



Pricing lifelong guarantees: mixed-exponential jump-diffusion risks

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Abstract

This paper focuses on the pricing of lifelong guarantees, to provide a stream of life-time income in a Variable Annuity contract, mitigating both longevity and market downside risks. The dynamics of the risky assets is described by a Mixed-Exponential Jump-Diffusion model to effectively capture key empirical characteristics of asset returns, including high kurtosis and leverage effects. The pricing problem is assessed with and without the surrender option. A sensitivity analysis is presented to investigate the different models studied and some numerical illustrations are provided.

Keywords Annuities · GLWB pricing · Jump-diffusion models · Lifetime income · Mortality models · Surrender · Variable annuity

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Abbreviations

$\mathbb{E}(\text{PV})$ Expected present value

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GBM	Geometric Brownian motion
GLWB	Guaranteed lifelong withdrawal benefit
GMAB	Guaranteed minimum accumulation benefit
GMDB	Guaranteed minimum death benefit
GMIB	Guaranteed minimum income benefit
GMWB	Guaranteed minimum withdrawal benefit
MEJD	Mixed-exponential jump-diffusion
NJD	Normal jump-diffusion
VA	Variable annuity

1 Introduction

A VA contract is a life insurance contract whose benefits are tied to a separate fund, consisting of bonds and stocks. These products are tax-deferred and provide protection against downward movements in the market. Under these contracts, the premium (usually a lump sum) is invested in a portfolio of mutual funds chosen by the policyholder. Depending on the type of contract, the policyholder receives a minimum benefit at maturity, a minimum death benefit, or a series of fixed payments. This insurance product typically includes an assurance benefit rider. The riders available with VA contracts fall into two categories: a death benefit rider, which provides the policyholder with a minimum guaranteed benefit in the event of death; and a living benefit rider, which offers certain guaranteed benefits for life.

Mainly, the available living benefit riders are: GMIB, GMAB, GMWB, and GLWB. In recent years, there has been growing interest in creating advanced models for pricing variable annuities, particularly those with GLWB. Traditional models, such as Black-Scholes model, are widely used but fail to represent the complex market dynamics of long-term contracts, resulting in errors in pricing and risk assessment. The GLWB rider is the most popular among retirees, [6], and it provides a lifetime income stream to a policyholder, regardless of the performance of the underlying investments, [19]. The cost for a GLWB guarantee is a fee, which is continuously deducted from the policyholder's fund. Therefore, pricing these guarantees involves calculating the fee at which the $\mathbb{E}(\text{PV})$ of inflows equals the $\mathbb{E}(\text{PV})$ of outflows.

Because GLWB is a very long-term contract, inspired by [19], this paper studies the pricing problem of a GLWB rider, modeling the dynamics of risky assets with the MEJD model, and modeling the risk-free asset using the Vasicek model correlated with the risky asset dynamics. A simplified pricing expression is obtained by a change of measure technique. Taking into account that the insured can die at any time of the year, although the GLWB literature considers annual mortality rates, using the Gompertz model, mortality becomes continuous.

The approach used in this paper to price GLWB riders using a MEJD model is new. The need for this study lies in the fact that, unlike traditional pricing models such as GBM or NJD, the MEJD captures key market characteristics such as high kurtosis and leverage effects on asset returns, providing a more realistic and nuanced pricing solution for long-term contracts. Furthermore, the paper incorporates stochastic interest rate modeling with the Vasicek model, resulting in more accurate

interest rate estimates over time. The inclusion of surrender and roll-up options in the analysis improves the model's realism while providing a comprehensive solution for measuring the value and risk of GLWB contracts.

In the first part, this paper evaluates a simple VA contract with a GLWB rider, constant withdrawal and no additional benefits. This model is then extended to incorporate surrender risk.

The three models considered, the MEJD model, NJD model and GBM model, are particular cases of Lévy processes. The GBM model is a sub-case of the MEJD model obtained by assuming $\lambda = 0$. Due to its analytical tractability, the GBM model is one of the most widely used models in literature for modeling stock prices. However, it ignores most of the characteristics present in stock prices. This is the reason why, to model the underlying index, we considered the MEJD model. For a simple GLWB contract, we obtain a continuous closed-form solution for the value of the death benefit conditional on the value of the fund at the last withdrawal date.

A numerical illustration is performed by comparing $\mathbb{E}(\text{PV})$ of GLWB using an NJD model and its approximated MEJD model. Furthermore, the proposed model is compared with the standard GBM model. A sensitivity analysis of the $\mathbb{E}(\text{PV})$ of a GLWB contract is then performed for various model parameters.

The paper is structured as follows. Section 2 presents a literature review related to the studied problem. Section 3 models the fund, the risk-free interest rate, and provides the mortality model. Section 4 models a simple VA fund value using the MEJD model with static policyholder behaviour and no additional benefits. Next, we priced a surrender option and roll-up option in a GLWB contract. Section 5 contains the numerical results and the sensitivity analysis related to the changes in parameters value. Some concluding remarks and future works are provided in Sect. 6.

2 Literature review

Recently, authors have focused on using comprehensive models for pricing GLWB guarantees. [9] proposed a regime-switching model, arguing that Black and Scholes model is inappropriate for such long maturity products. [4] computed the indifference price of a VA contract with a GLWB rider using the concept of equivalent utility. [12] applied the variance reduction technique to price a GLWB guarantee. [8] incorporated mortality risk where investment risk was modelled by Black and Scholes model. [1] extends the Cosine Fourier (COS) method to the pricing and hedging of variable annuities embedded with GMWB riders. The COS method facilitates the efficient computation of prices and hedge ratios of the GMWB riders, when the underlying fund dynamics evolve under the influence of the general class of Lévy processes. Equations are derived to value the contract at each withdrawal date using a backward recursive dynamic programming algorithm. Due to the long maturity of these guarantees, several recent papers have considered modelling interest rate risk. [10] proposed a stochastic interest rate and a stochastic volatility model for the pricing of GLWB. [5] proposed a 3D tree model for pricing GLWB contracts considering investment, interest rate and mortality risk.

A critical factor in determining the pricing of VA is the dynamic behavior of policyholders, particularly regarding surrender and withdrawal options. [17] emphasizes that insurers must take into account policyholder decisions, which can have a significant impact on risk management strategies and the pricing of GLWB contracts. [2] focuses on the valuation of surrender options within life insurance contracts, adding more depth by modeling surrender risk under dynamic conditions. To better capture volatility, [15] introduced complex multi-factor stochastic interest rate models. Hedging strategies for equity-linked annuities using jump-diffusion models have also proven effective in managing market risks, while Lévy processes have demonstrated versatility in pricing guaranteed benefits [14].

