{S}}^j; i = 1, 2, \dots, 25\right)$ as compared to that of $\hat{S}^*$ $\left(\frac{1}{i} \sum_{j=1}^i C_{\hat{S}^*}^j; i = 1, 2, \dots, 25\right)$ corresponding to 25 realization points. The realization of execution costs $C_{\hat{S}}^j$ and $C_{\hat{S}^*}^j$ depending upon the realizations of vectors ϵ and η are obtained from the Eqs. (36) and (37), respectively.

5.3 Real market data

The numerical analysis is conducted on the real market data drawn from NSE India database. We assume that for the trading of Reliance Industries Ltd. stock, its execution price can be modeled by the AR(2) price model. Bertsimas and Lo (1998) mentioned that the return on indices is a common component in the prices of the most equities. Since Reliance Industries Ltd. stock is one of the fifty stocks of Nifty 50 index, for its trading we consider Nifty 50 index’s daily returns as the information factor. With this assumption the parameters u_1 , u_2 , ρ_1 , ρ_2 , σ_ϵ and σ_η of the AR(2) price model are estimated using least squares estimation method.

The parameters u_1 and u_2 of AR(2) price model are estimated so that the sum of squares of error term ϵ_k plus the impact factor $\theta \hat{S}_k + \gamma X_k$ is minimum. Let M be the

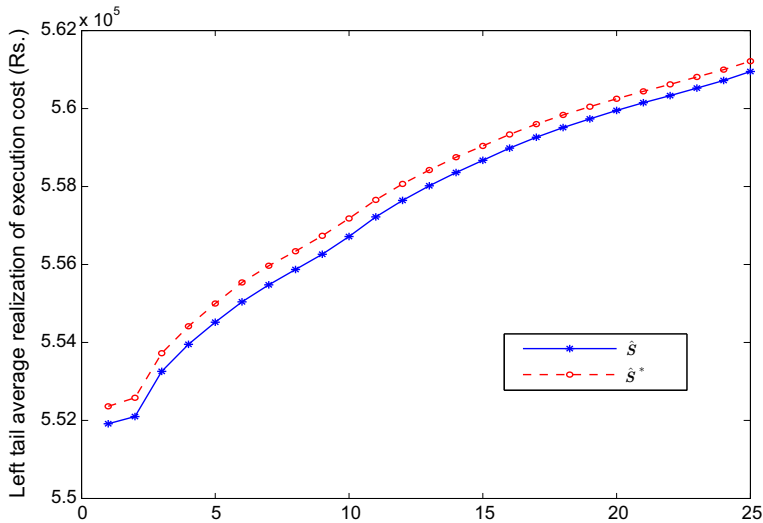


Fig. 2 The comparison of left tail average realization of execution cost of optimal static trading strategy $\hat{S}$ of QCQP problem with reference strategy $\hat{S}^*$. The strategies $\hat{S}$ and $\hat{S}^*$ are listed in Table 2

number of historical data values. Thus, we consider following system of equations.

$$\frac{\partial}{\partial u_1} \sum_{k=1}^M (P_k - u_1 P_{k-1} - u_2 P_{k-2})^2 = 0, \quad \frac{\partial}{\partial u_2} \sum_{k=1}^M (P_k - u_1 P_{k-1} - u_2 P_{k-2})^2 = 0. \quad (47)$$

Solving (47), following estimates for u_1 and u_2 are obtained.

$$\hat{u}_1 = \frac{\left(\sum_{k=1}^M P_k P_{k-2} \right) \left(\sum_{k=1}^M P_{k-1} P_{k-2} \right) - \left(\sum_{k=1}^M P_k P_{k-1} \right) \left(\sum_{k=1}^M P_{k-2}^2 \right)}{\left(\sum_{k=1}^M P_{k-1} P_{k-2} \right)^2 - \left(\sum_{k=1}^M P_{k-1}^2 \right) \left(\sum_{k=1}^M P_{k-2}^2 \right)}, \quad (48)$$

$$\hat{u}_2 = \frac{\left(\sum_{k=1}^M P_k P_{k-1} \right) \left(\sum_{k=1}^M P_{k-1} P_{k-2} \right) - \left(\sum_{k=1}^M P_k P_{k-2} \right) \left(\sum_{k=1}^M P_{k-1}^2 \right)}{\left(\sum_{k=1}^M P_{k-1} P_{k-2} \right)^2 - \left(\sum_{k=1}^M P_{k-1}^2 \right) \left(\sum_{k=1}^M P_{k-2}^2 \right)}. \quad (49)$$

The parameters ρ_1 and ρ_2 of the information factor X_k are estimated so that the sum of squares of error terms η_k for historical data values is minimum. Thus, we have

$$\frac{\partial}{\partial \rho_1} \sum_{k=1}^M (X_k - \rho_1 X_{k-1} - \rho_2 X_{k-2})^2 = 0, \quad \frac{\partial}{\partial \rho_2} \sum_{k=1}^M (X_k - \rho_1 X_{k-1} - \rho_2 X_{k-2})^2 = 0. \quad (50)$$

Solving system of Eq. (50), following estimates for ρ_1 and ρ_2 are obtained.

$$\hat{\rho}_1 = \frac{\left(\sum_{k=1}^M X_k X_{k-2}\right)\left(\sum_{k=1}^M X_{k-1} X_{k-2}\right) - \left(\sum_{k=1}^M X_k X_{k-1}\right)\left(\sum_{k=1}^M X_{k-2}^2\right)}{\left(\sum_{k=1}^M X_{k-1} X_{k-2}\right)^2 - \left(\sum_{k=1}^M X_{k-1}^2\right)\left(\sum_{k=1}^M X_{k-2}^2\right)}, \quad (51)$$

$$\hat{\rho}_2 = \frac{\left(\sum_{k=1}^M X_k X_{k-1}\right)\left(\sum_{k=1}^M X_{k-1} X_{k-2}\right) - \left(\sum_{k=1}^M X_k X_{k-2}\right)\left(\sum_{k=1}^M X_{k-1}^2\right)}{\left(\sum_{k=1}^M X_{k-1} X_{k-2}\right)^2 - \left(\sum_{k=1}^M X_{k-1}^2\right)\left(\sum_{k=1}^M X_{k-2}^2\right)}. \quad (52)$$

Next, we estimate the parameters σ_ϵ and σ_η which are the variances of noise factors ϵ_k and η_k in the dynamics of no-impact execution price P_k ($P_k = u_1 P_{k-1} + u_2 P_{k-2} + \epsilon_k$) and the information factor X_k ($X_k = \rho_1 X_{k-1} + \rho_2 X_{k-2} + \eta_k$) respectively. Consider following:

$$\hat{\sigma}_\epsilon = \frac{1}{M-1} \sum_{k=1}^M (\epsilon_k - \bar{\epsilon})^2 = \frac{1}{M-1} \sum_{k=1}^M \left\{ (P_k - \hat{a}_1 P_{k-1} - \hat{a}_2 P_{k-2}) - \bar{\epsilon} \right\}^2 \quad (53)$$

$$\text{where } \bar{\epsilon} = \frac{1}{M} \sum_{k=1}^M (P_k - \hat{a}_1 P_{k-1} - \hat{a}_2 P_{k-2}).$$

$$\hat{\sigma}_\eta = \frac{1}{M-1} \sum_{k=1}^M (\eta_k - \bar{\eta})^2 = \frac{1}{M-1} \sum_{k=1}^M \left\{ (X_k - \hat{\rho}_1 X_{k-1} - \hat{\rho}_2 X_{k-2}) - \bar{\eta} \right\}^2 \quad (54)$$

$$\text{where } \bar{\eta} = \frac{1}{M} \sum_{k=1}^M (X_k - \hat{\rho}_1 X_{k-1} - \hat{\rho}_2 X_{k-2}).$$

The real market data considered is consists of daily closing prices (P_k) of one share of Reliance Industries Ltd. stock, and the daily returns (X_k) on one share of Nifty index for the year 2014. The considered historical data has 241 values ($M = 241$). We take $P_0 = \text{Rs. } 895.2$, $P_{-1} = \text{Rs. } 887.1$ as the year 2013's last two trading days' prices of one share of Reliance Industries Ltd. stock. Further $X_1 = 0.0021$, $X_0 = -0.0036$ are considered as returns on the one share of Nifty index in the last two trading days of the year 2013.

