



Vulnerable power exchange options with liquidity risk

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ABSTRACT

Modeling the impact of liquidity risk and counterparty default risk on power exchange options is a crucial problem in finance. This paper presents a stochastic model that considers these risks for more accurate pricing of power exchange options. Liquidity risk is considered using a liquidity discount factor, whereas counterparty risk is assessed with firm value models. To derive the pricing formula, the proposed model is transformed into a risk-neutral measure, which is then simplified by applying the Fourier transform to the payoff function and then expressed in terms of the characteristic function. This characteristic function is derived by solving a system of differential equations. To verify the accuracy of the derived pricing formula, a comparison with the simulated option prices is provided. Through numerical analysis, the significance of considering market liquidity and its impact on options prices is illustrated. It has been observed that the liquidity risk increases the options premium. Furthermore, the sensitivity of options prices with respect to various parameters is presented.

1. Introduction

Power exchange options are a generalization of the power options and the exchange options. These options were introduced by Blenman and Clark [1]. The payoff of the power options is given by some predetermined positive power of the underlying asset price at the maturity of the option. These options allow the investors to implement a high-leverage strategy and hedge against nonlinear price risks. Additionally, for investors with a strong view of the market, writing power options yields significantly more premium income than a vanilla option. Exchange options are options that offer the holder the right to exchange the value of one asset for the value of another asset at a fixed maturity time. Exchange options help investors mitigate the risk against relative variation between the two assets.

Default risk is a major concern in over-the-counter (OTC) markets, which are a key part of the financial markets. Financial institutions actively trade derivative instruments in these markets, where transactions occur directly between counterparties rather than through a centralized exchange. Hence, unlike transactions on established exchanges, OTC transactions carry the risk that the counterparty may fail to fulfill its contractual commitments, resulting in the absence of guaranteed payments between the parties involved. Power exchange options are generally traded in the OTC markets hence are exposed to counterparty default risk. These options are vulnerable to credit risk therefore known as vulnerable options. The premium for these options depends on the writer's creditworthiness.

In literature, there are two main methods for modeling counterparty default risk: firm value models (also known as structural models) and reduced form models (also called intensity-based models). Firm value models are based on the Merton [2] model, which models the default based on the value of a firm's assets and its obligations at the time of maturity. The default occurs if the firm's asset values are less than its obligations. Reduced-form models do not explicitly model the price process of counterparty

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assets. Instead, they represent the default time as the time of the first jump of a jump process, where the default time depends on an intensity process. If the default time is less than the option's maturity time, the counterparty defaults.

Traditionally, vulnerable power exchange options have been modeled with the assumption that the market is frictionless and complete, implying that liquidity has no impact on the price of underlying assets. Wang et al. [3] extended the Wang [4] three Poisson jump process model, where one jump process models the correlated risk and the other two processes model the independent jump risk by considering the counterparty default risk. Pasricha and Goel [5] proposed an intensity-based framework to price vulnerable power exchange options under counterparty risk. Xu et al. [6] adopted a structural approach to model the default risk, allowing for the possibility of default before maturity while accounting for correlations between the two underlying assets and the issuer's assets. Yue et al. [7] modeled the vulnerable power exchange options with stochastic volatility and stochastic interest rate.

Liquidity significantly impacts the pricing of the financial assets [8]. Market liquidity refers to the level of ease with which a financial instrument can be traded or purchased in the market without significantly affecting its price. In a perfectly liquid market, it is assumed that transactions occur seamlessly at consistent prices. This assumption is unrealistic; consequently, assets in markets are highly vulnerable to liquidity risk. Several methodologies have been proposed for incorporating liquidity risk into the modeling of the price process of financial derivatives [9–11]. The liquidity risk premium is the compensation that investors demand for holding an asset with lower liquidity or that is more difficult to sell in the market.

Brunetti and Caldarera [12] examined the impact of liquidity risk on the asset price process by considering a liquidity discount factor that depends on overall market liquidity and asset sensitivity to market liquidity. Feng et al. [13] modeled the market liquidity function to be a mean-reverting stochastic process and developed the closed-form pricing formula for the European options. The liquidity discount factor, which depends on both stock and market liquidity, is motivated by the fact that “liquidity is risky and has commonality: it varies over time both for individual stocks and for the market as a whole” refer [14].

In financial markets, liquidity risk is important and significantly impacts asset pricing. Understanding its influence, numerous researchers have included liquidity risk to improve the accuracy of options pricing models. Li et al. [15] considered quanto options pricing with liquidity risk. Li et al. [16] analyzed the pricing of Asian options with fixed and floating strikes in illiquid markets. The asset price process is assumed to be driven by a liquidity-adjusted diffusion model. Li et al. [17] studied the valuation of discrete barrier options with liquidity risk. Xu et al. [18] considered the liquidity risk for pricing of variance and volatility swaps. Wang [19] considered the vulnerable options valuation within the framework of liquidity-adjusted asset price. Pasricha and He [20] derived the exchange options pricing formula by considering stochastic market liquidity. Wang et al. [21] studied the pricing of vulnerable basket options considering the liquidity risk. Cai et al. [22] modeled the spread options taking into account both liquidity and default risks. Zhang et al. [23] extended the American option pricing model by incorporating both liquidity risk and jump risk into the pricing framework.

Motivated by the importance of liquidity risk in options pricing, this article focuses on the pricing of vulnerable power exchange options with liquidity risk. To derive the pricing formula, the Fourier transform of the expectation function is calculated. The accuracy of the derived formula, the impact of liquidity, and the sensitivity of various other parameters on the options price are depicted numerically. It has been found that vulnerable power exchange options have lower premiums than non-vulnerable options. Furthermore, liquidity has been shown to have a significant impact on the price of power exchange options, with premiums being much higher when liquidity risk is taken into account.

The remainder of this paper is organized as follows: Section 2 provides a description of the model. Section 3 formulates the pricing problem and explains the pricing methodology. The numerical calculations are presented in Section 4, and the paper concludes with Section 5.

2. Model description

2.1. Empirical evidence

To demonstrate the structure of the market liquidity, a dataset of the S&P 500 index spanning from January 1, 2021, to December 31, 2024, obtained from Thomson Reuters, is considered. Following Feng et al. [13], it is assumed that market liquidity is the ratio of S&P 500 returns to the volume traded. All these computations are performed in Python using pandas, numpy, scipy.stats, matplotlib.pyplot, and statsmodels.tsa.stattools modules.

