A CHEMICAL QUEUE

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Abstract

A stochastic model originating in chemical physics is interpreted as a randomized random walk on the integers with an unusual exponential pattern for the inter-step time intervals. Some features of this are analyzed, and then attention is turned to the queue counterpart, where the walk is confined to the non-negative integers. The system state and busy period processes are analyzed in some detail, yielding modified Bessel function formulae. A continued fraction expansion technique is used, but not described, to give specimen values of the queueing probabilities. An interesting operational feature is that, whatever the parameters, the queue has no finite steady state.

Keywords: Stochastic processes; chemical physics; chain diffusion; randomized random walk; queueing counterpart; busy period

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1. The randomized random walk

The reason for the title of this paper is the origin of the model in branches of chemical physics where interest lies in chain molecular diffusion. In the apparently extremely simplified cases discussed in Stockmayer et al. (1977), which should be consulted for more background detail, a molecule is modelled as an infinitely long chain of atoms joined by links of equal length. The links are subjected to random shocks and this causes the atoms to move and the molecule to diffuse. Another physical application with a similar probabilistic formalism arises in statistical mechanics in connection with the Ising model, for which see Glauber (1963). In both cases the resulting stochastic behaviour is described by a set of what are essentially Chapman–Kolmogorov forward equations. In the simplest case the equations have a single parameter. In the more interesting case considered here the atoms are of two alternating kinds and the shock mechanism is different according to whether the atom occupies an odd or an even position on the chain. Interpreting the mechanism in terms of the random walk described by the same Chapman–Kolmogorov equations, we observe that it is governed by a variant of the exponential mechanisms that drive $M/M/1$ queueing systems and their unrestricted random walk counterpart, the latter being dealt with in extenso by Conolly (1971). In this paper we investigate first the random walk and then aspects of the corresponding queueing system.

The model can be described in random walk terminology as specifying the motion in time and space of a particle that moves on the integers $-\infty < k < \infty$ of the real line as follows.
Let \( N(t) \) be its coordinate at time \( t \). The rules for the next step, whatever happened before, are that for some \( \lambda, \mu > 0 \),

\[
\begin{align*}
P[N(t + h) = 2n + 1 \mid N(t) = 2n] &= \lambda h + o(h), \\
P[N(t + h) = 2n - 1 \mid N(t) = 2n] &= \mu h + o(h), \\
P[N(t + h) = 2n \mid N(t) = 2n + 1] &= \lambda h + o(h), \\
P[N(t + h) = 2n \mid N(t) = 2n - 1] &= \mu h + o(h).
\end{align*}
\]

(1)

This contrasts with the usual \( M/M^\gamma \) mechanism in that the rules governing the jump process depend on whether \( N(t) \) is even or odd. It is supposed, unless stated otherwise, that \( N(0) = 0 \). Whether or not he is interested in physical chemistry the stochastic analyst is delighted, claps his hands and, inspired by tradition, is likely to proceed as follows. Let

\[
p_k(t) = P[N(t) = k/N(0) = 0]
\]

(2)

for the positive and negative integers \(-\infty < k < \infty\). According to the rules one obtains

\[
\begin{align*}
\frac{dp_{2n}(t)}{dt} + (\lambda + \mu) p_{2n}(t) &= \mu p_{2n-1}(t) + \lambda p_{2n+1}(t), \\
\frac{dp_{2n+1}(t)}{dt} + (\lambda + \mu) p_{2n+1}(t) &= \lambda p_{2n}(t) + \mu p_{2n+2}(t),
\end{align*}
\]

(3)

for \( n = 0, \pm 1, \pm 2, \ldots \), with \( p_0(0) = 1 \).

Standard methods show that the even-integer probabilities \( p_{2n}(t) \) are generated by the hyperbolic cosine of \( \Omega t \) and the odd-integer \( p_{2n+1}(t) \) by the hyperbolic sine, \( \Omega \) being given below. Thus

\[
G(x, t) = \sum_{n=-\infty}^{\infty} x^n p_n(t) = e^{-at} \left[ \cosh \Omega t + \left( \frac{\lambda x + (\mu/x)}{\mu x + (\lambda/x)} \right)^{1/2} \sinh \Omega t \right]
\]

(4)

where

\[
a = \lambda + \mu, \quad \Omega^2 = \left( \lambda x + \frac{\mu}{x} \right) \left( \mu x + \frac{\lambda}{x} \right).
\]