Using Eqs. (48), (49) and (51–54), we obtain following estimates of the parameters.

$$\begin{aligned} \hat{u}_1 &= 1.1146, \quad \hat{u}_2 = -0.1147, \quad \hat{\rho}_1 = 0.2078, \quad \hat{\rho}_2 = -0.0971, \\ \hat{\sigma}_\epsilon &= \text{Rs. } 14.1844 \quad \text{and} \quad \hat{\sigma}_\eta = 0.0079. \end{aligned}$$

Apart from manual calculation, we fit the fourth order autoregressive equation to the considered real market data by using Econometric Views (EViews) statistical package. The output obtained is shown in Table 3. The last column of the table represents p value of the test. The underlying theory suggests the rejection of null hypothesis of

Table 3 Result of estimation of fourth order autoregressive equation fitted to the daily closing prices of Reliance Industries Ltd. stock ($P_t = u_1 P_{t-1} + u_2 P_{t-2} + u_3 P_{t-3} + u_4 P_{t-4}$), and daily returns of Nifty 50 index ($X_t = \rho_1 X_{t-1} + \rho_2 X_{t-2} + \rho_3 X_{t-3} + \rho_4 X_{t-4}$) for the each trading day of year 2014 using EViews statistical package

Lag-order (i)	Coefficient (u_i)	Standard error	t Statistics	Probability
Price of Reliance Industries Ltd. stock (P_t)				
1	1.120894	0.065143	17.20676	0.0000
2	-0.255558	0.097755	-2.614266	0.0095
3	0.160971	0.098036	1.641954	0.1019
4	-0.026269	0.065217	-0.402787	0.6875
Return on Nifty 50 index (X_t)				
1	0.227426	0.064832	3.507927	0.0005
2	-0.131268	0.066082	-1.986427	0.0481
3	0.114834	0.065761	1.746242	0.0821
4	-0.089054	0.064524	-1.380168	0.1688

Table 4 The comparison of optimal static trading strategy $\hat{S}$ of QCQP problem with reference strategy as $\hat{S}^*$ (optimal static trading strategy of Mean-variance QP problem with $\lambda = 10^{-7}$)

Trading day	$\hat{S}$	$\hat{S}^*$	Trading day	$\hat{S}$	$\hat{S}^*$
1	479.4	2246.0	12	480.9	136.9
2	480.4	1594.8	13	481.1	97.8
3	480.4	1105.3	14	481.4	118.9
4	480.3	825.4	15	481.7	197.8
5	480.3	570.1	16	482.0	211.9
6	480.3	398.6	17	482.3	290.4
7	480.3	314.5	18	482.9	365.1
8	480.4	208.9	19	485.0	436.9
9	480.5	155.4	20	859.1	517.4
10	480.6	141.4	Expected execution cost (Rs.)	8.9759×10^6	8.9833×10^6
11	480.7	66.5			

The parameters considered are $N = 20$, $\bar{S} = 10000$, $\hat{\sigma}_\epsilon = \text{Rs. } 14.1844$, $\hat{\sigma}_\eta = 0.0079$, $\theta = 5 \times 10^{-4}$, $\gamma = 5$, $\hat{a}_1 = 1.1146$, $\hat{a}_2 = -0.1147$, $\hat{\rho}_1 = 0.2078$, $\hat{\rho}_2 = -0.0971$, $P_0 = \text{Rs. } 895.2$, $P_{-1} = \text{Rs. } 887.1$, $X_0 = 0.0021$ and $X_{-1} = -0.0036$.

zero coefficient of a lag-order if the p value is smaller than the significance level. It is evident from Table 3 that in both financial data considered, the coefficient of the first and second lag-orders are significant at 5% level of significance. Whereas, coefficients corresponding to higher lag-orders, third and fourth are not significant. These results justify the presence of second order autoregressive behavior in no-impact price dynamics and information factor, and thus the choice of AR(2) price model.

Furthermore, we consider problem of trading of 10000 shares of Reliance Industries Ltd. stock in the first 20 trading days of the year 2014. That is, $T = 20$, $\bar{S} = 10000$. With the considered real market data and the estimated parameter values, the minimum

cost of execution corresponding to Mean QP problem is obtained as Rs. 8.9681×10^6 . In Table 4 the optimal static trading strategy $\hat{S}$ corresponding to QCQP problem with reference strategy as the optimal static trading strategy $\hat{S}^*$ of Mean–variance QP problem with $\lambda = 10^{-7}$, is given. The cost of execution with respect to $\hat{S}$ and $\hat{S}^*$ are also shown.

6 Conclusion and future work

This paper studied the problem of optimal trading under the proposed AR(2) price model. The AR(2) price dynamics is modeled as the generalization of AR(1) price dynamics of Bertsimas and Lo (1998) by incorporating the second order autoregressive behavior in latter. By DP method, the closed form solution of unconstrained optimal trading problem under AR(2) price model is derived. However, in view of computational difficulty associated with DP method to incorporate non-negativity constraints in the problem formulation, the optimal static trading problems under AR(2) price model are formulated, and solved by SA method. The optimal static trading problems for risk neutral and risk averse investors are formulated as quadratic programming problems, referred respectively as Mean QP and Mean–variance QP problems. Moreover, the significance of stochastic dominance constraints in optimal trading is mentioned, and the optimal static trading problem subject to these constraints is formulated as a quadratically constrained quadratic programming problem (QCQP problem). The algorithm based on SA method to solve proposed optimal static trading problems—Mean QP, Mean–variance QP and QCQP, is presented. Using simulated and real market financial data the extensive numerical illustrations are conducted which depict the significance of AR(2) price model and the optimal static trading problems. It is inferred that the AR(2) price model is able to capture the bearish, stable and bullish market scenarios, and the optimal execution cost corresponding to Mean QP problem depends upon these scenarios. Comparison of efficient trading frontiers corresponding to Mean–variance QP problem infers that an optimal static trading strategy under AR(2) price model is always preferable over that under the AR(1) price model.

In this paper single asset optimal trading problem is discussed. Thus this work can be further extended to the trading of portfolio of assets. Moreover, on lines of the work of Moazeni et al. (2013) the price path of this paper can be further generalized with inclusion of jump processes to model the other traders' large buy and sell trades.

Acknowledgements Authors are thankful to Prof. Suresh Chandra, former emeritus professor of Indian Institute of Technology Delhi, for his continuous encouragement and helpful suggestions in the preparation of this paper. Moreover, authors are grateful to the editor and two anonymous reviewers for their valuable suggestions and comments which helped improve the paper to great extent.

Appendix 1

In this Appendix, proof of Proposition 1 is presented.

Proof We use method of mathematical induction to prove (7) and (8) for $t = 0, 1, \dots, T - 1$.

In the last trading period T , optimal strategy is to execute all the remaining shares to complete the trading. Thus

$$\begin{aligned} S_T &= W_T \\ &= \delta_{P,0}P_{T-1} + \delta_{PP,0}P_{T-2} + \delta_{W,0}W_T + \delta_{X,0}X_T + \delta_{XX,0}X_{T-1} \end{aligned} \quad (55)$$

where $\delta_{P,0} = 0$, $\delta_{PP,0} = 0$, $\delta_{W,0} = 1$, $\delta_{X,0} = 0$, $\delta_{XX,0} = 0$.

By (5), we get

$$\begin{aligned} V_T(P_{T-1}, P_{T-2}, X_T, X_{T-1}, W_T) &= u_1P_{T-1}W_T + u_2P_{T-2}W_T + \theta W_T^2 + \gamma X_TW_T \\ &= a_0P_{T-1}^2 + b_0P_{T-2}^2 + c_0W_T^2 + d_0X_T^2 + e_0X_{T-1}^2 + f_0P_{T-1}P_{T-2} + g_0P_{T-1}W_T \\ &\quad + h_0P_{T-1}X_T + i_0P_{T-1}X_{T-1} + j_0P_{T-2}W_T + k_0P_{T-2}X_T + l_0P_{T-2}X_{T-1} \\ &\quad + m_0W_TX_T + n_0W_TX_{T-1} + o_0X_TX_{T-1} + p_0, \end{aligned} \quad (56)$$

where $a_0 = 0$, $b_0 = 0$, $c_0 = \theta$, $d_0 = 0$, $e_0 = 0$, $f_0 = 0$, $g_0 = u_1$, $h_0 = 0$, $i_0 = 0$, $j_0 = u_2$, $k_0 = 0$, $l_0 = 0$, $m_0 = \gamma$, $n_0 = 0$, $o_0 = 0$, $p_0 = 0$.