To find literature on modeling an index with a jump-diffusion process in a VA contract with a guaranteed minimum rider, we essentially need to consider GMDB, GMIB, GMAB, and GMWB riders. [13] investigated the pricing of GMDB and GMAB riders by modeling the index value as a jump-diffusion process with normal jumps. [16] provided a unified framework for pricing and hedging a VA contract with a GMIB, GMAB and GMDB rider, considering the Lévy process framework for index dynamics, concluding that the GBM assumption leads to undervaluation of the VA contract. In [3], the jumps are modeled as a mixture of exponential distributions with positive or negative weight. Therefore, the MEJD model generates a highly skewed leptokurtic distribution and is able to reproduce the key characteristics of index returns. Furthermore, recently, to assume dependence on past events, jumps are modeled by Hawkes processes with a general exciting function,[18].

3 GLWB contract description

A GLWB is a rider embedded to a VA contract that provides a lifetime annuity for the policyholder. In this contract, typically, the policyholder pays a lump-sum premium at the beginning of the contract, and chooses funds to invest in the premium from the insurer's list of fund options. The policyholder can then withdraw a fixed amount from the account at regular intervals throughout their lifetime. Withdrawals result in a deduction of fund at each withdrawal date. The remaining fund balance (if any) is paid to the beneficiaries at the time of policyholder's demise. At the same time, the insurance company charges a fee that is continuously deducted from the fund value.

For each time t , let W_t denote the fund value and $S(t)$ the risky asset's price. Initially, assume that $t = 0$, and that W_0 is the initial premium paid by the policyholder, and $S(0)$ is the initial asset price. Then, the number of asset units purchased is $\frac{W_0}{S(0)}$.

The rider's fee is assumed to be charged continuously as a δ percentage of the fund value. The withdrawals are assumed to take place at times $t_1, t_2, t_3, \dots$, and it can follow any regime, such as monthly, quarterly and yearly. Hence, $t_0 = 0$, and the times $t_1, t_2, t_3, \dots$ take values $\left\{\frac{1}{12}, \frac{2}{12}, \frac{3}{12}, \dots\right\}$, $\left\{\frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \dots\right\}$ and $\{1, 2, 3, \dots\}$, respectively, when withdrawals are monthly, quarterly and yearly. The policyholder, periodically, receives the G amount, and upon his demise, his beneficiaries will receive

the remaining fund value, W_t . Then, the evolution of policyholder's fund value, for $k = 0, 1, 2, \dots$, is given by

$$W_{t_{k+1}} = \left(W_{t_k} \exp(-\delta t_{k+1}) \frac{S(t_{k+1})}{S(t_k)} - G \right)^+,$$

where, as usual, $(y)^+ = \max(y, 0)$. Note that the fund value, W_t , remains zero once it reaches zero. Without loss of generality, in this paper, we assume annual withdrawals by the policyholder at the end of a year.

In the optimal scenario, the policyholder withdraws the amount G or surrenders the policy to get the fund value after the surrender penalty. The surrender penalty is applied as a percentage of the fund value and depends on the time of surrender. Typically, the penalty percentage decreases over time. Let $S_p(t)$ be the penalty percentage at any time t . Then, the surrender value, A_t , at any time t is

$$A_t = \left(1 - \frac{S_p(t)}{100} \right) W_t.$$

A_t is a one-time payment made upon surrender. The policyholder is entitled to either a death benefit (DB) or surrender benefit (SB). The contract terminates after one of the two benefits is paid.

A roll-up option is an option available to the policyholder, and added to the base contract with or without extra pay. With this option, GLWB withdrawals are delayed for a fixed period, and during this period, the guaranteed amount (G) increases at a constant rate until the insured begins withdrawals. Once the withdrawal begins, the guarantee is frozen and becomes a fixed amount for the remainder of the policy term.

Let i be the continuous rate at which the guarantee is compounded annually, and let y years be the roll-up period duration, i.e., the time for which withdrawals are not taken. Then, the new guarantee amount for the policyholder is $G_1 = G(1+i)^y$. Then, the evolution of W_t is given by

$$W_t = \begin{cases} W_0 \frac{S(t)}{S(0)} \exp(-\delta t) & \text{for } t < y \\ W_{t-1} \exp(-\delta t) \frac{S(t)}{S(t-1)} - G_1 & \text{for } t \geq y \end{cases}$$

where t is the time in years, the suffix $t-1$ is considered for fund value before a year, and, for $t = y$, then $t-1 = 0$.

At the beginning of the GLWB contract, let the policyholder be x year old, and let τ_x be a random variable denoting the policyholder's lifetime. Assuming that 110 is the maximum age lived by the policyholder, the maximum duration of the contract will be $T = 110 - x$.

Let ${}_t p_x$ denote the survival probability of a person of age x for a time t , and let μ_{x+t} be the instantaneous rate of mortality at time $x+t$. Then, the continuous-time distribution function of τ_x is

$$f_{\tau_x}(t) = {}_t p_x \mu_{x+t}, \quad \text{for } 0 \leq t \leq 110 - x.$$

Note that ${}_1 p_x$, also written as p_x , is the probability of a person of age x , survive to age $x + 1$. Let the mortality be modelled by the Gompertz model, namely, $\mu_x = Ae^{Bx}$, where A and B are given constants. Then, the survival and death probabilities are

$${}_t p_x = \exp \left\{ - \int_0^t \mu_{x+s} ds \right\}, \quad \text{and} \quad {}_t q_x = 1 - {}_t p_x,$$

where ${}_t q_x$ is the probability that a person of age x dying in next t years. For instance, ${}_1 q_x$, or q_x , is the probability that a person of age x dying before attaining the age $x + 1$. Substituting the values of μ_x , the survival probability function becomes

$${}_t p_x = \exp \left\{ - \frac{A}{B} e^{Bx} (e^{Bt} - 1) \right\}.$$

Consequently, the continuous lifetime distribution becomes

$$f_{\tau_x}(t) = {}_t p_x \mu_{x+t} = A \exp \left\{ - \frac{A}{B} e^{Bx} (e^{Bt} - 1) + B(x + t) \right\}. \quad (1)$$

4 Modelling fund

There are mainly two risks involved in a GLWB contract for the underlying fund value: the investment risk and the long-term interest rate risk.