Fig. 1 presents the plot of the market liquidity from January 1, 2021, to December 31, 2024. It can be observed that liquidity fluctuates around the mean, with both upward and downward deviations. This trait indicates that liquidity adheres to a mean-reverting process, wherein the data typically reverts to its long-term mean. To strengthen the claim that liquidity is mean-reverting, three statistical tests, the Augmented Dickey–Fuller (ADF) test, Kwiatkowski–Phillips–Schmidt–Shin (KPSS) and Hurst Exponent analysis are performed.

1. ADF test is a statistical test that determines if a time series is stationary or non-stationary. The null hypothesis is that the time series is not stationary. The computed ADF statistic is -31.3409 with p -value 0.00 . Since the p -value < 0.05 , the null hypothesis is rejected, indicating that liquidity is mean-reverting.
2. The KPSS test is performed to check the stationarity of a time series. In this test, the null hypothesis is that the time series is stationary. The calculated KPSS statistic is 0.1462 , and the p -value is 0.1 . Here, p -value > 0.05 . Hence, the authors fail to reject the null hypothesis, and the series is stationary.

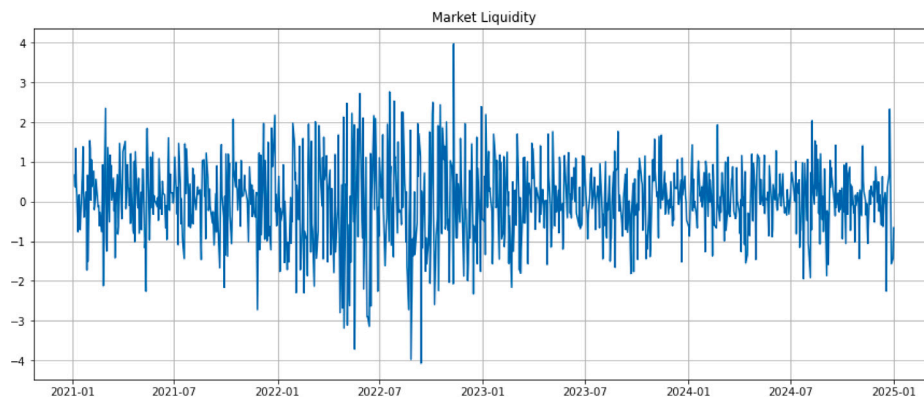


Fig. 1. Dynamics of liquidity level of S&P 500 from January 1, 2021 to December 31, 2024.

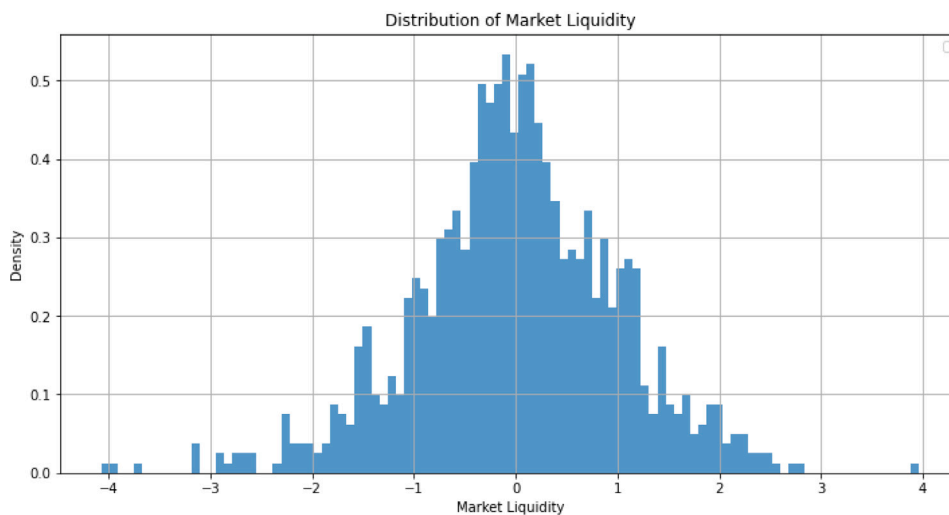


Fig. 2. Histogram depicting the distribution of the calculated market liquidity.

Table 1
Normality test results.

Test	Statistic	p-value	Result
Jarque–Bera Test	44.0871	0.0000	p-value < 0.05
Kolmogorov–Smirnov Test	0.0408	0.0686	p-value > 0.05

3. The Hurst Exponent is calculated to measure the long-term dependence of liquidity fluctuations. The estimated Hurst Exponent is 0.0059. Since Hurst Exponent < 0.5, liquidity exhibits anti-persistent (mean-reverting) behavior, consistent with the ADF and KPSS tests findings.

Fig. 2 plots the histogram of the market liquidity, depicting the frequency of various market liquidity levels. The form closely resembles a normal distribution, indicating that the liquidity data is approximately normally distributed. The Q–Q (Quantile–Quantile) plot to compare the distribution of the market liquidity data with normal distribution is plotted in Fig. 3. The graph depicts that there are only a few outliers. To test whether market liquidity follows a normal distribution or not, two statistical tests: Jarque–Bera test and Kolmogorov–Smirnov test are conducted. JB test assesses if the sample skewness and kurtosis match those of a normal distribution, whereas the KS test compares the cumulative distribution function (CDF) of the data with the CDF of a normal distribution. In both tests, the null hypothesis is that data follows a normal distribution. The null hypothesis for both tests is that the data follows a normal distribution. From Table 1, it can be observed that Jarque–Bera test rejects normality, whereas Kolmogorov–Smirnov test fails to reject normality. The results suggest that liquidity is not perfectly normal, but it is approximately Gaussian in its overall structure. As a result, it is reasonable to think of market liquidity as a mean-reverting process with a Gaussian distribution. For further details regarding the dependency structure of returns and liquidity, refer [23].