Extraction of the coefficients of \( x^n \) gives

\[
\begin{align*}
p_{2v+1}(t) &= e^{-at} \sum_{n=v}^{\infty} \frac{(\mu t)^{2n+1}}{(2n+1)!} \sum_{r=1}^{n} \binom{n}{r-v} \binom{n+1}{r+1} p^{2r+1-v} \quad (v = 0, 1, 2, \ldots), \\
p_{2v}(t) &= e^{-at} \sum_{n=v}^{\infty} \frac{(\mu t)^{2n}}{(2n)!} \sum_{r=1}^{n} \binom{n}{r} \binom{n}{r-v} p^{2n-2r+v} \quad (v = 0, \pm 1, \pm 2, \ldots),
\end{align*}
\]

(5)

where \( a = \lambda + \mu, \rho = \lambda/\mu \). Note that the formulae hold for non-negative subscripts and the formula for even subscripts is valid also when they are negative. For negative odd subscripts merely exchange \( \lambda \) and \( \mu \). When \( \lambda = \mu \) the formulae reduce to modified Bessel functions and so the right-hand sides could be described as ‘generalised modified Bessel functions’. Specifically, \( p_0(t) \) can be expressed as

\[
p_0(t) = e^{-at} \frac{d}{dt} \int_0^t I_0(as)I_0(b(t-s)) \, ds \quad (a = \lambda + \mu, b = \lambda - \mu),
\]
or

\[ p_0(t) = e^{-at} \left( b \int_0^t I_0(as)I_1(b(t-s)) ds + I_0(at) \right). \] (6)

Note also that the internal finite sums in (5) both enumerate and give the probabilities of the various paths the particle can follow. For example, \( p_0(t) \) involves all paths that, starting at \( N(0) = 0 \), have arrived at \( N(t) = 0 \), but not necessarily for the first time by time \( t \). To achieve this goal in the simplest case the particle remains at zero throughout \( (0, t) \) with probability \( \exp(-at) \). More generally, the path must incorporate an equal number \( n \) \((n \geq 1)\) of positive and negative steps. \( n = 1 \) requires one positive step followed by a negative step, or the reverse, during \( (0, t) \), giving probability contribution

\[ (\lambda e^{-at} \ast \mu e^{-at} + \mu e^{-at} \ast \mu e^{-at}) \ast e^{-at} = (\lambda^2 + \mu^2)e^{-at \frac{t^2}{2}} \quad (a = \lambda + \mu). \]

The asterisks denote convolution.

\( n = 2 \) requires two positive and two negative steps, which may be represented by \( + + -- \), \( + - + - \), \( + + + - \), \( + + - + \), \( + - + - \), \( - + + - \). There are \( (2n)!/[(n!)^2] = 6 \) (in this case) possible paths with probabilities

\[
\begin{align*}
\lambda e^{-at} \ast \mu e^{-at} \ast \mu e^{-at} \ast \mu e^{-at} \ast e^{-at}, & \quad \lambda e^{-at} \ast \lambda e^{-at} \ast \lambda e^{-at} \ast \lambda e^{-at} \ast e^{-at}, \\
\lambda e^{-at} \ast \mu e^{-at} \ast \mu e^{-at} \ast \mu e^{-at} \ast e^{-at}, & \quad \mu e^{-at} \ast \mu e^{-at} \ast \lambda e^{-at} \ast \lambda e^{-at} \ast e^{-at}, \\
\mu e^{-at} \ast \mu e^{-at} \ast \mu e^{-at} \ast \mu e^{-at} \ast e^{-at}, & \quad \mu e^{-at} \ast \lambda e^{-at} \ast \lambda e^{-at} \ast \mu e^{-at} \ast e^{-at},
\end{align*}
\]

i.e.

\[ e^{-at} \frac{t^4}{4!}(\lambda^2 \mu^2, \lambda^4, \lambda^2 \mu^2, \lambda^2 \mu^2, \mu^4, \lambda^2 \mu^2), \]

respectively. This can be seen to give the term in (5) corresponding to \( n = 2 \), but it would be very troublesome to identify the paths individually as well as their probabilities as \( n \) increases.

The generating function provides a quick and relatively pain-free way of performing a tiresome combinatorial task.