Since, the period T for the GLWB contract is finite, let us assume that the uncertainty in the economy is modelled by $(\Omega, \mathcal{F}, \mathbb{Q}, \mathcal{F}_{t \in [0, T]})$, a filtered probability space. On this space, let the market be comprised by only two traded assets: a risky asset $S = \{S(t)\}_{0 \leq t \leq T}$ and a risk-free asset $r = \{r(t)\}_{0 \leq t \leq T}$. As usual, let $\mathbb{E}_{\mathbb{Q}}$ be the expectation with respect to the risk-neutral measure $\mathbb{Q}$, and let the asset price dynamics, $S(t)$, under $\mathbb{Q}$, be a jump-diffusion process such that

$$\frac{dS(t)}{S(t-)} = r(t)dt + \sigma_1 dB_{1,t} + d \left[\sum_{j=1}^{N(t)} Y_j - m\lambda t \right], \quad (2)$$

where

- $S(0)$ is the initial price of the asset,
- σ_1 is the constant volatility,
- $B_{1,t}$, $0 \leq t \leq T$, is a standard Brownian motion,
- $N(t)$, $t \geq 0$, is a Poisson process with parameter λ ,
- $Y_j := \log(1 + Z_j)$, $j = 1, 2, \dots$, are independent and identically distributed random variables, following mixed exponential distribution, and m is the expected

value of the random variable Z_j .

As in [3], assume that all sources of randomness are given by the processes N and B , and by the random variables Y_j , which are assumed to be independent by each others, and, for each j , are assumed to be independent of N and B . Thence, the probability density function (PDF) of Y_j is given by

$$f_Y(y) = p_u \sum_{i=1}^m p_i \eta_i e^{-\eta_i y} I_{|y| \geq 0} + q_d \sum_{j=1}^n q_j \theta_j e^{\theta_j y} I_{|y| < 0}, \quad (3)$$

where

- $p_u \geq 0$ is the probability of an upward jump,
- $q_d = 1 - p_u \geq 0$ is the probability of a downward jump,
- $p_i \in (-\infty, +\infty)$, $\eta_i > 1$, $i = 1, \dots, m$, and $\sum_{i=1}^m p_i = 1$,
- $q_j \in (-\infty, +\infty)$, $\theta_j > 0$, $j = 1, \dots, n$, and $\sum_{j=1}^n q_j = 1$.

In order to guarantee that $f_Y(y)$ is a density function, let $p_1 > 0$, $q_1 > 0$,

$$\sum_{i=1}^k p_i \eta_i \geq 0, \quad k = 1, \dots, m \quad \text{and} \quad \sum_{i=1}^l q_i \theta_j \geq 0, \quad l = 1, \dots, n.$$

Therefore,

$$m = \mathbb{E}(Z_j) = \mathbb{E}(e^{Y_j} - 1) = p_u \sum_{i=1}^m \frac{p_i \eta_i}{\eta_i - 1} + q_d \sum_{j=1}^n \frac{q_j \theta_j}{\theta_j + 1} - 1.$$

Using Itô's Lemma to solve Eq. (2), for $0 \leq s \leq t$, we get

$$S(t) = S(s) e^{\int_s^t r(u) du - \left(m\lambda + \frac{\sigma_1^2}{2}\right)(t-s) + \sigma_1 \int_s^t dB_{1,u} + \sum_{k=1}^{N(t-s)} \ln(1+Z_{t_k})}. \quad (4)$$

Given a and b positive constants, and the volatility $\sigma_2 > 0$, let the dynamics of risk-free interest rate process $\{r(t), 0 \leq t \leq T\}$ be

$$dr(t) = a(b - r(t))dt + \sigma_2 dB_{2,t}, \quad (5)$$

where the process $\{B_{2,t}, 0 \leq t \leq T\}$ is a standard Brownian motion, correlated to $\{B_{1,t}, 0 \leq t \leq T\}$, with correlation coefficient ρ , and it is independent to $\{Y_j, j = 1, 2, \dots\}$ and $\{N(t), t \geq 0\}$. Hence, the solution of Eq. (5) is given by

$$r(t) = r(s)e^{-a(t-s)} + b(1 - e^{-a(t-s)}) + \sigma_2 \int_s^t e^{-a(t-v)} dB_{2,v}.$$

Furthermore, by Fubini's theorem, we get

$$\begin{aligned} \int_s^t r(u) du &= \\ &= b(t-s) + \frac{(r(s)-b)}{a} (1 - e^{-a(t-s)}) + \frac{\sigma_2}{a} \int_s^t (1 - e^{-a(t-u)}) dB_{2,u}. \end{aligned}$$

Recalling that $\mathbb{E}_{\mathbb{Q}}$ is the expectation under the risk-neutral measure $\mathbb{Q}$, let $X(t, T)$ denote the price of a zero coupon bond with maturity T , at time t , which, as it is well known, admits the representation

$$X(t, T) = \mathbb{E}_{\mathbb{Q}} \left(e^{-\int_t^T r(s) ds} \mid \mathcal{F}_t \right). \quad (6)$$

Finally, by noting that $\int_t^T r(u) du \sim N(m_{t,T}, \sigma_{t,T}^2)$, where

$$\begin{aligned} m_{t,T} &= b(T-t) + \frac{(r(t)-b)}{a} (1 - e^{-a(T-t)}), \\ \sigma_{t,T}^2 &= \frac{\sigma_2^2}{a^2} \int_t^T (1 - e^{-a(T-u)})^2 du \\ &= \frac{\sigma_2^2}{a^2} \left(T-t - \frac{1}{a} (1 - e^{-a(T-t)}) - \frac{1}{2a} (1 - e^{-a(T-t)})^2 \right), \end{aligned}$$

we have that

$$X(t, T) = -m_{t,T} + \frac{1}{2} \sigma_{t,T}^2. \quad (7)$$

5 Pricing model

The problem of pricing a VA contract is the problem of the insurers that want to minimise the value of losses, namely, the difference between $\mathbb{E}(\text{PV})$ of outflows and $\mathbb{E}(\text{PV})$ of inflows.

The cash inflows are the premium paid by the insured, generally a one-time payment for a GLWB rider, and the cash outflows are the death benefit and living benefits paid by the insurance company to the insured. Therefore, obtaining a price for a GLWB contract means getting the break-even fee, which is the value of δ for which the $\mathbb{E}(\text{PV})$ of outflows equals the $\mathbb{E}(\text{PV})$ of inflows. For the policyholder, the outflows are the premium paid, W_0 , and inflows are the living benefits and death benefits. Denoting by

- $\mathbb{E}(LB_0) := \mathbb{E}(\text{PV})$ of living benefits i.e., the time zero value of living benefits, and
- $\mathbb{E}(DB_0) := \mathbb{E}(\text{PV})$ of death benefits,

the break-even fee is the value of δ for which the following equation holds

$$W_0 = \mathbb{E}(LB_0) + \mathbb{E}(DB_0).$$

As usual, all over this paper, by computation of fee we mean computation of the break-even fee.