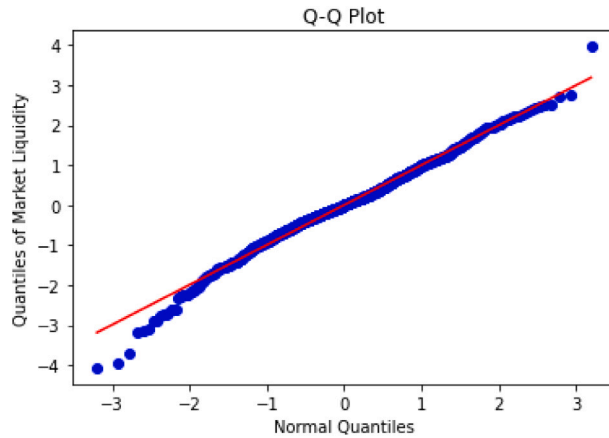


Fig. 3. Q-Q plot comparing market liquidity distribution to normal distribution.

2.2. Model specification

Let $(\Omega, \mathcal{F}, \mathbb{F}_{t \in [0, T]}, \mathbb{P})$ be a filtered probability space where $\mathbb{P}$ represents the real-world probability measure. It is assumed that there are two types of risky financial assets in the market, namely, asset 1 and asset 2. These assets have imperfect liquidity, which occurs when there is an excess or lack of assets in the market. Let $\tilde{S}_i(t)$ represent the price process of i th asset when the asset has perfect liquidity, i.e., demand is equal to the supply. Assume $\tilde{S}_i(t)$ follow the following stochastic differential equation (SDE) for $i = 1, 2$,

$$\frac{d\tilde{S}_i(t)}{\tilde{S}_i(t)} = \mu_i dt + \sigma_i d\tilde{W}_{i,S}(t), \quad (1)$$

where $\{\tilde{W}_{1,S}(t)\}_{t \geq 0}$ and $\{\tilde{W}_{2,S}(t)\}_{t \geq 0}$ are standard Wiener processes with $d\tilde{W}_{1,S}(t) d\tilde{W}_{2,S}(t) = \rho dt$. The expected return and volatility of the asset i are μ_i and σ_i respectively.

Let $S_1(t)$ and $S_2(t)$ be the price processes of asset 1 and asset 2 after considering the liquidity risk. The perfectly liquid asset price process is discounted using a liquidity discount factor process. Let $\{D_1(t)\}_{t \geq 0}$ and $\{D_2(t)\}_{t \geq 0}$ be the liquidity discount factors of asset 1 and asset 2. Let $\{L(t)\}_{t \geq 0}$ measure the market illiquidity and β_i measures the impact of market illiquidity on the return and volatility of i th asset. It is assumed that liquidity impacts the asset returns as well as volatility, hence the structure of the liquidity discount factor is assumed to be

$$D_i(t) = \exp \int_0^t \left(-\beta_i L(s) ds - \beta_i L(s) d\tilde{W}_{i,D}(s) \right).$$

Using Itô's formula it satisfies the following SDE,

$$\frac{dD_i(t)}{D_i(t)} = \left(-\beta_i L(t) + \frac{1}{2} \beta_i^2 L(t)^2 \right) dt - \beta_i L(t) d\tilde{W}_{i,D}(t), \quad (2)$$

where following Feng et al. [13],

$$dL(t) = \tilde{\kappa} (\tilde{\theta} - L(t)) dt + \xi d\tilde{W}_L(t),$$

where $d\tilde{W}_{1,D}(t) d\tilde{W}_L(t) = \rho_1 dt$ and $d\tilde{W}_{2,D}(t) d\tilde{W}_L(t) = \rho_2 dt$. If $L(t) > 0$ market is in short supply, and if $L(t) < 0$, then the market is in surplus. Fig. 1 also depicts that market liquidity fluctuates around taking both positive and negative values. In both cases, the market has imperfect liquidity. Market has perfect liquidity when $L(t) = 0$.

Assume that the supply for both the assets is fixed and is equal to $\tilde{S}_i$, and the demand functions $\tilde{D}_i(S(t), D_i(t), I_i(t))$ are given by

$$\tilde{D}_i(S_i(t), D_i(t), I_i(t)) = \Phi \left(\frac{I_i(t)^\nu}{D_i(t) S_i(t)} \right),$$

where $I_i(t)$ is the information process, Φ is a smooth, increasing function and ν is a positive constant. Under the market clearing condition, i.e., when supply and demand are equal the following equation holds,

$$\Phi \left(\frac{I_i(t)^\nu}{D_i(t) S_i(t)} \right) = \tilde{S}_i,$$

which implies

$$S_i(t) = \frac{1}{D_i(t)} \left(\frac{I_i(t)^\nu}{\Phi^{-1}(\tilde{S}_i)} \right).$$

If market is at perfect liquidity level, then $D_i(t) = 1$ which implies that

$$\tilde{S}_i(t) = \frac{I_i(t)^\nu}{g^{-1}(\tilde{S})}.$$

Hence,

$$S_1(t) = \frac{1}{D_1(t)} \tilde{S}_1(t) \text{ and } S_2(t) = \frac{1}{D_2(t)} \tilde{S}_2(t). \quad (3)$$

Therefore $S_1(t)$ and $S_2(t)$ follows

$$\begin{aligned} \frac{dS_1(t)}{S_1(t)} &= \left(\mu_1 + \beta_1 L(t) + \frac{1}{2} \beta_1^2 L(t)^2 \right) dt + \sigma_1 d\tilde{W}_{1,S}(t) + \beta_1 L(t) d\tilde{W}_{1,D}(t), \\ \frac{dS_2(t)}{S_2(t)} &= \left(\mu_2 + \beta_2 L(t) + \frac{1}{2} \beta_2^2 L(t)^2 \right) dt + \sigma_2 d\tilde{W}_{2,S}(t) + \beta_2 L(t) d\tilde{W}_{2,D}(t). \end{aligned}$$

To model the default risk, Klein [24] structural method is considered in which the value of counterparty assets is modeled. A default occurs when the value of the counterparty assets has a market value lower than the total claims D that are outstanding at maturity. Assume the counterparty assets price process considering the liquidity risk follows the following SDE

$$\frac{dV(t)}{V(t)} = \left(\mu_3 + \beta_3 L(t) + \frac{1}{2} \beta_3^2 L(t)^2 \right) dt + \sigma_3 d\tilde{W}_S(t) + \beta_3 L(t) d\tilde{W}_{3,D}(t), \quad (4)$$

where $d\tilde{W}_S(t)d\tilde{W}_{2,S}(t) = \tilde{\rho}dt$ and $d\tilde{W}_{3,D}(t)d\tilde{W}_L(t) = \rho_3dt$.