As an alternative to generating function analysis in the time domain we can take Laplace transforms of the differential–difference equations, and using the conventional notation

\[ f^*(z) = \int_0^\infty e^{-zt} f(t) \, dt \]

we obtain

\[
\begin{align*}
Zp_0^*(z) &= 1 + \mu p_{-1}^*(z) + \lambda p_1^*(z) \quad \text{(since } p_0(0) = 1), \\
Zp_{2n}^*(z) &= \mu p_{2n-1}^*(z) + \lambda p_{2n+1}^*(z) \quad \text{(} n = \pm 1, \pm 2, \ldots), \\
Zp_{2n+1}^*(z) &= \lambda p_{2n}^*(z) + \mu p_{2n+2}^*(z) \quad \text{(} n = 0, \pm 1, \pm 2, \ldots),
\end{align*}
\] (7)

where \( Z = z + a, a = \lambda + \mu \). This set of linear difference equations is solved by standard methods and entails the roots \( a^2(\geq) \beta^2 \) of the biquadratic equation

\[ \lambda \mu x^4 - (Z^2 - \lambda^2 - \mu^2)x^2 + \lambda \mu = 0. \]
Using \( b = \lambda - \mu \) (in addition to \( a = \lambda + \mu \)), \( A^2 = Z^2 - a^2, B^2 = Z^2 - b^2 \), we get

\[
\lambda \mu = (a^2 - b^2)/4, \quad Z^2 - \lambda^2 - \mu^2 = (A^2 + B^2)/2,
\]

and the biquadratic becomes

\[
(a^2 - b^2)x^4 - 2(A^2 + B^2)x^2 + (a^2 - b^2) = 0.
\]

The roots are

\[
\alpha^2, \beta^2 = \frac{(A \pm B)^2}{a^2 - b^2},
\]

the minus sign being taken for \( \beta^2 \). This leads to the formulae

\[
\begin{align*}
P_{2n}^*(z) &= \frac{Z \beta^{2n}}{AB} \quad (n = 0, \pm 1, \pm 2, \ldots), \\
P_{2n+1}^*(z) &= \frac{(aB + bA)(B - A)\beta^{2n}}{(a^2 - b^2)AB}, \\
P_{-(2n+1)}^* &= \frac{(aB - bA)(B - A)\beta^{2n}}{(a^2 - b^2)AB},
\end{align*}
\]

the last two being for \( n = 0, 1, 2, \ldots \) The particular form of \( \beta^2 \), namely

\[
\beta^2 = \frac{[(Z^2 - b^2)^{1/2} - (Z^2 - a^2)^{1/2}]^2}{a^2 - b^2},
\]

reduces the Laplace transform formulae to a recognizably invertible shape, and, indeed, that is how (6) was obtained. As a check, note that when \( \lambda = \mu \) (\( b = 0 \)) these formulae reduce to the known ‘randomised random walk’ forms. For example, \( p_0(t) \) is the probability that during \( (0, t) \) the particle takes an equal number of \( +1 \) and \( -1 \) steps. As these are Poisson distributed with parameter \( \mu \) we have

\[
p_0(t) = \sum_{m \geq 0} \left( e^{-\mu t} \frac{(\mu t)^m}{m!} \right)^2 = e^{-2\mu t} I_0(2\mu t),
\]

which is (6) when \( \lambda = \mu \).

2. The queueing process

2.1. System state

A central topic of interest in the study of queueing systems is the so-called ‘system state’, i.e. the number of customers present, waiting and receiving service. The system state is also a random walk, but its nature requires the particle to ‘avoid’ negative states. The state is increased by one upon an arrival (instruction by the mechanism to the particle to take a step in the positive direction), and decreased by one when a customer completes service and leaves the system. The original, and probably the most widely studied queueing system (usually described as an \( M/M/1 \) system) has a generating mechanism which increases or decreases the system state by one with infinitesimal probability densities \( \lambda, \mu \), whatever the system state.
The chemical rules are an interesting variant in that different rules operate according as the system state is even or odd. They reduce to the $M/M/\cdot$ form when $\lambda = \mu$, and this provides checks on the analysis.