Proposition 1 below allows us to derive the valuation of GLWB contract by a change of measure. The proof follows by applying Girsanov's Theorem.

Proposition 1 *Let the measure $\mathbb{Q}^{(r)}$, equivalent to $\mathbb{Q}$, be defined by the Radon-Nikodym derivative*

$$\begin{aligned} \eta(T) &= \frac{d\mathbb{Q}^{(r)}}{d\mathbb{Q}} = \frac{e^{-\int_t^T r(s)ds}}{\mathbb{E}\left(e^{-\int_t^T r(s)ds}\right)} \\ &= \exp\left\{-\int_t^T \frac{\sigma_2}{a} (1 - e^{-a(T-u)}) dB_{2,u} - \frac{1}{2} \int_t^T \frac{\sigma_2^2}{a^2} (1 - e^{-a(T-u)})^2 du\right\}. \end{aligned}$$

Then,

$$\begin{aligned} \widehat{B}_{1,t} &:= B_{1,t} + \rho \int_0^t \frac{\sigma_2}{a} (1 - e^{-a(T-u)}) du \\ \widehat{B}_{2,t} &:= B_{2,t} + \int_0^t \frac{\sigma_2}{a} (1 - e^{-a(T-u)}) du, \end{aligned}$$

are $\mathbb{Q}^{(r)}$ -Wiener processes with correlation coefficient ρ .

Hence, under the measure $\mathbb{Q}^{(r)}$, the dynamics of $S(t)$, the risky asset's price given in Eq. (4), admits the representation

$$\begin{aligned} S(t) = S(s) \exp \bigg\{ & b(t-s) - \left(m\lambda + \frac{\sigma_1^2}{2}\right)(t-s) + \frac{(r(s)-b)}{a} (1 - e^{-a(t-s)}) \\ & - \frac{\sigma_2^2}{a^2} \int_s^t (1 - e^{-a(t-u)}) (1 - e^{-a(T-u)}) du - \rho\sigma_1\sigma_2 \frac{1}{a} \int_s^t (1 - e^{-a(T-u)}) du \\ & + \sigma_1 \int_s^t d\widehat{B}_{1,t} + \frac{\sigma_2}{a} \int_s^t (1 - e^{-a(t-u)}) d\widehat{B}_{2,t} + \sum_{k=1}^{N(t-s)} \ln(1 + Z_{t_k}) \bigg\}. \end{aligned}$$

To evaluate $\mathbb{E}(LB_0)$ and $\mathbb{E}(DB_0)$, the expected time zero values of living benefits and death benefits, recall that the living benefits consists of a fixed lifetime guarantee of amount G , and the amount remaining in the policyholder's fund at the policyholder's demise is provided to the beneficiaries as death benefit. To avoid too many technicali-

ties, for the rest of the paper, we assume that withdrawals occur yearly at time points $\{1, 2, 3, \dots, T\}$. Note that $\mathbb{E}(LB_0)$ is given by

$$\mathbb{E}(LB_0) = G \sum_{k=1}^T e^{-\int_0^k r(s)ds} {}_kP_x.$$

To evaluate $\mathbb{E}(DB_0)$, note that the fund value after fee deduction is

$$S^\delta(t) = S^\delta(s) \exp(D_s^t + J_s^t), \quad (8)$$

where the diffusion component and the jump component are respectively given by

$$D_s^t = \int_s^t r(u)du - \left(\frac{\sigma_1^2}{2} - \delta\right)(t-s) + \sigma_1 \int_s^t dB_{1,u} \quad (9)$$

and

$$J_s^t = \sum_{k=1}^{N(t-s)} \ln(1 + Z_{t_k}). \quad (10)$$

Deducting constant withdrawals amounts of G from W_t , for GLWB we get that

$$W_1 = \left(W_0 \frac{S^\delta(1)}{S^\delta(0)} - G\right)^+, \quad W_{k+1} = \left(t_k \frac{S^\delta(k+1)}{S^\delta(k)} - G\right)^+, \quad k = 1, 2, 3, \dots, T.$$

Between successive withdrawal dates, k to $k+1$, the dynamics of the fund evolution must take into account Eq. (8) with initial value W_k .

Thence, the death benefit is the positive fund value available at τ , the time of death of the insured, then

$$DB_\tau = \max\{W_\tau, 0\}. \quad (11)$$

Theorem 2 *The present value of death benefit $\mathbb{E}(DB_0)$ of a plain GLWB contract is*

$$\mathbb{E}(DB_0) = A \sum_{k=0}^{T-1} W_k \int_{\tau=k}^{k+1} X(0, \tau) R(\tau, k, x) d\tau,$$

where $X(0, \tau)$ is given in Eq. (7),

$$R(\tau, k, x) = \exp\left(\lambda m(\tau - k) + \mu_\tau + \frac{1}{2}\sigma_\tau^2 - \frac{A}{B}e^{Bx}(e^{B\tau} - 1) + B(x + \tau)\right), \quad (12)$$

$$\begin{aligned}\mu_\tau &= b(\tau - k) + \frac{(r(k) - b)}{a} (1 - e^{-a(\tau-k)}) - \int_k^\tau (1 - e^{-a(\tau-u)}) (1 - e^{-a(T-u)}) du \\ &\quad - \left(\lambda + \frac{\sigma_1^2}{2} - \delta \right) (\tau - k) - \rho \sigma_1 \sigma_2 \frac{1}{a} \int_k^\tau (1 - e^{-a(T-u)}) du,\end{aligned}$$

and

$$\sigma_\tau^2 = \frac{\sigma_2^2}{a} \int_k^\tau (1 - e^{-a(\tau-u)})^2 dt + 2\rho\sigma_1\sigma_2 \frac{1}{a} \int_k^\tau (1 - e^{-a(\tau-u)}) du + \sigma_1^2(\tau - k).$$

Proof. Let τ be the time of death, then the expected value of death benefit is

$$\mathbb{E}[DB_\tau | \tau_x = \tau] = \mathbb{E}[\max\{W_\tau, 0\} | \tau_x = \tau].$$

Therefore, the $\mathbb{E}(\text{PV})$ of death benefit at inception is given by

$$\mathbb{E}(DB_0) = \mathbb{E} \left[\mathbb{E} \left[e^{-\int_0^\tau r(t)dt} DB_\tau | \tau_x = \tau \right] \right].$$