To calculate the prices of vulnerable options considering liquidity risk within the given framework, it is necessary to work under an equivalent martingale measure, where the discounted prices are martingales. Under the proposed model, the market is incomplete; hence, many equivalent martingale measures exist. To preserve the tractability of the model, it is assumed that the liquidity risk premium is $c_i \beta_i L(t)$. Assume,

$$\begin{aligned} \frac{d\mathbb{Q}}{d\mathbb{P}} &= \exp \left\{ - \int_0^t \lambda_1(s) d\tilde{W}_{1,S}(s) - \int_0^t \lambda_2(s) d\tilde{W}_{2,S}(s) - \int_0^t \lambda_3(s) d\tilde{W}_S(s) - \int_0^t c_1 \beta_1 L(s) d\tilde{W}_{1,D}(s) - \int_0^t c_2 \beta_2 L(s) d\tilde{W}_{2,D}(s) \right. \\ &\quad - \int_0^t c_3 \beta_3 L(s) d\tilde{W}_{3,D}(s) - \frac{1}{2} \int_0^t (\lambda_1^2(s) + \lambda_2^2(s) + \lambda_3^2(s)) ds - \frac{1}{2} (c_1^2 \beta_1^2 + c_2^2 \beta_2^2 + c_3^2 \beta_3^2) \int_0^t L^2(s) ds - \frac{1}{2} \int_0^t \rho \lambda_1(s) \lambda_2(s) ds \\ &\quad \left. - \frac{1}{2} \int_0^t \tilde{\rho} \lambda_2(s) \lambda_3(s) ds \right\}. \end{aligned}$$

To obtain an equivalent martingale measure, it is assumed

$$\begin{aligned} r + \sigma_1 \lambda_1(t) + \sigma_1 \rho \lambda_2(t) + c_1 \beta_1^2 L(t)^2 &= \mu_1 + \beta_1 L(t) + \frac{1}{2} \beta_1^2 L(t)^2, \\ r + \sigma_2 \lambda_2(t) + \sigma_2 \rho \lambda_1(t) + \sigma_2 \tilde{\rho} \lambda_3(t) + c_2 \beta_2^2 L(t)^2 &= \mu_2 + \beta_2 L(t) + \frac{1}{2} \beta_2^2 L(t)^2, \\ r + \sigma_3 \lambda_3(t) + \sigma_3 \tilde{\rho} \lambda_2(t) + c_3 \beta_3^2 L(t)^2 &= \mu_3 + \beta_3 L(t) + \frac{1}{2} \beta_3^2 L(t)^2. \end{aligned}$$

Applying Girsanov's theorem, the dynamics of $S_1(t)$, $S_2(t)$, $V(t)$ and $L(t)$ under the risk-neutral probability measure $\mathbb{Q}$ is given by,

$$\frac{dS_1(t)}{S_1(t)} = rdt + \sigma_1 dW_{1,S}(t) + \beta_1 L(t) dW_{1,D}(t), \quad (5)$$

$$\frac{dS_2(t)}{S_2(t)} = rdt + \sigma_2 dW_{2,S}(t) + \beta_2 L(t) dW_{2,D}(t), \quad (6)$$

$$\frac{dV(t)}{V(t)} = rdt + \sigma_3 dW_S(t) + \beta_3 L(t) dW_{3,D}(t), \quad (7)$$

$$dL(t) = \kappa (\theta - L(t)) dt + \xi dW_L(t), \quad (8)$$

where

$$\theta = \tilde{\theta} + \xi(\rho_1 c_1 \beta_1 + \rho_2 c_2 \beta_2 + \rho_3 c_3 \beta_3), \quad \kappa = \frac{\tilde{\kappa} \tilde{\theta}}{\theta}.$$

3. Options price

The proposed model is now defined; hence the pricing of power exchange options can be discussed under the proposed model. Section 3.1 presents the pricing of power exchange options with liquidity risk and counterparty default risk whereas in Section 3.2 the pricing formula for power exchange options with liquidity risk is presented. The pricing formula is presented as the function of the joint characteristic function.

3.1. Vulnerable power exchange options price with liquidity risk

The payoff of vulnerable power exchange options depends on the occurrence of default by the counterparty within the duration of the option. Thus, the payoff of power exchange options comprises two components. Let the maturity time of the power exchange option be T , then the default occurs if $V(T) < D$. In the absence of default, the payoff of power exchange options will be

$$\mathbb{1}_{\{V(T) \geq D\}} \left(\lambda_1 S_1^{(\alpha_1)}(T) - \lambda_2 S_2^{(\alpha_2)}(T) \right)^+ . \quad (9)$$

In case default occurs, then certain expenditures, known as dead-weight costs, must be paid due to bankruptcy which is a percentage of the value of the counterparty's assets. Let γ represent the recovery rate. In case of default, a proportion of claim $\frac{\gamma V(t)}{D}$ is recovered. Let $\lambda_1, \lambda_2, \alpha_1, \alpha_2$, and D be positive constants. The payoff function for power exchange option in case of default will be

$$\frac{\gamma V(t) \mathbb{1}_{\{V(T) < D\}}}{D} \left(\lambda_1 S_1^{(\alpha_1)}(T) - \lambda_2 S_2^{(\alpha_2)}(T) \right)^+ . \quad (10)$$

The vulnerable power exchange option price will be given by,

$$\begin{aligned} C(S_1(t), S_2(t), L(t), V(t), t, T) &= \mathbb{E}_{\mathbb{Q}} \left[e^{-r(T-t)} \left(\lambda_1 S_1^{(\alpha_1)}(T) - \lambda_2 S_2^{(\alpha_2)}(T) \right)^+ \left(\mathbb{1}_{\{V(T) \geq D\}} \frac{\gamma V(t)}{D} \mathbb{1}_{\{V(T) < D\}} \right) \middle| \mathcal{F}_t \right] \\ &= \lambda_1 e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}} \left[S_2^{(\alpha_2)}(T) \left(\frac{S_1^{(\alpha_1)}(T)}{S_2^{(\alpha_2)}(T)} - \frac{\lambda_2}{\lambda_1} \right)^+ \left(\mathbb{1}_{\{V(T) \geq D\}} + \frac{\gamma V(t) \mathbb{1}_{\{V(T) < D\}}}{D} \right) \middle| \mathcal{F}_t \right]. \end{aligned}$$