We study first the state process generated by the 'chemical' rules, i.e.

$$q_n(t) = P[N(t) = n|N(0) = 0] \quad (n \geq 0).$$

(11)

The differential–difference equations corresponding to (3) are

$$\frac{dq_0(t)}{dt} + \lambda q_0(t) = \lambda q_1(t)$$

(12)

for $n = 0$, and otherwise just as (3), namely:

$$\frac{dq_{2n}(t)}{dt} + (\lambda + \mu)q_{2n}(t) = \mu q_{2n-1}(t) + \lambda q_{2n+1}(t) \quad (n > 0),$$

$$\frac{dq_{2n+1}(t)}{dt} + (\lambda + \mu)q_{2n+1}(t) = \lambda q_{2n}(t) + \mu q_{2n+2}(t) \quad (n \geq 0).$$

(13)

These equations show that, as a queueing process, this system does not in the long run ever settle down to a finite steady state, whatever $\lambda$ and $\mu$, for as $t \rightarrow \infty$, $q_n(t) \rightarrow 0$ for all finite $n$.

Some numerical values of queue state probabilities with notes are given in Figure 1 in the Appendix. Here we look at the solution of (13) by Laplace transforms and the return to the time domain.

The Laplace transformation gives, after some algebra akin to that used for $p_n^*(z)$, the following results in which $q_n^*(z)$ is the Laplace transform of $q_n(t)$:

$$q_{2n}^*(z) = R \beta^{2n}, \quad q_{2n+1}^*(z) = S \beta^{2n} \quad (n = 0, 1, \ldots).$$

(14)

where

$$R = q_0^*(z) = \frac{[(Z + a)(Z - b)]^{1/2} - [(Z - a)(Z + b)]^{1/2}}{(a - b)[(Z - a)(Z + b)]^{1/2}},$$

$$S = \frac{R(Z^2 + ab - AB)}{Z(a + b)},$$

$$a = \lambda + \mu, \quad b = \lambda - \mu, \quad A^2 = Z^2 - a^2, \quad B^2 = Z^2 - b^2.$$

The inversion formula (Erdélyi et al. (1954), p. 233, (24))

$$\left(\frac{z + \alpha}{z - \alpha}\right)^{1/2} - 1 = \alpha \int_0^\infty e^{-zt}(I_0(\alpha t) + I_1(\alpha t))\, dt$$

enables us to recover $q_0(t)$ and leads to

$$q_0(t) = \frac{e^{-at}}{a - b} \left( aF(at) - bF(-bt) - ab \int_0^t F(as)F(-b(t - s))\, ds \right)$$

(15)

where

$$F(x) = I_0(x) + I_1(x), \quad a = \lambda + \mu, b = \lambda - \mu.$$

Similar methods are the key to modified Bessel function formulae for the queue state probabilities $q_n(t)$.
2.2. The busy period

The expression 'busy period' is used in queuing theory to denote the time a service point is occupied without a break and the number of customers served during that time. Let \( k_n(t) \) be the joint probability and pdf of the time interval between time zero when the particle (i.e. system state) is at 1, and time \( t > 0 \) when it makes first passage to 0, \( n \) customers having been served during this interval. To complement this the analysis requires \( l_n(t) \), similar to \( k \) except that the particle starts at 2 and makes first passage to 1. In random walk terminology the busy period concerns the first passage time \( T \), usually from \( N(0) = 1 \) to \( N(T) = 0 \), and \( v(T) \), the number of downward ladder points during \( T \). Using the symbol \( * \) for convolution and following the rules of the 'chemical' mechanism we get

\[
\begin{align*}
k_1(t) &= \lambda e^{-\lambda t}, \\
k_2(t) &= \mu e^{-\lambda t} \ast l_1(t) \ast k_1(t), \\
k_3(t) &= \mu e^{-\lambda t} \ast (l_1(t) \ast k_2(t) + l_2(t) \ast k_1(t)) \\
& \quad \vdots \\
k_n(t) &= \mu e^{-\lambda t} \ast (l_1(t) \ast k_{n-1}(t) + l_2(t) \ast k_{n-2}(t) + \cdots + l_{n-1}(t) \ast k_1(t)).
\end{align*}
\] (16)

By symmetry \( l_n(t) \) is \( k_n(t) \) with interchange of \( \lambda \) and \( \mu \). Let

\[
K(x, t) = \sum_{n \geq 1} k_n(t)x^n, \quad L(x, t) = \sum_{n \geq 1} l_n(t)x^n
\] (17)

and let \( K^*(x, z) \), \( L^*(x, z) \) be their Laplace transforms. Then

\[
\begin{align*}
K^*(x, z) &= \frac{\lambda x}{Z} + \frac{\mu}{Z} K^*(x, z)L^*(x, z) \\
L^*(x, z) &= \frac{\mu x}{Z} + \frac{\lambda}{Z} L^*(x, z)K^*(x, z).
\end{align*}
\] (18)