Let following equation holds true.

$$\begin{aligned} V_{T-t+1}(P_{T-t}, P_{T-t-1}, X_{T-t+1}, X_{T-t}, W_{T-t+1}) &= a_{t-1}P_{T-t}^2 + b_{t-1}P_{T-t-1}^2 + c_{t-1}W_{T-t+1}^2 + d_{t-1}X_{T-t+1}^2 + e_{t-1}X_{T-t}^2 \\ &\quad + f_{t-1}P_{T-t}P_{T-t-1} + g_{t-1}P_{T-t}W_{T-t+1} + h_{t-1}P_{T-t}X_{T-t+1} \\ &\quad + i_{t-1}P_{T-t}X_{T-t} + j_{t-1}P_{T-t-1}W_{T-t+1} + k_{t-1}P_{T-t-1}X_{T-t+1} + l_{t-1}P_{T-t-1}X_{T-t} \\ &\quad + m_{t-1}W_{T-t+1}X_{T-t+1} + n_{t-1}W_{T-t+1}X_{T-t} + o_{t-1}X_{T-t+1}X_{T-t} + p_{t-1} \end{aligned} \quad (57)$$

By Eq. (6), we have

$$\begin{aligned} V_{T-t}(P_{T-t-1}, P_{T-t-2}, X_{T-t}, X_{T-t-1}, W_{T-t}) &= \min_{S_{T-t}} E_{T-t}[P_{T-t}S_{T-t} + V_{T-t+1}(P_{T-t}, P_{T-t-1}, X_{T-t+1}, X_{T-t}, W_{T-t+1})] \\ &\quad (\text{Putting value of } V_{T-t+1} \text{ from Eq. (57)}) \\ &= \min_{S_{T-t}} E_{T-t}[P_{T-t}S_{T-t} + a_{t-1}P_{T-t}^2 + b_{t-1}P_{T-t-1}^2 + c_{t-1}W_{T-t+1}^2 \\ &\quad + d_{t-1}X_{T-t+1}^2 + e_{t-1}X_{T-t}^2 + f_{t-1}P_{T-t}P_{T-t-1} + g_{t-1}P_{T-t}W_{T-t+1} \\ &\quad + h_{t-1}P_{T-t}X_{T-t+1} + i_{t-1}P_{T-t}X_{T-t} + j_{t-1}P_{T-t-1}W_{T-t+1} \\ &\quad + k_{t-1}P_{T-t-1}X_{T-t+1} + l_{t-1}P_{T-t-1}X_{T-t} + m_{t-1}W_{T-t+1}X_{T-t+1} \\ &\quad + n_{t-1}W_{T-t+1}X_{T-t} + o_{t-1}X_{T-t+1}X_{T-t} + p_{t-1}] \\ &\quad (\text{Using Eqs. (3) and (4), first we put values of } P_{T-t} \text{ and } X_{T-t+1}, \text{ respectively.}) \\ &\quad \text{Further we use relation } W_{T-t+1} = W_{T-t} - S_{T-t}, \text{ and rearrange terms as a} \\ &\quad \text{quadratic function in } S_{T-t}.) \\ &= \min_{S_{T-t}} E_{T-t} \left[\left\{ \theta + a_{t-1}\theta^2 + c_{t-1} - g_{t-1}\theta \right\} S_{T-t}^2 + \left\{ (2a_{t-1}u_1\theta + f_{t-1}\theta \right. \right. \end{aligned}$$

$$\begin{aligned}
 & +u_1 - j_{t-1} - g_{t-1}u_1)P_{T-t-1} + (2a_{t-1}u_2\theta + u_2 - g_{t-1}u_2)P_{T-t-2}\gamma \\
 & + (g_{t-1}\theta - 2c_{t-1})W_{T-t} + (2a_{t-1}\theta + \gamma - g_{t-1}\gamma + i_{t-1}\theta - m_{t-1}\rho_1 - n_{t-1})X_{T-t} \\
 & + (h_{t-1}\theta\rho_2 - m_{t-1}\rho_2)X_{T-t-1} \Big\} S_{T-t} + (a_{t-1}u_1^2 + b_{t-1} + f_{t-1}u_1)P_{T-t-1}^2 \\
 & + (a_{t-1}u_2^2)P_{T-t-2}^2 + (c_{t-1})W_{T-t}^2 + (a_{t-1}\gamma^2 + d_{t-1}\rho_1^2 + e_{t-1} \\
 & + h_{t-1}\gamma\rho_1 + i_{t-1}\gamma + o_{t-1}\rho_1)X_{T-t}^2 + (d_{t-1}\rho_2^2)X_{T-t-1}^2 + (2a_{t-1}u_1u_2 \\
 & + f_{t-1}u_2)P_{T-t-1}P_{T-t-2} + (g_{t-1}u_1 + j_{t-1})P_{T-t-1}W_{T-t} + (2a_{t-1}u_1\gamma + f_{t-1}\gamma \\
 & + h_{t-1}u_1\rho_1 + i_{t-1}u_1 + k_{t-1}\rho_1 + l_{t-1})P_{T-t-1}X_{T-t} \\
 & + (h_{t-1}u_1\rho_2 + k_{t-1}\rho_2)P_{T-t-1}X_{T-t-1} \\
 & + (g_{t-1}u_2)P_{T-t-2}W_{T-t} + (2a_{t-1}u_2\gamma + h_{t-1}u_2\rho_1 + i_{t-1}u_2)P_{T-t-2}X_{T-t} \\
 & + (h_{t-1}u_2\rho_2)P_{T-t-2}X_{T-t-1} + (g_{t-1}\gamma + m_{t-1}\rho_1 + n_{t-1})W_{T-t}X_{T-t} \\
 & + (m_{t-1}\rho_2)W_{T-t}X_{T-t-1} + (2d_{t-1}\rho_1\rho_2 + h_{t-1}\gamma\rho_2 \\
 & + o_{t-1}\rho_2)X_{T-t}X_{T-t-1} + (a_{t-1}\sigma_\epsilon^2 + d_{t-1}\sigma_\eta^2 + p_{t-1}) \Big] \quad (58)
 \end{aligned}$$

Thus at time period $T - t$ the unrestricted minimization of (58) occurs at following point:

$$\begin{aligned}
 S_{T-t} &= \left(\frac{g_{t-1}u_1 + j_{t-1} - u_1 - f_{t-1}\theta - 2a_{t-1}u_1\theta}{2(\theta + a_{t-1}\theta^2 + c_{t-1} - g_{t-1}\theta)} \right) P_{T-t-1} \\
 &+ \left(\frac{g_{t-1}u_2 - 2a_{t-1}u_2\theta - u_2}{2(\theta + a_{t-1}\theta^2 + c_{t-1} - g_{t-1}\theta)} \right) P_{T-t-2} \\
 &+ \left(\frac{2c_{t-1} - g_{t-1}\theta}{2(\theta + a_{t-1}\theta^2 + c_{t-1} - g_{t-1}\theta)} \right) W_{T-t} \\
 &+ \left(\frac{n_{t-1} + m_{t-1}\rho_1 - i_{t-1}\theta + g_{t-1}\gamma - 2a_{t-1}\theta\gamma - \gamma - h_{t-1}\theta\rho_1}{2(\theta + a_{t-1}\theta^2 + c_{t-1} - g_{t-1}\theta)} \right) X_{T-t} \\
 &+ \left(\frac{m_{t-1}\rho_2 - h_{t-1}\theta\rho_1}{2(\theta + a_{t-1}\theta^2 + c_{t-1} - g_{t-1}\theta)} \right) X_{T-t-1} \\
 &= \delta_{P,t}P_{T-t-1} + \delta_{PP,t}P_{T-t-2} + \delta_{W,t}W_{T-t} + \delta_{X,t}X_{T-t} + \delta_{XX,t}X_{T-t-1}. \quad (59)
 \end{aligned}$$

Putting point of minima S_{T-t} from (59) into (58), we get

$$\begin{aligned}
 V_{T-t} &= a_tP_{T-t-1}^2 + b_tP_{T-t-2}^2 + c_tW_{T-t}^2 + d_tX_{T-t}^2 + e_tX_{T-t-1}^2 + f_tP_{T-t-1}P_{T-t-2} \\
 &+ g_tP_{T-t-1}W_{T-t} + h_tP_{T-t-1}X_{T-t} + i_tP_{T-t-1}X_{T-t-1} + j_tP_{T-t-2}W_{T-t} \\
 &+ k_tP_{T-t-2}X_{T-t} + l_tP_{T-t-2}X_{T-t-1} + m_tW_{T-t}X_{T-t} + n_tW_{T-t}X_{T-t-1} \\
 &+ o_tX_{T-t}X_{T-t-1} + p_t. \quad (60)
 \end{aligned}$$

In view of (55)–(60), Proposition 1 is proved. $\square$

Appendix 2

In this Appendix, the proof of Lemma 1 is presented.