Let $\lambda = \left(\frac{\lambda_2}{\lambda_1} \right)$, $X_1(T) = \ln(S_1(T))$, $X_2(T) = \ln(S_2(T))$ and $X_3(T) = \ln(V(T))$. Then

$$\begin{aligned} C(S_1(t), S_2(t), L(t), V(t), t, T) &= \lambda_1 e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}} \left[e^{(\alpha_2 X_2(T))} \left(e^{(\alpha_1 X_1(T) - \alpha_2 X_2(T))} - \lambda \right)^+ \left(\mathbb{1}_{\{e^{X_3(T)} \geq D\}} + \frac{\gamma V(t)}{D} \mathbb{1}_{\{e^{X_3(T)} < D\}} \right) \middle| \mathcal{F}_t \right] \\ &= \lambda_1 e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}} \left[e^{(\alpha_2 X_2(T))} \left(e^{(\alpha_1 X_1(T) - \alpha_2 X_2(T))} - \lambda \right)^+ \mathbb{1}_{\{e^{X_3(T)} \geq D\}} \middle| \mathcal{F}_t \right] \end{aligned} \quad (11)$$

$$\begin{aligned} &+ \lambda_1 e^{-r(T-t)} \frac{\gamma V(t)}{D} \mathbb{E}_{\mathbb{Q}} \left[e^{(\alpha_2 X_2(T))} \left(e^{(\alpha_1 X_1(T) - \alpha_2 X_2(T))} - \lambda \right)^+ \mathbb{1}_{\{e^{X_3(T)} < D\}} \middle| \mathcal{F}_t \right] \\ &= \lambda_1 e^{-r(T-t)} \left(C_1(\lambda, D) + \frac{\gamma V(t)}{D} C_2(\lambda, D) \right) \end{aligned} \quad (12)$$

where $C_1(\lambda, D) = \mathbb{E}_{\mathbb{Q}} \left[e^{(\alpha_2 X_2(T))} \left(e^{(\alpha_1 X_1(T) - \alpha_2 X_2(T))} - \lambda \right)^+ \mathbb{1}_{\{e^{X_3(T)} \geq D\}} \middle| \mathcal{F}_t \right]$, $C_2(\lambda, D) = \mathbb{E}_{\mathbb{Q}} \left[e^{(\alpha_2 X_2(T))} \left(e^{(\alpha_1 X_1(T) - \alpha_2 X_2(T))} - \lambda \right)^+ \mathbb{1}_{\{e^{X_3(T)} < D\}} \middle| \mathcal{F}_t \right]$.

Let $Z(T) = \alpha_1 X_1(T) - \alpha_2 X_2(T)$. To exploit the affine structure of $X_1(t)$, $X_2(t)$ and $X_3(t)$, the pricing problem is transferred into Fourier space allowing it to be expressed in terms of the characteristic function. The Fourier transform of

$(e^{Z(T)} - \lambda)^+ \mathbb{1}_{\{e^{X_3(T)} \geq D\}}$ and $(e^{Z(T)} - \lambda)^+ \mathbb{1}_{\{e^{X_3(T)} < D\}}$ will be considered.

Let $f_1(z, x_3) = (e^z - \lambda)^+ \mathbb{1}_{\{e^{x_3} \geq D\}}$, then for $\epsilon_1 < -1$ and $\epsilon_2 < 0$ the Fourier transform of $f_1(z, x_3) e^{\epsilon_1 z + \epsilon_2 x_3}$ will exist and let it be denoted by $F_1(\omega_1, \omega_2)$. Then

$$\begin{aligned} F_1(\omega_1, \omega_2) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i\omega_1 z - i\omega_2 x_3} e^{\epsilon_1 z + \epsilon_2 x_3} (e^z - \lambda)^+ \mathbb{1}_{\{e^{x_3} \geq D\}} dz dx_3 \\ &= \int_{\ln(D)}^{\infty} \int_{\ln(\lambda)}^{\infty} e^{-i\omega_1 z - i\omega_2 x_3} e^{\epsilon_1 z + \epsilon_2 x_3} (e^z - \lambda) dz dx_3 \\ &= \int_{\ln(D)}^{\infty} \int_{\ln(\lambda)}^{\infty} (e^{(-i\omega_1 + \epsilon_1 + 1)z} - \lambda e^{(-i\omega_1 + \epsilon_1)z}) e^{-i\omega_2 + \epsilon_2 x_3} dz dx_3 \\ &= \frac{\lambda e^{(-i\omega_1 + \epsilon_1) \ln(\lambda) - (i\omega_2 - \epsilon_2) \ln(D)}}{(-i\omega_1 + \epsilon_1)(-i\omega_1 + \epsilon_1 + 1)(i\omega_2 - \epsilon_2)}. \end{aligned}$$

The inverse Fourier transform will give,

$$e^{\epsilon_1 z + \epsilon_2 x_3} f_1(z, x_3) = \frac{\lambda}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{iz\omega_1 + ix_3\omega_2} e^{(-i\omega_1 + \epsilon_1) \ln(\lambda) - (i\omega_2 - \epsilon_2) \ln(D)}}{(-i\omega_1 + \epsilon_1)(-i\omega_1 + \epsilon_1 + 1)(i\omega_2 - \epsilon_2)} d\omega_1 d\omega_2. \quad (13)$$

Similarly assume $f_2(z, x_3) = (e^z - \lambda)^+ \mathbb{1}_{\{e^{x_3} < D\}}$ then the Fourier transform of $f_2(z, x_3) e^{\epsilon_1 z + \epsilon_2 x_3}$ will exist and is

$$\begin{aligned} F_2(\omega_1, \omega_2) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{\epsilon_1 z - \epsilon_2 x_3} (e^z - \lambda)^+ \mathbb{1}_{\{e^{x_3} < D\}} e^{-i\omega_1 z - i\omega_2 x_3} dz dx_3 \\ &= \frac{\lambda e^{(-i\omega_1 + \epsilon_1) \ln(\lambda) - (i\omega_2 + \epsilon_2) \ln(D)}}{(-i\omega_1 + \epsilon_1)(-i\omega_1 + \epsilon_1 + 1)(i\omega_2 + \epsilon_2)}. \end{aligned}$$