By eliminating \( L^* \) we find that \( K^* \) satisfies the quadratic equation

\[
\frac{1}{2}(a + b)Z(K^*)^2 - (Z^2 + abx)K^* + \frac{1}{2}(a + b)Zx = 0,
\] (19)

giving

\[
K^*(x, z) = \frac{Z^2 + abx - (Z^2 - a^2x)^{1/2}(Z^2 - b^2x)^{1/2}}{(a + b)Z}.
\] (20)

Here, \( a = \lambda + \mu, b = \lambda - \mu \), as usual and the minus sign is chosen so that \( K^*(1, 0) \leq 1 \) (since it is a probability). By using \( S(x) = 1 - (1 - x^2)^{1/2} \) we can reduce (20) to

\[
K^*(x, z) = \frac{abx}{(a + b)Z} + \frac{Z}{a + b} \left[ S\left(\frac{a\sqrt{x}}{Z}\right) + S\left(\frac{b\sqrt{x}}{Z}\right)\right] \\
- \frac{1}{(a + b)Z} S\left(\frac{a\sqrt{x}}{Z}\right) S\left(\frac{b\sqrt{x}}{Z}\right).
\] (21)

Using Erdélyi et al. (1954), p. 235, (28) in the form

\[
ZS\left(\frac{\Omega}{Z}\right) = \int_0^\infty e^{-zt} I_1(\Omega t) \frac{\Omega}{t} dt,
\] (22)
we obtain the modified Bessel function formula

\[ K(x, t) = \frac{abxe^{-at}}{a+b} + \frac{e^{-at}}{a+b} \left( \frac{a\sqrt{x}}{t} I_1(at\sqrt{x}) + \frac{b\sqrt{x}}{t} I_1(bt\sqrt{x}) \right) - \frac{e^{-at}abx}{a+b} \int_0^t ds \int_0^{t-s} \frac{I_1(au\sqrt{x})I_1(b(t-s-u)\sqrt{x})}{u(t-s-u)} \, du \]  

(23)

with \( a = \lambda + \mu, \ b = \lambda - \mu. \)

This may be checked against the known result for the \( M/M/1 \) queue (see, e.g., Conolly (1975)) by putting \( \lambda = \mu. \) Further comments on the busy period are made in the Appendix.

The coefficients \( k_n(t) \) of \( x^n \) in (25) are of particular practical as well as theoretical interest. They are the aggregate of the joint probabilities and pdfs of all the possible paths of the particle (system state) making its first passage from initial state 1 to final state 0 at time \( t, \) with \( n \) downward ladder epochs on the way corresponding to \( n \) served customers. To achieve this, \( n - 1 \) arrivals are needed in such a way that the system does not become empty before time \( t. \) It is known from Bertrand’s ballot theorem (see, for example, Feller (1957), Chapter III) that the total number of such paths is

\[ \frac{1}{n} \binom{2n-2}{n-1} \]

and, because of the underlying exponential distributions, it is intuitively clear that the form of \( k_n(t) \) is

\[ k_n(t) = \lambda \mu^2 e^{-at} \frac{t^{2n-2}}{(2n-2)!} d_n \quad (n \geq 2), \]

(24)

with

\[ k_1(t) = \lambda e^{-at}, \quad d_n = \sum_{m=0}^{n-2} d_{nm} \mu^{2m} \lambda^{2n-2m-4}, \]

\[ d_{n0} = d_{n,n-2} = 1, \quad \sum_{m=0}^{n-2} d_{nm} = \frac{1}{n} \binom{2n-2}{n-1}. \]

We see that the coefficients \( d_{nm} \) satisfy the recurrence

\[ d_{nm} = \frac{(n-m)(n-m-1)}{m(m+1)} d_{n-1} \quad (m \geq 1). \]

(25)

This gives

\[ d_{nm} = \frac{(n-1)(n-2)^2(n-3)^2 \cdots (n-m-2)^2(n-m-1)}{1 \cdot 2^2 \cdot 3^2 \cdots m^2 \cdot (m+1)} \]

\[ = \frac{1}{n-1} \binom{n-1}{m} \binom{n-1}{m+1} \quad (0 \leq m \leq n-2). \]

(26)
That these coefficients sum to the value given in (24) follows from the fact that the coefficient of $x^{n-2}$ in $(1+x)^{n-1}(1+x)^{n-1}$ is $(n-1)d_n$, and also the coefficient of $x^{n-2}$ in $(1+x)^{2n-2}$, giving

$$d_n = \frac{1}{n-1} \binom{2n-2}{n-2} = \frac{1}{n} \binom{2n-2}{n-1}. \quad (27)$$

As remarked in the earlier discussion on paths in Section 1, it would be extremely interesting to arrive at this result by identifying the paths that the particle can follow, but this seems more complicated than in the relatively simple case of the queue $M/M/1$.