Proof First we prove the Eq. (13) of Lemma 1.

Consider the case when t is an even positive integer.

That is let $t = 2m$ for some positive integer m .

$$\begin{aligned}
 u_1 L_t + u_2 L_{t-1} &= u_1 L_{2m} + u_2 L_{2m-1} \\
 &= u_1 \left[\sum_{k=0}^m \binom{2m-k}{k} u_1^{2m-2k} u_2^k \right] + u_2 \left[\sum_{k=0}^{m-1} \binom{2m-1-k}{k} u_1^{2m-1-2k} u_2^k \right] \\
 &= \left[\sum_{k=0}^m \binom{2m-k}{k} u_1^{2m+1-2k} u_2^k \right] + \left[\sum_{k=0}^{m-1} \binom{2m-1-k}{k} u_1^{2m-1-2k} u_2^{k+1} \right] \\
 &= \left[\sum_{k=0}^m \binom{2m-k}{k} u_1^{2m+1-2k} u_2^k \right] + \left[\sum_{k=1}^m \binom{2m-k}{k-1} u_1^{2m+1-2k} u_2^k \right] \\
 &= \left[u_1^{2m+1} + \sum_{k=1}^m \binom{2m-k}{k} u_1^{2m+1-2k} u_2^k \right] + \left[\sum_{k=1}^m \binom{2m-k}{k-1} u_1^{2m+1-2k} u_2^k \right] \\
 &= u_1^{2m+1} + \sum_{k=1}^m \left[\binom{2m-k}{k} + \binom{2m-k}{k-1} \right] u_1^{2m+1-2k} u_2^k \\
 &= u_1^{2m+1} + \sum_{k=1}^m \binom{2m+1-k}{k} u_1^{2m+1-2k} u_2^k \\
 &= \sum_{k=0}^m \binom{2m+1-k}{k} u_1^{2m+1-2k} u_2^k \\
 &= L_{2m+1} \\
 &= L_{t+1}
 \end{aligned}$$

Similarly Eq. (13) can be proved for the case when t is an odd positive integer.

Similar to the above proof, the proof of Eq. (14) follows. $\square$

Appendix 3

The proof of Lemma 2 is given in this Appendix C.

Proof We will use method of mathematical induction to prove the Eq. (15) of Lemma 2.

Consider

$$\begin{aligned}
 X_2 &= \rho_1 X_1 + \rho_2 X_0 + \eta_2 \\
 &= X_1(M_1) + \rho_2 X_0(M_0) + \sum_{k=0}^{2-2} M_k \eta_{2-k}.
 \end{aligned}$$

Thus Eq. (15) is true for $t = 2$.

Let Eq. (15) be true for all t such that $2 \leq t \leq m$ for some positive integer m where $2 < m < T$.

To prove Lemma 2, next we need to show that Eq. (15) is true for $t = m + 1$.

$$\begin{aligned}
 X_{m+1} &= \rho_1 X_m + \rho_2 X_{m-1} + \eta_{m+1} \\
 &= \rho_1 \left[X_1(M_{m-1}) + \rho_2 X_0(M_{m-2}) + \sum_{k=0}^{m-2} M_k \eta_{m-k} \right] \\
 &\quad + \rho_2 \left[X_1(M_{m-2}) + \rho_2 X_0(M_{m-3}) + \sum_{k=0}^{m-3} M_k \eta_{m-1-k} \right] + \eta_{m+1} \\
 &= X_1(\rho_1 M_{m-1} + \rho_2 M_{m-2}) + \rho_2 X_0(\rho_1 M_{m-2} + \rho_2 M_{m-3}) \\
 &\quad + \left[(\rho_1 M_0) \eta_m + \sum_{k=1}^{m-2} (\rho_1 M_k + \rho_2 M_{k-1}) \eta_{m-k} \right] + \eta_{m+1} \\
 &\quad \text{(By Eq. (14) of Lemma 1 and by equation } \rho_1 M_0 = M_1) \\
 &= X_1(M_m) + \rho_2 X_0(M_{m-1}) + \left[M_1 \eta_m + \sum_{k=1}^{m-2} M_{k+1} \eta_{m-k} \right] + \eta_{m+1} \\
 &= X_1(M_m) + \rho_2 X_0(M_{m-1}) + \sum_{k=0}^{m-1} M_k \eta_{m+1-k}
 \end{aligned}$$

Thus claim is proved. $\square$

Appendix 4

In this Appendix, proof of Lemma 3 is presented.

Proof The Eq. (16) of Lemma 3 is obvious.

We will use method of mathematical induction to prove the Eq. (17) of Lemma 3.

Consider

$$\begin{aligned}
 P_2 &= u_1 P_1 + u_2 P_0 + \theta \hat{S}_2 + \gamma X_2 + \epsilon_2 \\
 &\quad \text{(Putting values of } P_2 \text{ and } X_2 \text{ from Eqs. (3) and (4), respectively, we get)} \\
 &= u_1 (u_1 P_0 + u_2 P_{-1} + \theta \hat{S}_1 + \gamma X_1 + \epsilon_1) + u_2 P_0 + \theta \hat{S}_2 + \gamma (\rho_1 X_1 + \rho_2 X_0 + \eta_2) + \epsilon_2 \\
 &= P_0(L_2) + u_2 P_{-1}(L_1) + \gamma X_1(L_0 M_1 + L_1 M_0) + \gamma \rho_2 X_0(L_0 M_0) \\
 &\quad + (L_0 \epsilon_2 + L_1 \epsilon_1) + \gamma (L_0 M_0 \eta_2)
 \end{aligned}$$

Thus Eq. (17) is true for $t = 2$.

Let Eq. (17) be true for all t such that $2 \leq t \leq m$ for some positive integer m where $2 < m < T$.

To prove Lemma 3, next we need to show that Eq. (17) is true for $t = m + 1$.