The inverse Fourier transform will give,

$$e^{\epsilon_1 z - \epsilon_2 x_3} f_2(z, x_3) = \frac{\lambda}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{iz\omega_1 + ix_3\omega_2} e^{(-i\omega_1 + \epsilon_1) \ln(\lambda) - (i\omega_2 + \epsilon_2) \ln(D)}}{(-i\omega_1 + \epsilon_1)(-i\omega_1 + \epsilon_1 + 1)(i\omega_2 + \epsilon_2)} d\omega_1 d\omega_2. \quad (14)$$

Let $\tau = T - t$. Substitute the values of $f_1(z, x_3)$ and $f_2(z, x_3)$ from Eqs. (13) and (14) in (12) to calculate the value of $C_1(\lambda, D)$ and $C_2(\lambda, D)$.

$$\begin{aligned} C_1(\lambda, D) &= \mathbb{E}_{\mathbb{Q}} \left[e^{(a_2 X_2(T))} (e^{(a_1 X_1(T) - a_2 X_2(T))} - \lambda)^+ \mathbb{1}_{\{e^{X_3(T)} \geq D\}} \middle| \mathcal{F}_t \right] \\ &= \mathbb{E}_{\mathbb{Q}} \left[e^{(a_2 X_2(T))} (e^{Z(T)} - \lambda)^+ \mathbb{1}_{\{e^{X_3(T)} \geq D\}} \middle| \mathcal{F}_t \right] \\ &= \mathbb{E}_{\mathbb{Q}} \left[\frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\lambda e^{(-i\omega_1 + \epsilon_1) \ln(\lambda) - (i\omega_2 - \epsilon_2) \ln(D)}}{(-i\omega_1 + \epsilon_1)(-i\omega_1 + \epsilon_1 + 1)(i\omega_2 - \epsilon_2)} e^{(i\omega_1 - \epsilon_1) \alpha_1 X_1(T) - (i\omega_1 - \epsilon_1 - 1) \alpha_2 X_2(T) + (i\omega_2 - \epsilon_2) X_3(T)} d\omega_2 d\omega_1 \middle| \mathcal{F}_t \right] \\ &= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\lambda e^{(-i\omega_1 + \epsilon_1) \ln(\lambda) - (i\omega_2 - \epsilon_2) \ln(D)}}{(-i\omega_1 + \epsilon_1)(-i\omega_1 + \epsilon_1 + 1)(i\omega_2 - \epsilon_2)} \varphi((\omega_1 + i\epsilon_1)\alpha_1, -(\omega_1 + i(\epsilon_1 + 1))\alpha_2, (\omega_2 + i\epsilon_2), \tau) d\omega_2 d\omega_1, \quad (15) \end{aligned}$$

$$\begin{aligned} C_2(\lambda, D) &= \mathbb{E}_{\mathbb{Q}} \left[e^{(a_2 X_2(T))} (e^{(a_1 X_1(T) - a_2 X_2(T))} - \lambda)^+ \mathbb{1}_{\{e^{X_3(T)} < D\}} \middle| \mathcal{F}_t \right] \\ &= \mathbb{E}_{\mathbb{Q}} \left[e^{(a_2 X_2(T))} (e^{Z(T)} - \lambda)^+ \mathbb{1}_{\{e^{X_3(T)} < D\}} \middle| \mathcal{F}_t \right] \\ &= \mathbb{E}_{\mathbb{Q}} \left[\frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\lambda e^{(-i\omega_1 + \epsilon_1) \ln(\lambda) - (i\omega_2 + \epsilon_2) \ln(D)}}{-(-i\omega_1 + \epsilon_1)(-i\omega_1 + \epsilon_1 + 1)(i\omega_2 + \epsilon_2)} e^{(i\omega_1 - \epsilon_1) \alpha_1 X_1(T) - (i\omega_1 - \epsilon_1 - 1) \alpha_2 X_2(T) + (i\omega_2 + \epsilon_2) X_3(T)} d\omega_2 d\omega_1 \middle| \mathcal{F}_t \right] \\ &= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\lambda e^{(-i\omega_1 + \epsilon_1) \ln(\lambda) - (i\omega_2 + \epsilon_2) \ln(D)}}{-(-i\omega_1 + \epsilon_1)(-i\omega_1 + \epsilon_1 + 1)(i\omega_2 + \epsilon_2)} \varphi((\omega_1 + i\epsilon_1)\alpha_1, -(\omega_1 + i(\epsilon_1 + 1))\alpha_2, (\omega_2 - i\epsilon_2), \tau) d\omega_2 d\omega_1, \quad (16) \end{aligned}$$

where $\varphi(\cdot)$ is the conditional joint characteristic function of $X_1(T)$, $X_2(T)$, and $X_3(T)$ under risk-neutral probability measure $\mathbb{Q}$. Now the options pricing formula can be obtained by solving the double integration on the joint characteristic function. The double integration is solved numerically using packages available in Python/R. The characteristic function can be computed by using the Feynman–Kac formula and affine structure of $X_1(T)$, $X_2(T)$, and $X_3(T)$. By definition, $\varphi(\cdot)$ is given by

$$\varphi(\delta_1, \delta_2, \delta_3, \tau) = \mathbb{E}_{\mathbb{Q}} [e^{i\delta_1 X_1(T) + i\delta_2 X_2(T) + i\delta_3 X_3(T)} | X_1(t) = x_1, X_2(t) = x_2, X_3(t) = x_3, L(t) = L]. \quad (17)$$

By Feynman–Kac formula $\varphi(\cdot)$ satisfies the following partial differential equation (PDE)

$$\begin{aligned} \frac{\partial \varphi}{\partial \tau} + \sum_{i=1}^3 \left[\left(r - \frac{1}{2} (\sigma_i^2 + \beta_i^2 L^2) \right) \frac{\partial \varphi}{\partial x_i} + \frac{1}{2} (\sigma_i^2 + \beta_i^2 L^2) \frac{\partial^2 \varphi}{\partial x_i^2} + \rho_i \beta_i \xi L \frac{\partial^2 \varphi}{\partial x_i \partial L} \right] + \kappa(\theta - L) \frac{\partial \varphi}{\partial L} \\ + \frac{1}{2} \xi^2 \frac{\partial^2 \varphi}{\partial L^2} + \sigma_1 \sigma_2 \rho \frac{\partial^2 \varphi}{\partial x_1 \partial x_2} + \sigma_2 \sigma_3 \tilde{\rho} \frac{\partial^2 \varphi}{\partial x_2 \partial x_3} = 0, \quad (18) \end{aligned}$$

with boundary condition $\varphi(\delta_1, \delta_2, \delta_3, 0) = e^{i\delta_1 X_1(T) + i\delta_2 X_2(T) + i\delta_3 X_3(T)}$.