By the results reached in Proposition 1, Eq. (6) and the distribution of lifetime random variable, τ_x , given in Eq. (1), we have that

$$\begin{aligned}\mathbb{E}(DB_0) &= \int_0^T \mathbb{E} \left[e^{-\int_0^\tau r(t)dt} \right] \mathbb{E}_{\mathbb{Q}(r)} [DB_\tau] f_{\tau_x}(\tau) d\tau \\ &= \int_0^T X(0, \tau) \mathbb{E}_{\mathbb{Q}(r)} [DB_\tau] A e^{\left(-\frac{A}{B} e^{Bx} (e^{B\tau} - 1) + B(x+\tau)\right)} d\tau \\ &= \sum_{k=0}^{T-1} \int_k^{k+1} X(0, \tau) \mathbb{E}_{\mathbb{Q}(r)} [DB_\tau] A e^{\left(-\frac{A}{B} e^{Bx} (e^{B\tau} - 1) + B(x+\tau)\right)} d\tau.\end{aligned}$$

Between successive withdrawal dates, $k < \tau \leq k+1$, taking into account Eq. (8), the fund evolution W_τ is positive for $W_k > 0$, and $\max\{W_\tau, 0\} \equiv W_\tau$. Hence,

$$\mathbb{E}_{\mathbb{Q}(r)} [DB_\tau] = W_k \mathbb{E}_{\mathbb{Q}(r)} \left[e^{D_k^\tau + J_k^\tau} \right], \quad k < \tau \leq k+1,$$

where D_k^τ and J_k^τ are given, respectively, by Eqs. (9) and (10) and are supposed to be independent, then

$$\mathbb{E}_{\mathbb{Q}(r)} [DB_\tau] = W_k \mathbb{E}_{\mathbb{Q}(r)} \left[e^{D_k^\tau} \right] \mathbb{E}_{\mathbb{Q}(r)} \left[e^{J_k^\tau} \right] = W_k e^{\mu_\tau + \frac{1}{2}\sigma_\tau^2 + \lambda m(\tau-k)}. \quad \square$$

Let ${}_1q_{x+t}^s$ denotes probability of surrendering the policy between the time points t and $t+1$ for the policyholder aged x . Assume that the surrender can happen only at

the end of a year. Then, for the probability of survival and no surrender till time t , denoted by ${}_t p_x^s$, we get that

$${}_t p_x^s = {}_{t-1} p_x^s * {}_1 p_{x+t-1} (1 - {}_1 q_{x+t-1}^s), \quad \text{for } t > 1, \quad \text{and} \quad {}_1 p_x^s = {}_1 p_x (1 - {}_1 q_x^s).$$

First, we compute an expression for living benefits when a surrender option is available to the insured. Recall that, in this case, an insured get living benefits if he/she is alive and has not surrendered the contract. Therefore, the expression for $\mathbb{E}(LB_0)$ with surrender option is obtained analogously to a contract with no surrender, since, here, we use the probability of survival and no surrender (${}_k p_x^s$) instead of the survival probabilities(${}_k p_x$). Hence, the living benefit value for a contract with a surrender option is given by

$$\mathbb{E}(LB_0) = \sum_{k=1}^T {}_k p_x^s G X(0, k).$$

An insured gets either a surrender benefit or a death benefit, but cannot get both of them. The insurer gives a death benefit to the insured if the contract is not surrendered. Now we compute the surrender benefit and the death benefit together, and we denote the sum of them $\mathbb{E}(DB_0)$. The $\mathbb{E}(\text{PV})$ of the death benefit and surrender benefit is

$$\mathbb{E}(DB_0) = \sum_{k=0}^{T-1} \int_k^{k+1} X(0, \tau) E^{Q^r} [DB_\tau] {}_\tau p_x^s \mu_{x+\tau} d\tau + \sum_{k=1}^T {}_k p_x^s {}_1 q_{x+k}^s (1 - S_p(k)) W_k. \quad (13)$$

The integral term in Eq. (13) is the death benefit value, and the summation term is the surrender benefit value.

Let y be the roll-up period duration and i the continuous rate at which the guarantee is compounded annually. Then, the living benefit is a lifelong annuity of G_1 starting at time y , and the death benefit is the fund value W_t^R . Recalling that $W_{y-1}^R = W_0$ and that $S^\delta(y-1) = S^\delta(0)$, the evolution of the fund in case of roll-up option is given as

$$W_t^R = W_0 \frac{S^\delta(t)}{S^\delta(0)}, \quad \text{for } t = 1, 2, 3, \dots, y-1,$$

$$W_t^R = \left(W_{t-1}^R \frac{S^\delta(t)}{S^\delta(t-1)} - G_1 \right)^+, \quad \text{for } t = y, y+1, y+2, y+3, \dots, T.$$

The $\mathbb{E}(\text{PV})$ of living benefit is given by

$$\mathbb{E}(LB_0) = \sum_{k=y}^T e^{-\int_0^k r(s) ds} G_1 {}_k p_x.$$

Theorem 3 *The present value of death benefit $\mathbb{E}(DB_0)$ of a GLWB contract with a roll-up option is*

$$\mathbb{E}(DB_0) = AW_0 \int_0^y X(0, \tau) R(\tau, k, x) d\tau + A \sum_{k=y}^T W_y^R \int_k^{k+1} X(0, \tau) R(\tau, k, x) d\tau,$$

where $X(0, \tau)$ and $R(\tau, k, x)$ are given respectively by Eqs. (7) and (12).

The proof follows the same procedure used to prove Theorem 2.

6 Numerical illustrations and sensitivity analysis

This section analyses the GLWB model numerically, by considering the premium paid in the contract to be a lump sum premium of 100, paid at the inception of the GLWB contract. The mortality model is a Gompertz Model with parameters $A = 0.0001$ and $B = 0.0824$. To obtain the death benefit value, we perform 10,000 simulations for yearly values of W_k , $k = 1, 2, \dots$. Then, Theorem 2 allows us to get $\mathbb{E}(PV)$ of the continuous death benefit value.