$$\begin{aligned}
 P_{m+1} &= u_1 P_m + u_2 P_{m-1} + \theta \hat{S}_{m+1} + \gamma X_{m+1} + \epsilon_{m+1} \\
 &= u_1 \left[P_0(L_m) + u_2 P_{-1}(L_{m-1}) + \gamma X_1 \left(\sum_{k=0}^{m-1} L_k M_{m-1-k} \right) + \gamma \rho_2 X_0 \left(\sum_{k=0}^{m-2} L_k M_{m-2-k} \right) \right. \\
 &\quad \left. + \theta \left(\sum_{k=0}^{m-1} L_k \hat{S}_{m-k} \right) + \left(\sum_{k=0}^{m-1} L_k \epsilon_{m-k} \right) + \gamma \left\{ \sum_{k=0}^{m-2} \left(\sum_{l=0}^k L_l M_{k-l} \right) \eta_{m-k} \right\} \right] \\
 &\quad + u_2 \left[P_0(L_{m-1}) + u_2 P_{-1}(L_{m-2}) + \gamma X_1 \left(\sum_{k=0}^{m-2} L_k M_{m-2-k} \right) + \gamma \rho_2 X_0 \left(\sum_{k=0}^{m-3} L_k M_{m-3-k} \right) \right. \\
 &\quad \left. + \theta \left(\sum_{k=0}^{m-2} L_k \hat{S}_{m-1-k} \right) + \left(\sum_{k=0}^{m-2} L_k \epsilon_{m-1-k} \right) + \gamma \left\{ \sum_{k=0}^{m-3} \left(\sum_{l=0}^k L_l M_{k-l} \right) \eta_{m-1-k} \right\} \right] \\
 &\quad + \theta \hat{S}_{m+1} + \gamma X_{m+1} + \epsilon_{m+1} \\
 &= P_0(u_1 L_m + u_2 L_{m-1}) + u_2 P_{-1}(u_1 L_{m-1} + u_2 L_{m-2}) + \gamma X_1 \{ (u_1 L_0) M_{m-1} \\
 &\quad + \sum_{k=1}^{m-1} (u_1 L_k + u_2 L_{k-1}) M_{m-1-k} \} + \gamma \rho_2 X_0 \{ (u_1 L_0) M_{m-2} \\
 &\quad + \sum_{k=1}^{m-2} (u_1 L_k + u_2 L_{k-1}) M_{m-2-k} \} + \theta \left\{ (u_1 L_0) \hat{S}_m + \sum_{k=1}^{m-1} (u_1 L_k + u_2 L_{k-1}) \hat{S}_{m-k} \right\} \\
 &\quad + \left\{ (u_1 L_0) \epsilon_m + \sum_{k=1}^{m-1} (u_1 L_k + u_2 L_{k-1}) \epsilon_{m-k} \right\} \\
 &\quad + \gamma \left[(u_1 L_0) M_0 \eta_m + \sum_{k=1}^{m-2} \left\{ (u_1 L_0) M_k + \sum_{l=1}^k (u_1 L_l + u_2 L_{l-1}) M_{k-l} \right\} \eta_{m-k} \right] + \theta \hat{S}_{m+1} \\
 &\quad + \gamma X_{m+1} + \epsilon_{m+1}
 \end{aligned}$$

(By using Eq. (13) of Lemma 1 and the equation $u_1 L_0 = L_1$.)

$$\begin{aligned}
 &= P_0(L_{m+1}) + u_2 P_{-1}(L_m) + \gamma X_1 \left\{ L_1 M_{m-1} + \sum_{k=1}^{m-1} L_{k+1} M_{m-1-k} \right\} + \gamma \rho_2 X_0 \{ L_1 M_{m-2} \\
 &\quad + \sum_{k=1}^{m-2} L_{k+1} M_{m-2-k} \} + \theta \left\{ L_1 \hat{S}_m + \sum_{k=1}^{m-1} L_{k+1} \hat{S}_{m-k} \right\} + \left\{ L_1 \epsilon_m + \sum_{k=1}^{m-1} L_{k+1} \epsilon_{m-k} \right\} \\
 &\quad + \gamma \left[(L_1 M_0) \eta_m + \sum_{k=1}^{m-2} \left\{ L_1 M_k + \sum_{l=1}^k L_{l+1} M_{k-l} \right\} \eta_{m-k} \right] + \theta \hat{S}_{m+1} + \gamma X_{m+1} + \epsilon_{m+1}
 \end{aligned}$$

(Simplifying above equation further and by using Eq. (15) of Lemma 2.)

$$\begin{aligned}
 &= P_0(L_{m+1}) + u_2 P_{-1}(L_m) + \gamma X_1 \left\{ \sum_{k=1}^m L_k M_{m-k} \right\} + \gamma \rho_2 X_0 \left\{ \sum_{k=1}^{m-1} L_k M_{m-1-k} \right\} \\
 &\quad + \theta \left\{ \sum_{k=1}^m L_k \hat{S}_{m+1-k} \right\} + \left\{ \sum_{k=1}^m L_k \epsilon_{m+1-k} \right\} + \gamma \left[\sum_{k=0}^{m-2} \left\{ \sum_{l=0}^k L_{l+1} M_{k-l} \right\} \eta_{m-k} \right] \\
 &\quad + \theta \hat{S}_{m+1} + \gamma \left[X_1(M_m) + \rho_2 X_0(M_{m-1}) + \sum_{k=0}^{m-1} M_k \eta_{m+1-k} \right] + \epsilon_{m+1}
 \end{aligned}$$

$$\begin{aligned}
 &= P_0(L_{m+1}) + u_2 P_{-1}(L_m) + \gamma X_1 \left\{ \sum_{k=0}^m L_k M_{m-k} \right\} + \gamma \rho_2 X_0 \left\{ \sum_{k=0}^{m-1} L_k M_{m-1-k} \right\} \\
 &\quad + \theta \left\{ \sum_{k=0}^m L_k \hat{S}_{m+1-k} \right\} + \left\{ \sum_{k=0}^m L_k \epsilon_{m+1-k} \right\} + \gamma \left[\sum_{k=0}^{m-1} \left\{ \sum_{l=0}^k L_l M_{k-l} \right\} \eta_{m+1-k} \right]
 \end{aligned}$$

Thus Lemma 3 is proved. $\square$

Appendix 5

The proof of Lemma 4 is presented in this Appendix.

Proof We use method of mathematical induction to prove Eq. (23) of Lemma 4.

Consider

$$\begin{aligned}
 Cov_1(X_3, X_2) &= Cov_1(\rho_1 X_2 + \rho_2 X_1 + \eta_3, X_2) \\
 &= \rho_1 Var_1(X_2).
 \end{aligned}$$

Thus Eq. (23) is true for $t = 3$.

Let Eq. (23) be true for all t such that $3 \leq t \leq m$ for some positive integer m where $3 < m < T$.

To prove Lemma 4, next we need to show that Eq. (23) is true for $t = m + 1$.

$$\begin{aligned}
 Cov_1(X_{m+1}, X_m) &= Cov_1(\rho_1 X_m + \rho_2 X_{m-1} + \eta_{m+1}, X_m) \\
 &= \rho_1 Var_1(X_m) + \rho_2 Cov_1(X_m, X_{m-1}) \\
 &\quad (\text{By the assumption that Eq. (23) is true for all } t \text{ such that } 3 \leq t \leq m) \\
 &= \rho_1 Var_1(X_m) + \rho_2 [\rho_1 \{Var_1(X_{m-1}) + \rho_2 Var_1(X_{m-2}) + \cdots + \rho_2^{m-3} Var_1(X_2)\}] \\
 &= \rho_1 [Var_1(X_m) + \rho_2 Var_1(X_{m-1}) + \rho_2^2 Var_1(X_{m-2}) + \cdots + \rho_2^{m-2} Var_1(X_2)]
 \end{aligned}$$

Thus Lemma 4 is proved. $\square$

Appendix 6

The proof of Lemma 5 is presented in this Appendix.

Proof We consider the Eq. (24) of Lemma 5 as the statement corresponding to t . With this we prove (24) by the method of mathematical induction.

First we will show that Eq. (24) is true for $t = 4$.

For $t = 4$, we have $2 \leq k \leq 4 - 2$, that is $k = 2$.

$$\begin{aligned}
 Cov_1(X_4, X_{4-2}) &= Cov_1(X_4, X_2) \\
 &= Cov_1(\rho_1 X_3 + \rho_2 X_2 + \eta_4, X_2) \\
 &= \rho_1 Cov_1(X_3, X_2) + \rho_2 Cov_1(X_2, X_2)
 \end{aligned}$$

$$= M_1 Cov_1(X_3, X_2) + \rho_2 M_0 Var_1(X_2)$$

Assume that Eq. (24) is true for $4 \leq t \leq m$ for some positive integer m where $4 < m < T$.

In particular, we have following equations.

For $t = m$ and corresponding $2 \leq k \leq m - 2$

$$Cov_1(X_m, X_{m-k}) = M_{k-1} Cov_1(X_{m-k+1}, X_{m-k}) + \rho_2 M_{k-2} Var_1(X_{m-k}) \quad (61)$$

For $t = m - 1$ and corresponding $2 \leq k \leq m - 3$

$$Cov_1(X_{m-1}, X_{m-1-k}) = M_{k-1} Cov_1(X_{m-k}, X_{m-1-k}) + \rho_2 M_{k-2} Var_1(X_{m-1-k}). \quad (62)$$

To prove the Lemma 5, next we need to show that Eq. (24) is true for $t = m + 1$.