According to Duffie et al. [25] $\varphi(\delta_1, \delta_2, \delta_3, \tau)$ has the following form

$$\varphi(\delta_1, \delta_2, \delta_3, \tau) = \exp\{A(\tau) + B(\tau)L + C(\tau)L^2 + i\delta_1 x_1 + i\delta_2 x_2 + i\delta_3 x_3\}.$$

Substituting $\varphi(\delta_1, \delta_2, \delta_3, \tau)$ in the above PDE, the following system of ordinary differential equations is obtained,

$$\begin{aligned} \frac{dC(\tau)}{d\tau} &= 2C(-\kappa + i\xi(\delta_1 \rho_1 \beta_1 + \delta_2 \rho_2 \beta_2 + \delta_3 \rho_3 \beta_3)) - \frac{1}{2} \sum_{j=1}^3 \beta_j^2 (i\delta_j + \delta_j^2) + 2\xi^2 C^2, \\ \frac{dB(\tau)}{d\tau} &= (2\xi^2 C + i\xi \delta_1 \rho_1 \beta_1 + i\xi \delta_2 \rho_2 \beta_2 + i\xi \delta_3 \rho_3 \beta_2 - \kappa) B + 2\kappa \theta C, \\ \frac{dA(\tau)}{d\tau} &= \kappa \theta B + \frac{1}{2} \xi^2 B^2 + \xi^2 C - \rho \sigma_1 \sigma_2 \delta_1 \delta_2 - \tilde{\rho} \sigma_2 \sigma_3 \delta_2 \delta_3 + \sum_{j=1}^3 \left(i r \delta_j - \frac{1}{2} \sigma_j^2 (i\delta_j + \delta_j^2) \right). \end{aligned}$$

Solving for the functions $A(\tau)$, $B(\tau)$ and $C(\tau)$ with initial condition

$$A(0) = B(0) = C(0) = 0,$$

$$\begin{aligned} C(\tau) &= \frac{\zeta - \Gamma}{2\xi^2} \frac{1 - e^{-2\xi\tau}}{\mathbb{N} - e^{-2\xi\tau}}, \\ B(\tau) &= \frac{-\kappa\theta(\zeta - \Gamma)(1 - e^{-\xi\tau})^2}{\zeta\xi^2} \frac{1}{\mathbb{N} - e^{-2\xi\tau}}, \\ A(\tau) &= \sum_{j=1}^3 \left(i r \delta_j - \frac{1}{2} \sigma_j^2 (i\delta_j + \delta_j^2) \right) \tau + \frac{1}{2} \left[-(\zeta + \Gamma)\tau - \ln \left(\frac{e^{-2\xi\tau} - \mathbb{N}}{1 - \mathbb{N}} \right) \right] + \frac{(\kappa\theta(\Gamma - \zeta))^2}{2\xi^2 \zeta^2} \left\{ \frac{\tau}{\mathbb{N}^2} + \frac{2\Gamma}{(\Gamma - \zeta)^2} \ln \left(\frac{e^{-2\xi\tau} + \mathbb{N}}{1 + \mathbb{N}} \right) \right. \\ &\quad \left. + \frac{2\Gamma(e^{-2\xi\tau} - 1)}{(e^{-2\xi\tau} + \mathbb{N})} + \frac{2(\mathbb{N} + 1)}{\sqrt{\mathbb{N}\mathbb{N}\zeta}} \left[\tan^{-1} \left(\frac{e^{-\xi\tau}}{\sqrt{\mathbb{N}}} \right) - \tan^{-1} \left(\frac{1}{\sqrt{\mathbb{N}}} \right) \right] - \frac{(\mathbb{N}^2 - 6\mathbb{N} + 1)(1 - e^{-2\xi\tau})}{2\xi\mathbb{N}} \right\} + \tilde{\rho} \sigma_2 \sigma_3 \delta_2 \delta_3 \tau \end{aligned}$$

$$+ \frac{(\kappa\theta)^2(\zeta - \Gamma)}{\xi^2\zeta} \left\{ \frac{1}{\Gamma - \zeta} \ln \left(\frac{e^{-2\zeta\tau} + N}{1 + N} \right) + \frac{\tau}{N} + \frac{2}{\sqrt{N}\zeta} \left[\tan^{-1} \left(\frac{e^{-\zeta\tau}}{\sqrt{N}} \right) - \tan^{-1} \left(\frac{1}{\sqrt{N}} \right) \right] \right\} - \rho\sigma_1\sigma_2\delta_1\delta_2\tau,$$

where

$$\Gamma = i\xi(\delta_1\rho_1\beta_1 + \delta_2\rho_2\beta_2 + \delta_3\rho_3\beta_3) - \kappa, \quad \zeta = \sqrt{\Gamma^2 + \xi^2(\beta_1^2(i\delta_1 + \delta_1^2) + \beta_2^2(i\delta_2 + \delta_2^2) + \beta_3^2(i\delta_3 + \delta_3^2))}, \quad N = \frac{\Gamma - \zeta}{\Gamma + \zeta}.$$

3.2. Power exchange options price with liquidity risk

In this subsection, the pricing formula is given in the case that the probability of counterparty default is zero, i.e., $P(V(T) \geq D) = 1$.