First, $N(0, 0.01^2)$ is approximated by a mixture of ten exponential or five double exponential random variables. Following [3], the parameters are given by $p_u = 0.5$, $q_d = 0.5$, and

$$p = (8.7303, 2.1666, -10, 0.0622, 0.0409),$$

$$q = (8.7303, 2.1666, -10, 0.0622, 0.0409),$$

$$\eta = (213.0215, 236.0406, 237.1139, 939.7441, 939.8021),$$

$$\theta = (213.0215, 236.0406, 237.1139, 939.7441, 939.8021).$$

Then, we show the quality of approximation by comparing the $\mathbb{E}(PV)$ of a GLWB contract (denoted by GLWB value) under the two models NJD and MEJD. Recalling that the GLWB value is the sum of living benefits and death benefits, we assume $\lambda = 5$ for the rate of jumps, and diffusion parameters equal for both models (see Table 2). Table 1 contains the GLWB values for the two models corresponding to different δ values, note that the maximum difference is 0.11.

As the age at inception varies from 50 to 80 years, Fig. 1 shows the behaviour of these two models with age. Note that the GLWB value decreases with age at inception. Also, the GLWB value coincides for the two models after 60 years of age.

For numerical analysis, consider the pricing of a GLWB contract under a MEJD model with the same parameters given in Table 2. Here, we consider the price as the

Table 1 $\mathbb{E}(PV)$ of plain GLWB contract for a Normal Jump (NJ) diffusion model and for a MEJD model approximation of NJ model

Fee(δ)	$x = 60$		$x = 70$	
	NJD	MEJD	NJD	MEJD
0	121.32	121.21	113.09	113.05
100	108.05	107.96	103.74	103.71
200	98.14	98.06	96.11	96.08
300	90.70	90.65	89.86	89.84
400	85.09	85.05	84.72	84.70
500	80.83	80.80	80.48	80.46

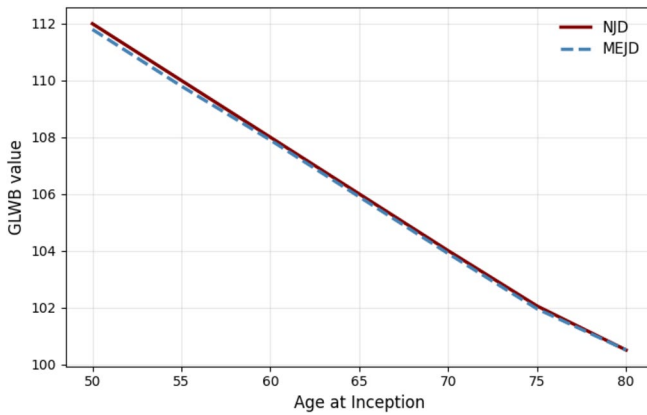


Fig. 1 Comparison of GLWB value for NJD and MEJD models as an approximation of NJD model with the parameters used in Table 1

Table 2 Default parameter values

Parameters	Values	Parameters	Values
a	0.85837	p	$(1.2, -0.2)$
b	0.089102	q	$(1.3, -0.3)$
ρ	0.1	η	$(30, 50)$
σ_1	0.2	θ	$(30, 40)$
σ_2	0.1	g	7%
λ	0.5	δ	1%
p_u	0.4	μ_N	0.036
q_d	0.6	σ_N	$\sqrt{0.0012}$
Age(x)	60	Roll-up rate	5% and 7%
T	$110 - x$	${}_1q_{x+t}^s$	0.01
g	0.03	$S_p(t)$	$0.03(1 - .07)^{t-1}$

Table 3 Break-even fee rate for MEJD model where unspecified parameters values are taken from Table (2)

Age	$\sigma_1 = 20\%$			$\sigma_1 = 30\%$		
	$g = 6$	$g = 7$	$g = 8$	$g = 6$	$g = 7$	$g = 8$
50	74.35	102.11	159.45	124.90	174.70	262.84
60	62.18	81.91	119.38	105.76	144.46	205.96
70	44.83	55.49	75.29	78.76	103.50	140.40
80	20.66	23.80	30.53	39.70	50.87	67.61

break-even fee. Table 3 shows the value of break-even fee for different values of age at inception, guarantee, and volatility of the risky asset. Analyzing these results we are able to deduce that the break-even fee is an increasing function of σ_1 , and, opposite to this, that increase in age results in a reduction in the fee. This behaviour is because of the shorter withdrawal periods for older policyholders than younger ones.

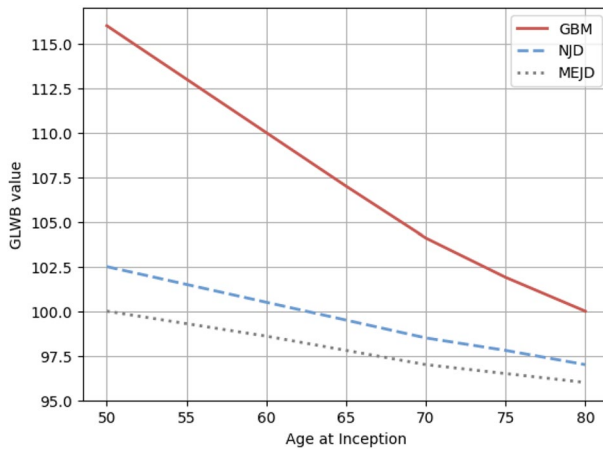
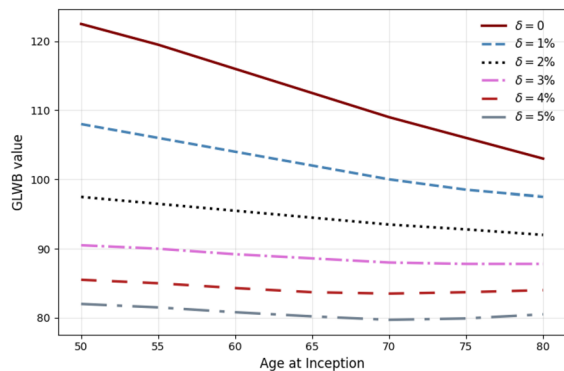


Fig. 2 Comparison between GLWB value for GBM, NJD and MEJD model with common diffusion parameters and jump parameters (μ_N, σ_N^2) for NJD model and default parameters for MEJD model as mentioned Table 2

Fig. 3 $\mathbb{E}(\text{PV})$ of GLWB corresponding to different fee, δ , values as age at inception increases



A comparison among GLWB values obtained from the GBM model, NJD model and MEJD model are shown in Fig. 2. The diffusion parameters for all three models are the same. The parameters for the jump part of the NJD and MEJD models are chosen such that the mean and variance of the jump size distribution are equal. The parameter values are mentioned in Table 2. Figure 2 shows that the three models generate completely different GLWB values. For a stock with high Kurtosis and leverage effect, the GBM and NJD models give misleading results. For instance, in our numerical illustration, the GBM model or NJD model give higher GLWB values. Hence, a higher fee than the MEJD model is required resulting in overpricing for the product. This observation becomes more evident by seeing the sensitivity of GLWB value with variations in δ values.