That is we need to show that $\forall 2 \leq k \leq m - 1$.

$$Cov_1(X_{m+1}, X_{m+1-k}) = M_{k-1} Cov_1(X_{m+1-k+1}, X_{m+1-k}) + \rho_2 M_{k-2} Var_1(X_{m+1-k}). \quad (63)$$

We prove Eq. (63) separately for $k = 2, k = 3$ and for $4 \leq k \leq m - 1$.

For $k = 2$, consider

$$\begin{aligned} Cov_1(X_{m+1}, X_{m+1-2}) &= Cov_1(X_{m+1}, X_{m-1}) \\ &= Cov_1(\rho_1 X_m + \rho_2 X_{m-1} + \eta_{m+1}, X_{m-2}) \\ &= \rho_1 Cov_1(X_m, X_{m-1}) + \rho_2 Cov_1(X_{m-1}, X_{m-1}) \\ &= M_1 Cov_1(X_m, X_{m-1}) + \rho_2 M_0 Var_1(X_{m-1}). \end{aligned} \quad (64)$$

For $k = 3$, consider

$$\begin{aligned} Cov_1(X_{m+1}, X_{m+1-3}) &= Cov_1(X_{m+1}, X_{m-2}) \\ &= Cov_1(\rho_1 X_m + \rho_2 X_{m-1} + \eta_{m+1}, X_{m-2}) \\ &= \rho_1 Cov_1(X_m, X_{m-2}) + \rho_2 Cov_1(X_{m-1}, X_{m-2}) \\ &= \rho_1 Cov_1(\rho_1 X_{m-1} + \rho_2 X_{m-2} + \eta_m, X_{m-2}) + \rho_2 Cov_1(X_{m-1}, X_{m-2}) \\ &= \rho_1 [\rho_1 Cov_1(X_{m-1}, X_{m-2}) + \rho_2 Cov_1(X_{m-2}, X_{m-2})] + \rho_2 Cov_1(X_{m-1}, X_{m-2}) \\ &= (\rho_1^2 + \rho_2) Cov_1(X_{m-1}, X_{m-2}) + \rho_2 \rho_1 Cov_1(X_{m-2}, X_{m-2}) \\ &= M_2 Cov_1(X_{m-1}, X_{m-2}) + \rho_2 M_1 Var_1(X_{m-2}). \end{aligned} \quad (65)$$

For $4 \leq k \leq m - 1$, we consider

$$\begin{aligned} Cov_1(X_{m+1}, X_{m+1-k}) &= Cov_1(\rho_1 X_m + \rho_2 X_{m-1} + \eta_{m+1}, X_{m+1-k}) \\ &= \rho_1 Cov_1(X_m, X_{m+1-k}) + \rho_2 Cov_1(X_{m-1}, X_{m+1-k}) \end{aligned} \quad (66)$$

We evaluate terms $Cov_1(X_m, X_{m+1-k})$ and $Cov_1(X_{m-1}, X_{m+1-k})$ of above Eq. (66) as follows:

$$\begin{aligned}
 & Cov_1(X_m, X_{m+1-k}) \\
 &= Cov_1(X_m, X_{m-(k-1)}) \\
 & \quad (\text{Let } k' = k - 1. \text{ Thus } 4 \leq k \leq m - 1 \Rightarrow 3 \leq k' \leq m - 2.) \\
 &= Cov_1(X_m, X_{m-k'}) \\
 & \quad (\text{Applying Eq. (61), in particular for } 3 \leq k' \leq m - 2, \text{ we get}) \\
 &= M_{k'-1} Cov_1(X_{m-k'+1}, X_{m-k'}) + \rho_2 M_{k'-2} Var_1(X_{m-k'}) \\
 & \quad (\text{Put } k' = k - 1.) \\
 &= M_{k-2} Cov_1(X_{m-k+2}, X_{m-k+1}) + \rho_2 M_{k-3} Var_1(X_{m-k+1}) \quad (67) \\
 & \quad Cov_1(X_{m-1}, X_{m+1-k}) = Cov_1(X_{m-1}, X_{m-1-(k-2)}) \\
 & \quad (\text{Let } k' = k - 2. \text{ Thus } 4 \leq k \leq m - 1 \Rightarrow 2 \leq k' \leq m - 3.) \\
 &= Cov_1(X_{m-1}, X_{m-1-k'}) \\
 & \quad (\text{Applying Eq. (62)}) \\
 &= M_{k'-1} Cov_1(X_{m-1-k'+1}, X_{m-1-k'}) + \rho_2 M_{k'-2} Var_1(X_{m-1-k'}) \\
 &= M_{k'-1} Cov_1(X_{m-k'}, X_{m-k'-1}) + \rho_2 M_{k'-2} Var_1(X_{m-k'-1}) \\
 & \quad (\text{Put } k' = k - 2.) \\
 &= M_{k-3} Cov_1(X_{m-k+2}, X_{m-k+1}) + \rho_2 M_{k-4} Var_1(X_{m-k+1}) \quad (68)
 \end{aligned}$$

Using Eqs. (67) and (68) in Eq. (66), we get $\forall 4 \leq k \leq m - 1$

$$\begin{aligned}
 & Cov_1(X_{m+1}, X_{m+1-k}) \\
 &= \rho_1 [M_{k-2} Cov_1(X_{m-k+2}, X_{m-k+1}) + \rho_2 M_{k-3} Var_1(X_{m-k+1})] + \\
 & \quad \rho_2 [M_{k-3} Cov_1(X_{m-k+2}, X_{m-k+1}) + \rho_2 M_{k-4} Var_1(X_{m-k+1})] \\
 &= (\rho_1 M_{k-2} + \rho_2 M_{k-3}) Cov_1(X_{m-k+2}, X_{m-k+1}) \\
 & \quad + \rho_2 (\rho_1 M_{k-3} + \rho_2 M_{k-4}) Var_1(X_{m-k+1}) \\
 & \quad (\text{By Eq. (14) of Lemma 1}) \\
 &= M_{k-1} Cov_1(X_{m-k+2}, X_{m-k+1}) + \rho_2 M_{k-2} Var_1(X_{m-k+1}) \\
 &= M_{k-1} Cov_1(X_{m+1-k+1}, X_{m+1-k}) + \rho_2 M_{k-2} Var_1(X_{m+1-k}) \quad (69)
 \end{aligned}$$

In view of (64), (65) and (69), the Eq. (63) is proved.

Thus Lemma 5 is proved. $\square$

Appendix 7

The proof of Lemma 6 is presented in this Appendix.

Proof We consider the Eq. (30) of Lemma 6 as the statement corresponding to t . With this we prove the Eq. (30) by the method of mathematical induction.

Consider

$$\begin{aligned} Cov_1(P_2, X_k) &= Cov_1(u_1 P_1 + u_2 P_0 + \theta \hat{S}_2 + \gamma X_2 + \epsilon_2, X_k) \\ &= \gamma Cov_1(X_2, X_k) \\ &= \gamma [L_0 Cov_1(X_2, X_k)]. \end{aligned}$$

Thus Eq. (30) is true for $t = 2$.

Let Eq. (30) be true for all t such that $2 \leq t \leq m$ for some positive integer m where $2 < m < T$.

To prove the Lemma 6, next we need to show that Eq. (30) is true for $t = m + 1$.

$$\begin{aligned} Cov_1(P_{m+1}, X_k) &= Cov_1(u_1 P_m + u_2 P_{m-1} + \theta \hat{S}_{m+1} + \gamma X_{m+1} + \epsilon_{m+1}, X_k) \\ &= u_1 Cov_1(P_m, X_k) + u_2 Cov_1(P_{m-1}, X_k) + \gamma Cov_1(X_{m+1}, X_k) \\ &\quad (\text{By the assumption that Eq. (30) is true for all } t \text{ such that } 2 \leq t \leq m) \\ &= u_1 \gamma [L_0 Cov_1(X_m, X_k) + L_1 Cov_1(X_{m-1}, X_k) + \cdots + L_{m-2} Cov_1(X_2, X_k)] \\ &\quad + u_2 \gamma [L_0 Cov_1(X_{m-1}, X_k) + L_1 Cov_1(X_{m-2}, X_k) + \cdots + L_{m-3} Cov_1(X_2, X_k)] \\ &\quad + \gamma Cov_1(X_{m+1}, X_k) \\ &= \gamma [(u_1 L_0) Cov_1(X_m, X_k) + (u_1 L_1 + u_2 L_0) Cov_1(X_{m-1}, X_k) \\ &\quad + (u_1 L_2 + u_2 L_1) Cov_1(X_{m-2}, X_k) + \cdots + (u_1 L_{m-2} + u_2 L_{m-3}) Cov_1(X_2, X_k)] \\ &\quad + \gamma Cov_1(X_{m+1}, X_k) \\ &\quad (\text{By using Eq. (13) of Lemma 1 and the equation } u_1 L_0 = L_1) \\ &= \gamma [L_0 Cov_1(X_{m+1}, X_k) + L_1 Cov_1(X_m, X_k) + \cdots + L_{m-1} Cov_1(X_2, X_k)] \end{aligned}$$

Thus Lemma 6 is proved. $\square$

Appendix 8

In this Appendix, we present the proof of Lemma 7.