Here is a particular example:

$$k_6(t) = \lambda \mu^2 e^{-at} \frac{t^{10}}{10!} \left( \frac{1}{\lambda^8} + \frac{10}{\lambda^6} \mu^2 + \frac{20\lambda^4}{\mu^4} \mu^4 + \frac{10\lambda^2}{\mu^2} \mu^6 + \mu^8 \right).$$

Appendix

A.1. Values of $q_n(t)$

Figure 1 shows the behaviour with time of $q_n(t)$ for $n = 0, 1, 5, 10, 15, 20, 25$. The initial system size is 1, i.e. $q_1(0) = 1$. The parameter values are $\lambda = 2, \mu = 1$. Time extends to 1200 units, when all the probabilities are for practical purposes equal to zero, as predicted by the theory. There is no significant difference in behaviour for other values of the parameters. Perhaps the most interesting feature of the curves is the way in which they were calculated. This is a widely applicable procedure due in detail here to Parthasarathy and Vijayalakshmi (1996), consisting in developing the Laplace transforms of the state probabilities as continued fractions and thence to series of rapidly converging exponentials. In this case, 25 terms of the series were retained until the remainder was negligible.

A.2. Further remarks on the busy period

The marginal pdf $k(t) = \sum_{n \geq 1} k_n(t) = K(1, t)$ can be found as follows: $k(t)$ is the pdf of the first passage time from $N(0) = 1$ to $N(t+) = 0$ for the randomized random walk. Consider $p_{10}(t)$, the probability that, starting from initial state $N(0) = 1$, $N(t) = 0$, not necessarily for the first time. To achieve this the particle must make a first passage to 0 in the interval $(0, t)$ and then be still, or again, at $N(t) = 0$. Thus $p_{10}(t) = k(t) \ast p_{00}(t)$, the asterisk denoting convolution. Under Laplace transformation this gives $p_{10}^*(z) = k^*(z) p_{00}^*(z)$, and so

$$k^*(z) = \frac{p_{10}^*(z)}{p_{00}^*(z)}.$$

But $p_{10}(t)$ is probabilistically identical to $p_{00-1}(t)$ upon interchange of $\lambda$ and $\mu$. Moreover, that interchange converts $p_{00-1}(t)$ into $p_{01}(t)$. Thus

$$k^*(z) = \frac{p_{01}^*(z)}{p_{00}^*(z)} = \frac{(aB + bA)(B - A)}{(a^2 - b^2)AB} \times \frac{AB}{Z}$$

$$= \frac{aB^2 - bA^2 - AB(a - b)}{(a^2 - b^2)Z} = \frac{(a - b)Z^2 - ab^2 + ba^2 - AB(a - b)}{(a^2 - b^2)Z}$$

$$= \frac{Z^2 + ab - AB}{(a + b)Z}.$$
**Figure 1:** Values of $q_n(t)$ for $n = 0, 1, 5, 10, 15, 20, 25$ starting from the top downwards. The starting point of the queueing process is $N(0) = 1, \lambda = 2, \mu = 1$.

It is also clear algebraically from (14) and (20) with $x = 1$ that

$$k^*(z) = \frac{S}{R} = \frac{q_{01}^*(z)}{q_{00}^*(z)},$$

the first subscript again denoting the starting point, this time of the restricted queueing random walk. But the argument used above in the framework of the randomized random walk should be valid and should give $q_{10}^*(z) = k^*(z)q_{00}^*(z)$. This means that $q_{10}^*(z)$ is formally identical to $q_{01}^*(z)$. The simplest proof is that the reverse of every $q$-path from 1 to 0 is a $q$-path from 0 to 1 having, because of the exponential mechanisms, the same probability. The result can be verified algebraically but it is very tedious.

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**References**