By performing a sensitivity analysis on the GLWB, looking at Fig. 3, we note that the parameter δ have no impact on the value of the living benefits, but the death benefit value decreases with increasing values of δ . Therefore, an increase in δ leads to a decrease in the GLWB value. To get the break-even fee, we are required to find

Fig. 4 Effect of σ_1 on GLWB value with increasing age

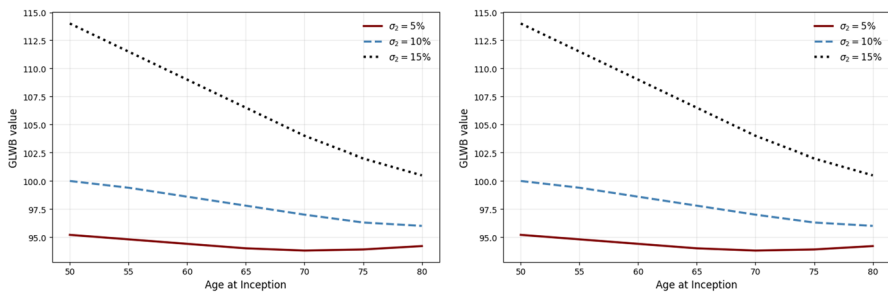
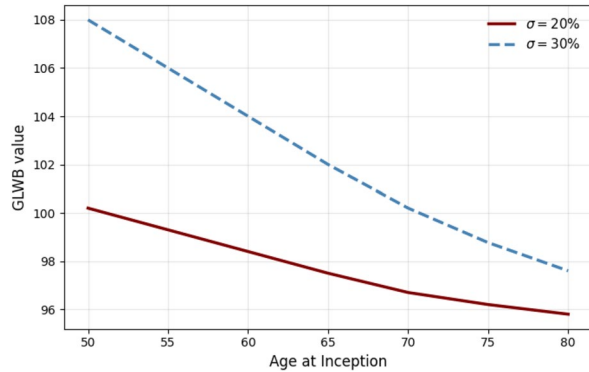


Fig. 5 Effect of changes in values of σ_2 and ρ on the GLWB value with respect to age

that value of δ for which $\mathbb{E}(\text{PV})$ of inflows is equal to $\mathbb{E}(\text{PV})$ of outflows, for instance, by a method of interpolation. Fixing δ , if GLWB value at δ is greater than W_0 , then δ should be increased to reduce the GLWB value. Similarly, if the GLWB value at δ is less than W_0 , then δ should be decreased to increase the GLWB value.

For increasing values of σ_1 , Fig. 4, the GLWB value increases. The parameter σ_1 does not affect the living benefit value, but it affects the GLWB value by changing the death benefit value only. This means that the contract becomes more volatile, implying an increase in the GLWB value (Fig. 5).

The high benefit value requires a higher fee to equate the benefit amount to the initial premium. The effect of the variations in the guarantee on GLWB value is shown in Fig. 6. Unlike σ_1 and δ , the guarantee, g , affects both the living and death benefit, since an increase in g , increases the living benefit value, and decreases the death benefit value. Further, as age increases, the living benefit values drop because of a reduced number of withdrawals compared to a younger policyholder. However, the decrease in death benefit value dominates the increase in living benefit value, and the overall impact on the GLWB value decreases. In addition, as age at inception increases and is close to ages 80 and above, the effect of variation in GLWB value corresponding to g value diminishes. As for older policyholders, a crucial part in pricing is played by the death benefit and not by the living benefit. Insurance firms can use this impact of g on older people to attract older customers by offering them a high guarantee value. The interest rate model parameters, σ_2 and ρ , affect both the living benefit and death

Fig. 6 $\mathbb{E}(\text{PV})$ of GLWB corresponding to different guarantee, g , values as age at inception increases

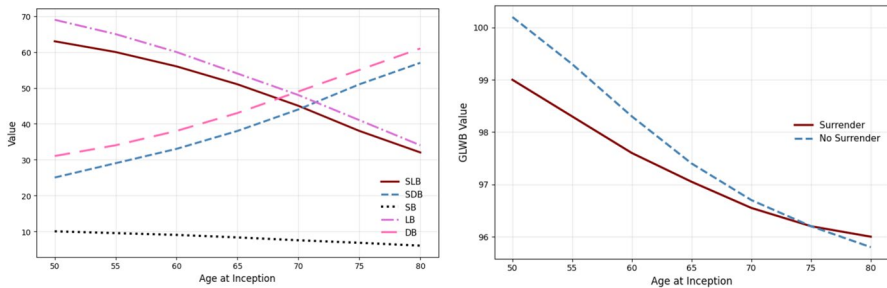
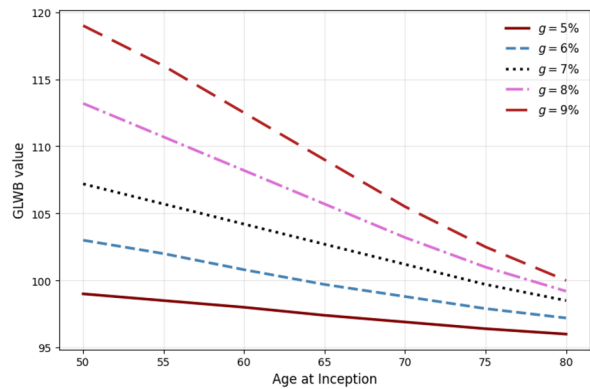


Fig. 7 Behaviour of $\mathbb{E}(\text{PV})$ of Living Benefit (LB), Death Benefit (DB), Surrender Benefit (GB), and GLWB value of contracts with and without surrender benefit as age at inception increases

benefit. Interest rates are used in discounting the living benefit values and affect the death benefit as they are part of the risky asset model. Figure 5 shows that an increase in σ_2 increases the volatility of the risky asset and, hence, gives a higher death benefit value dominating the decrease in living benefit value. Contrary to this, an increase in ρ leads to a reduction in the GLWB value.

Considering the impact of the surrender and roll-up option on the GLWB value, we assume the surrender rate in a year to be 1% with a surrender penalty as given in Table 2. As defined, the surrender penalty reduces as the contract term increases. Figure 7 shows a comparison of living benefit, death benefit and GLWB value of a contract with and without surrender. By Fig. 7a, note that the living benefit and death benefit value of a contract with surrender is always less than that with no surrender. But, this difference is more significant for the younger policyholders, since it decreases with increasing age at the beginning. The effect of this vanishing difference in living benefits and death benefits and the presence of surrender benefit becomes prominent Fig. 7b. Like surrender, a roll-up contract with 5 or 10 deferment years and a roll-up rate of 5% provides a lesser GLWB value than a GLWB contract with no roll-up.