Proof We will use method of mathematical induction to prove the Eq. (31) of Lemma 7.

Consider

$$\begin{aligned} Cov_1(P_2, P_1) &= Cov_1(u_1 P_1 + u_2 P_0 + \theta \hat{S}_2 + \gamma X_2 + \epsilon_2, P_1) \\ &= u_1 Var_1(P_1) + \gamma Cov_1(X_2, P_1) \\ &= u_1 Var_1(P_1) + \gamma Cov_1(P_1, X_2). \end{aligned}$$

Thus Eq. (31) is true for $t = 2$.

Let Eq. (31) be true for all n such that $2 \leq t \leq m$ for some positive integer m where $2 < m < N$.

To prove the Lemma 5, next we need to show that Eq. (31) is true for $t = m + 1$.

$$\begin{aligned}
 & Cov_1(P_{m+1}, P_m) \\
 &= Cov_1(u_1 P_m + u_2 P_{m-1} + \theta \hat{S}_{m+1} + \gamma X_{m+1} + \epsilon_{m+1}, P_m) \\
 &= u_1 Var_1(P_m) + u_2 Cov_1(P_{m-1}, P_m) + \gamma Cov_1(X_{m+1}, P_m) \\
 &= u_1 Var_1(P_m) + u_2 Cov_1(P_m, P_{m-1}) + \gamma Cov_1(X_{m+1}, P_m) \\
 &\quad (\text{By the assumption that Eq. (31) is true for all } n \text{ such that } 2 \leq n \leq m) \\
 &= u_1 Var_1(P_m) + u_2 \left[u_1 \left\{ Var_1(P_{m-1}) + u_2 Var_1(P_{m-2}) + \cdots + u_2^{m-2} Var_1(P_1) \right\} \right. \\
 &\quad \left. + \gamma \left\{ Cov_1(P_{m-1}, X_m) + u_2 Cov_1(P_{m-2}, X_{m-1}) + \cdots + u_2^{m-2} Cov_1(P_1, X_2) \right\} \right] \\
 &\quad + \gamma Cov_1(X_{m+1}, P_m) \\
 &= u_1 [Var_1(P_m) + u_2 Var_1(P_{m-1}) + u_2^2 Var_1(P_{m-2}) + \cdots + u_2^{m-1} Var_1(P_1)] \\
 &\quad + \gamma [Cov_1(P_m, X_{m+1}) + u_2 Cov_1(P_{m-1}, X_m) + \cdots + u_2^{m-1} Cov_1(P_1, X_2)]
 \end{aligned}$$

Thus Lemma 7 is proved. $\square$

Appendix 9

In this Appendix, proof of Lemma 8 is presented.

Proof We consider the Eq. (32) of Lemma 8 as the statement corresponding to n . With this we prove the Eq. (32) by the method of mathematical induction.

First we will show that Eq. (32) is true for $n = 3$.

For $n = 3$, we have $2 \leq k \leq 3 - 1$, that is $k = 2$.

$$\begin{aligned}
 Cov_1(P_3, P_{3-2}) &= Cov_1(P_3, P_1) \\
 &= Cov_1(u_1 P_2 + u_2 P_1 + \theta \hat{S}_3 + \gamma X_3 + \epsilon_3, P_1) \\
 &= u_1 Cov_1(P_2, P_1) + u_2 Cov_1(P_1, P_1) + \gamma Cov_1(X_3, P_1) \\
 &= L_1 Cov_1(P_2, P_1) + u_2 L_0 Var_1(P_1) + \gamma [L_0 Cov_1(P_1, X_3)]
 \end{aligned}$$

Assume that Eq. (32) is true for $3 \leq n \leq m$ for some positive integer m where $3 < m < N$. In particular we have following equations.

For $n = m$ and corresponding $2 \leq k \leq m - 1$,

$$\begin{aligned}
 Cov_1(P_m, P_{m-k}) &= L_{k-1} Cov_1(P_{m-k+1}, P_{m-k}) + u_2 L_{k-2} Var_1(P_{m-k}) \\
 &\quad \gamma [L_0 Cov_1(P_{m-k}, X_m) + L_1 Cov_1(P_{m-k}, X_{m-1}) + \cdots \\
 &\quad + L_{k-2} Cov_1(P_{m-k}, X_{m-k+2})].
 \end{aligned} \tag{70}$$

For $n = m - 1$ and corresponding $2 \leq k \leq m - 2$,

$$\begin{aligned}
 Cov_1(P_{m-1}, P_{m-1-k}) &= L_{k-1} Cov_1(P_{m-k}, P_{m-1-k}) + u_2 L_{k-2} Var_1(P_{m-1-k}) \\
 &\quad + \gamma [L_0 Cov_1(P_{m-1-k}, X_{m-1}) + L_1 Cov_1(P_{m-1-k}, X_{m-2})]
 \end{aligned}$$

$$+ \cdots + L_{k-2} \text{Cov}_1(P_{m-1-k}, X_{m-k+1})]. \quad (71)$$

To prove the Lemma 8, next we need to show that Eq. (32) is true for $n = m + 1$. That is we need to show that $\forall 2 \leq k \leq m$.

$$\begin{aligned} \text{Cov}_1(P_{m+1}, P_{m+1-k}) &= L_{k-1} \text{Cov}_1(P_{m+1-k+1}, P_{m+1-k}) + u_2 L_{k-2} \text{Var}_1(P_{m+1-k}) \\ &\quad + \gamma [L_0 \text{Cov}_1(P_{m+1-k}, X_{m+1}) \\ &\quad + L_1 \text{Cov}_1(P_{m+1-k}, X_m) \\ &\quad + \cdots + L_{k-2} \text{Cov}_1(P_{m+1-k}, X_{m-k+3})] \end{aligned} \quad (72)$$

We prove the Eq. (72) separately for $k = 2, k = 3$ and for $4 \leq k \leq m$. For $k = 2$, we consider

$$\begin{aligned} \text{Cov}_1(P_{m+1}, P_{m+1-2}) &= \text{Cov}_1(P_{m+1}, P_{m-1}) \\ &= \text{Cov}_1(u_1 P_m + u_2 P_{m-1} + \theta \hat{S}_{m+1} + \gamma X_{m+1} + \epsilon_{m+1}, P_{m-1}) \\ &= u_1 \text{Cov}_1(P_m, P_{m-1}) + u_2 \text{Cov}_1(P_{m-1}, P_{m-1}) \\ &\quad + \gamma \text{Cov}_1(X_{m+1}, P_{m-1}) \\ &= L_1 \text{Cov}_1(P_m, P_{m-1}) + u_2 L_0 \text{Var}_1(P_{m-1}) \\ &\quad + \gamma [L_0 \text{Cov}_1(P_{m-1}, X_{m+1})] \end{aligned} \quad (73)$$

For $k = 3$, we consider

$$\begin{aligned} \text{Cov}_1(P_{m+1}, P_{m+1-3}) &= \text{Cov}_1(P_{m+1}, P_{m-2}) \\ &= \text{Cov}_1(u_1 P_m + u_2 P_{m-1} + \theta \hat{S}_{m+1} + \gamma X_{m+1} + \epsilon_{m+1}, P_{m-2}) \\ &= u_1 \text{Cov}_1(P_m, P_{m-2}) + u_2 \text{Cov}_1(P_{m-1}, P_{m-2}) + \gamma \text{Cov}_1(X_{m+1}, P_{m-2}) \\ &= u_1 [u_1 \text{Cov}_1(P_{m-1}, P_{m-2}) + u_2 \text{Cov}_1(P_{m-2}, P_{m-2}) + \gamma \text{Cov}_1(X_m, P_{m-2})] \\ &\quad + u_2 \text{Cov}_1(P_{m-1}, P_{m-2}) + \gamma \text{Cov}_1(X_{m+1}, P_{m-2}) \\ &= (u_1^2 + u_2) \text{Cov}_1(P_{m-1}, P_{m-2}) + u_2 u_1 \text{Cov}_1(P_{m-2}, P_{m-2}) \\ &\quad + \gamma [\text{Cov}_1(X_{m+1}, P_{m-2}) + u_1 \text{Cov}_1(X_m, P_{m-2})] \\ &= L_2 \text{Cov}_1(P_{m-1}, P_{m-2}) + u_2 L_1 \text{Var}_1(P_{m-2}) \\ &\quad + \gamma [L_0 \text{Cov}_1(P_{m-2}, X_{m+1}) + L_1 \text{Cov}_1(P_{m-2}, X_m)]. \end{aligned} \quad (74)$$

For $4 \leq k \leq m$, we consider

$$\begin{aligned} \text{Cov}_1(P_{m+1}, P_{m+1-k}) &= \text{Cov}_1(u_1 P_m + u_2 P_{m-1} + \theta \hat{S}_{m+1} + \gamma X_{m+1} + \epsilon_{m+1}, P_{m+1-k}) \\ &= u_1 \text{Cov}_1(P_m, P_{m+1-k}) + u_2 \text{Cov}_1(P_{m-1}, P_{m+1-k}) \\ &\quad + \gamma \text{Cov}_1(X_{m+1}, P_{m+1-k}) \end{aligned} \quad (75)$$

We evaluate terms $Cov_1(P_m, P_{m+1-k})$ and $Cov_1(P_{m-1}, P_{m+1-k})$ of above Eq. (75) as follows.

$$\begin{aligned}
 Cov_1(P_m, P_{m+1-k}) &= Cov_1(P_m, P_{m-(k-1)}) \\
 &\quad (\text{Let } k' = k - 1. \text{ Thus } 4 \leq k \leq m \Rightarrow 3 \leq k' \leq m - 1.) \\
 &= Cov_1(P_m, P_{m-k'}) \\
 &\quad (\text{Applying Eq. (70), in particular for } 3 \leq k' \leq m - 1, \text{ we get}) \\
 &= L_{k'-1} Cov_1(P_{m-k'+1}, P_{m-k'}) + u_2 L_{k'-2} Var_1(P_{m-k'}) \\
 &\quad + \gamma [L_0 Cov_1(P_{m-k'}, X_m) + L_1 Cov_1(P_{m-k'}, X_{m-1}) + \cdots \\
 &\quad + L_{k'-2} Cov_1(P_{m-k'}, X_{m-k'+2})] \\
 &\quad (\text{Put } k' = k - 1.) \\
 &= L_{k-2} Cov_1(P_{m-k+2}, P_{m-k+1}) + u_2 L_{k-3} Var_1(P_{m-k+1}) \\
 &\quad + \gamma [L_0 Cov_1(P_{m-k+1}, X_m) + L_1 Cov_1(P_{m-k+1}, X_{m-1}) \\
 &\quad + \cdots + L_{k-3} Cov_1(P_{m-k+1}, X_{m-k+3})] \tag{76}
 \end{aligned}$$

$$\begin{aligned}
 Cov_1(P_{m-1}, P_{m+1-k}) &= Cov_1(P_{m-1}, P_{m-1-(k-2)}) \\
 &\quad (\text{Let } k' = k - 2. \text{ Thus } 4 \leq k \leq m \Rightarrow 2 \leq k' \leq m - 2.) \\
 &= Cov_1(P_{m-1}, P_{m-1-k'}) \\
 &\quad (\text{Applying Eq. (71)}) \\
 &= L_{k'-1} Cov_1(P_{m-k'}, P_{m-1-k'}) + u_2 L_{k'-2} Var_1(P_{m-1-k'}) \\
 &\quad + \gamma [L_0 Cov_1(P_{m-1-k'}, X_{m-1}) + L_1 Cov_1(P_{m-1-k'}, X_{m-2}) \\
 &\quad + \cdots + L_{k'-2} Cov_1(P_{m-1-k'}, X_{m-k'+1})] \\
 &\quad (\text{Put } k' = k - 2.) \\
 &= L_{k-3} Cov_1(P_{m-k+2}, P_{m-k+1}) + u_2 L_{k-4} Var_1(P_{m-k+1}) \\
 &\quad + \gamma [L_0 Cov_1(P_{m-k+1}, X_{m-1}) + L_1 Cov_1(P_{m-k+1}, X_{m-2}) \\
 &\quad + \cdots + L_{k-4} Cov_1(P_{m-k+1}, X_{m-k+3})] \tag{77}
 \end{aligned}$$

Using Eqs. (76) and (77) in Eq. (75), we get $\forall 4 \leq k \leq m$

$$\begin{aligned}
 Cov_1(P_{m+1}, P_{m+1-k}) &= u_1 \left\{ L_{k-2} Cov_1(P_{m-k+2}, P_{m-k+1}) + u_2 L_{k-3} Var_1(P_{m-k+1}) \right. \\
 &\quad + \gamma [L_0 Cov_1(P_{m-k+1}, X_m) + L_1 Cov_1(P_{m-k+1}, X_{m-1}) + \cdots \\
 &\quad + L_{k-3} Cov_1(P_{m-k+1}, X_{m-k+3})] \left. \right\} \\
 &\quad + u_2 \left\{ L_{k-3} Cov_1(P_{m-k+2}, P_{m-k+1}) + u_2 L_{k-4} Var_1(P_{m-k+1}) \right. \\
 &\quad + \gamma [L_0 Cov_1(P_{m-k+1}, X_{m-1}) + L_1 Cov_1(P_{m-k+1}, X_{m-2}) + \cdots \\
 &\quad + L_{k-4} Cov_1(P_{m-k+1}, X_{m-k+3})] \left. \right\} + \gamma Cov_1(X_{m+1}, P_{m+1-k}) \\
 &= (u_1 L_{k-2} + u_2 L_{k-3}) Cov_1(P_{m-k+2}, P_{m-k+1}) + u_2 (u_1 L_{k-3} \\
 &\quad + u_2 L_{k-4}) Var_1(P_{m-k+1}) + \gamma [u_1 L_0 Cov_1(P_{m-k+1}, X_m)
 \end{aligned}$$

$$\begin{aligned}
 & + (u_1 L_1 + u_2 L_0) \text{Cov}_1(P_{m-k+1}, X_{m-1}) + \cdots \\
 & + (u_1 L_{k-3} + u_2 L_{k-4}) \text{Cov}_1(P_{m-k+1}, X_{m-k+3})] \\
 & + \gamma \text{Cov}_1(X_{m+1}, P_{m+1-k}) \\
 & \text{(By Eq. (13) Lemma 1 and the equation } u_1 L_0 = L_1) \\
 & = L_{k-1} \text{Cov}_1(P_{m-k+2}, P_{m-k+1}) + u_2 L_{k-2} \text{Var}_1(P_{m-k+1}) \\
 & \quad + \gamma [L_0 \text{Cov}_1(X_{m+1}, P_{m+1-k}) \\
 & \quad + L_1 \text{Cov}_1(P_{m-k+1}, X_m) + L_2 \text{Cov}_1(P_{m-k+1}, X_{m-1}) + \cdots \\
 & \quad + L_{k-2} \text{Cov}_1(P_{m-k+1}, X_{m-k+3})] \\
 & = L_{k-1} \text{Cov}_1(P_{m+1-k+1}, P_{m+1-k}) + u_2 L_{k-2} \text{Var}_1(P_{m+1-k}) \\
 & \quad + \gamma [L_0 \text{Cov}_1(P_{m+1-k}, X_{m+1}) \\
 & \quad + L_1 \text{Cov}_1(P_{m+1-k}, X_m) + \cdots + L_{k-2} \text{Cov}_1(P_{m+1-k}, X_{m-k+3})] \\
 & \hspace{15em} (78)
 \end{aligned}$$

In view of (73), (74) and (78), the Eq. (72) is proved.

Thus Lemma 8 is proved. $\square$

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