Observing the living and death benefits, Fig. 8a shows that the living benefit for a contract with no roll-up is higher than a contract with a 5-years roll-up. Similarly, the LB value for the contract with a 5-years roll-up is greater than the 10-years roll-up

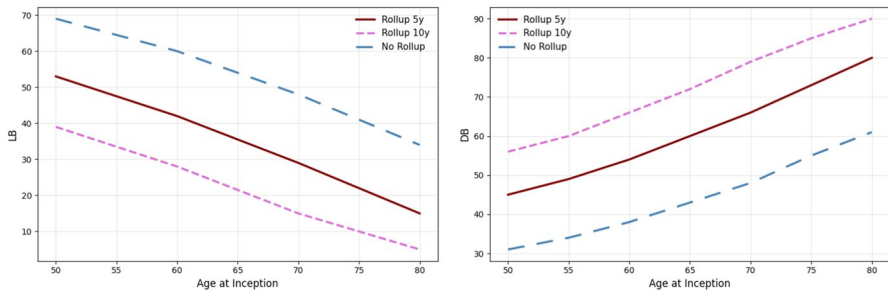
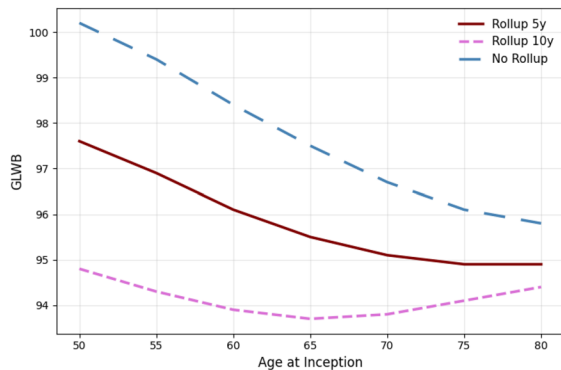


Fig. 8 $E(PV)$ of Living Benefit (LB) and Death Benefit (DB) value with respect to Age, with 5 years and 10 years deferred roll-up, and without roll-up contract

Fig. 9 GLWB value with respect to Age, with 5 years and 10 years deferred roll-up, and without roll-up contract



contract value. Whereas in the case of the death benefit, the value of contract with 10-years roll-up is highest, 5-years roll-up is second highest, and no roll-up option is lowest (See Fig. 8b). In the combined LB and DB values (GLWB value), the LB values dominate these contracts' total GLWB value. Therefore, the relation between the GLWB value of the three contracts is the same as the relation between the LB values of the three contracts Fig. 8b. Hence, it is not advisable to delay the withdrawals for a roll-up rate of 5%. Furthermore, as the age at inception comes closer to 80, the variations in the GLWB values due to these additional benefits vanish as the negative difference of values for living benefits is compensated by the positive difference in death benefit, Fig. 9. Now, we set the roll-up rate to be 8%. The graph of the living benefit and death benefit curve of the three contracts, i.e. contract with roll-up 5 years, 10 years and no roll-up, seems similar as in the case of 5% increment rate (See Fig. 10). Whereas the death benefit graph has changed substantially. Looking at the behaviour of the GLWB value for the three contracts, see Fig. 11, we get that, in case of high roll-up rates, it can be feasible for the younger policyholder to surrender the contract in the initial years.

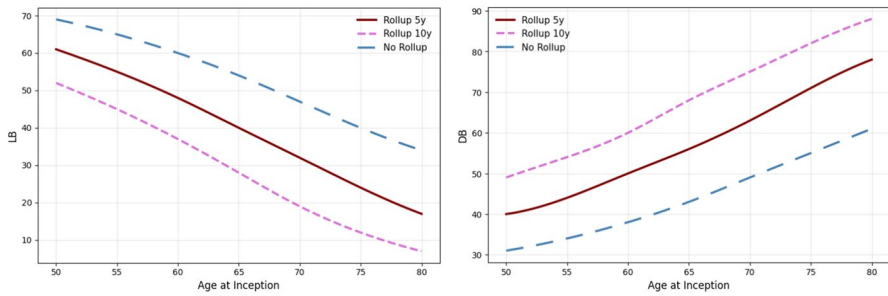
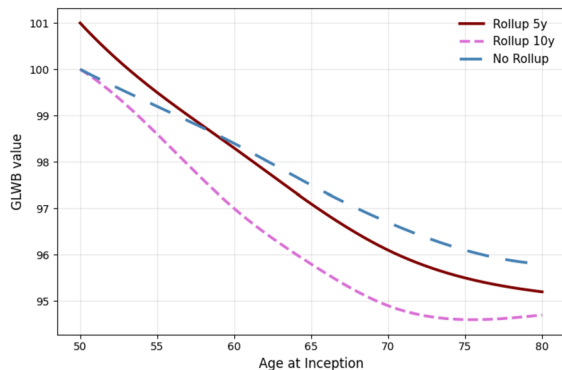


Fig. 10 $E(PV)$ of Living Benefit and Death Benefit value with respect to Age, with 5 years and 10 years deferred roll-up, and without roll-up contract with roll-up rate 8%

Fig. 11 GLWB value with respect to Age, with 5 years and 10 years deferred roll-up, and without roll-up contract with roll-up rate 8%



7 Conclusion and future work

The numerical illustration shows that the GLWB values generated by the three models differ significantly: the MEJD model outperforms both the GBM and NJD models. Additionally, we included the consequences of the surrender and roll-up options to improve the realism of our modeling method. The results are especially noteworthy when compared to a conventional VA contract, demonstrating that the incorporation of these elements has a major impact on the annuity's fees.

Future works can take into account more general assumptions making the model closer to the real life scenario, such as considering a time-varying fund volatility instead of a constant fund volatility, and also modelling fund volatility in a stochastic framework. Previous steps in this direction are given by [21–23] that modelled fund volatility using time series models, fractional Brownian motion environments with jumps and correlated jump-diffusion processes.

Furthermore, the proposed model can be enhanced by include a surrender probability based on the fund's performance. Specifically, when the fund is performing well, policyholders are more likely to choose to renounce their assets. In such cases, investors may be enticed to cash out their earnings, increasing the likelihood of abandoning the annuity. Accounting for the relationship between fund performance and

surrender behavior allows the model to better capture the dynamics of investor decision-making, resulting in more precise estimates and judgments.

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Consent for publication Not applicable.

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