Let $\tilde{C}(S_1(t), S_2(t), L(t), t, T)$ denotes the price of the power exchange options at time t with maturity T under risk-neutral probability measure $\mathbb{Q}$. Then

$$\begin{aligned} \tilde{C}(S_1(t), S_2(t), L(t), t, T) &= \mathbb{E}_{\mathbb{Q}} \left[e^{-r(T-t)} \left(\lambda_1 S_1^{(a_1)}(T) - \lambda_2 S_2^{(a_2)}(T) \right)^+ \middle| \mathcal{F}_t \right] \\ &= \lambda_1 e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}} \left[e^{(a_2 X_2(T))} \left(e^{(a_1 X_1(T) - a_2 X_2(T))} - \lambda \right)^+ \middle| \mathcal{F}_t \right]. \end{aligned}$$

In this case, $f(z) = (e^z - \lambda)^+$ and for $\epsilon < -1$,

$$f(z) = \frac{\lambda}{2\pi} \int_{-\infty}^{\infty} \frac{e^{iz\omega} e^{(-i\omega + \epsilon) \ln(\lambda)}}{(-i\omega + \epsilon)(-i\omega + \epsilon + 1)} d\omega.$$

Then the options price $\tilde{C}(\lambda, t, T) := \tilde{C}(S_1(t), S_2(t), L(t), t, T)$

$$\begin{aligned} \tilde{C}(\lambda, t, T) &= \lambda_1 e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}} \left[e^{a_2 X_2(T)} \frac{\lambda}{2\pi} \int_{-\infty}^{\infty} \frac{e^{iz\omega} e^{-\epsilon z} e^{(-i\omega + \epsilon) \ln(\lambda)}}{(-i\omega + \epsilon)(-i\omega + \epsilon + 1)} d\omega \middle| \mathcal{F}_t \right] \\ &= \frac{\lambda_2 e^{-r(T-t)}}{2\pi} \int_{-\infty}^{\infty} \frac{e^{(-i\omega + \epsilon) \ln(\lambda)} \tilde{\varphi}((\omega + i\epsilon)\alpha_1, -(\omega + i(\epsilon + 1))\alpha_2, \tau)}{(-i\omega + \epsilon)(-i\omega + \epsilon + 1)} d\omega, \end{aligned} \quad (19)$$

where $\tilde{\varphi}(\cdot) = \varphi(\delta_1, \delta_2, 0, \tau)$ is the conditional joint characteristic function of $X_1(T)$ and $X_2(T)$ under probability measure $\mathbb{Q}$.

4. Numerical and sensitivity analysis

Algorithm 1 Monte Carlo Simulation for Asset Pricing

- 1: **Set** parameters according to values in [Table 2](#)
- 2: **Set** $iterations = 100000$, $timesteps = 500$, $\Delta t = \frac{T}{timesteps}$
- 3: Create arrays S_1, S_2, V, L of size $(timesteps + 1, iterations)$
- 4: Initialize S_1, S_2, V, L
- 5: Generate correlated Brownian motions using Cholesky decomposition
- 6: **for** each timestep $i = 1$ to $timesteps$ **do**
- 7: Update S_1, S_2, V, L using stochastic process equations [\(5\)](#), [\(6\)](#), [\(7\)](#), [\(8\)](#)
- 8: **end for**
- 9: Initialize array *price*
- 10: **for** each iteration $i = 1$ to $iterations$ **do**
- 11: **if** $V[timesteps - 1, i] > D$ **then**
- 12: Compute *price*[i] using formula [\(9\)](#)
- 13: **else**
- 14: Compute *price*[i] using formula [\(10\)](#)
- 15: **end if**
- 16: **end for**
- 17: Compute final option price: $\exp(-rT) \times \text{mean}(\text{price})$
- 18: **Output** option price

This section presents the numerical analysis of the derived pricing formula of power exchange options. To verify the accuracy of the derived formula, firstly, the comparison with Monte Carlo simulations generated price is presented. The double integration in the derived formula is computed using the `dblquad` function of the SciPy module in Python, which performs adaptive quadrature to evaluate the double integral. The sensitivity analysis of various parameters to power exchange options price is also provided. Based on the literature, the values of the parameters are provided in [Table 2](#).

Table 2
The value of parameters.

Parameter	Value	Parameter	Value	Parameter	Value	Parameter	Value
$S_1(0)$	10	$S_2(0)$	10	$V(0)$	15	$L(0)$	0.3
σ_1	0.2	σ_2	0.2	σ_3	0.2	ξ	0.9
β_1	0.5	β_2	0.5	β_3	0.5	κ	0.2
ρ_1	-0.5	ρ_2	-0.5	ρ_3	-0.5	θ	0.3
α_1	2	α_2	2	λ_1	2	λ_2	2
r	0.02	ϵ_1	-1.5	ϵ_2	-0.2	ϵ	-1.1
D	7.5	γ	0.4	ρ	-0.25	$\bar{\rho}$	0.25

Algorithm 2 Exact Option Pricing

- 1: Compute cf characteristic function using the formula
- 2: Define function $integrand(u, v)$:
- 3:
$$\frac{\lambda e^{(-iu+\epsilon_1)\ln(\lambda)-(iv-\epsilon_2)\ln(D)}}{(-iu+\epsilon_1)(-iu+\epsilon_1+1)(iv-\epsilon_2)} cf((u+i\epsilon_1)\alpha_1, -(u+i(\epsilon_1+1))\alpha_2, (v+i\epsilon_2), \tau)$$
- 4: Perform double integral to get C_1
- 5: Define function $integrand1(u, v)$:
- 6:
$$\frac{\lambda e^{(-iu+\epsilon_1)\ln(\lambda)-(iv-\epsilon_2)\ln(D)}}{-(-iu+\epsilon_1)(-iu+\epsilon_1+1)(iv-\epsilon_2)} cf((u+i\epsilon_1)\alpha_1, -(u+i(\epsilon_1+1))\alpha_2, (v-i\epsilon_2), \tau)$$
- 7: Perform double integral to get C_2
- 8: Compute final option price
- 9:
$$\frac{\lambda_1}{(2\pi)^2} \cdot \exp(-r\tau) \cdot (C_1 - \gamma VC_2/D)$$
- 10: **return** Call option price

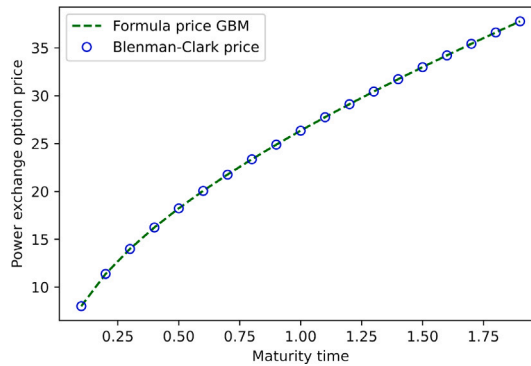


Fig. 4. Comparison of power exchange options prices under GBM model calculated from proposed formula and from Theorem 1 of Blenman and Clark [1].

4.1. Accuracy test

Monte Carlo Comparison: The comparison of the vulnerable power exchange options prices with Monte Carlo simulation-generated prices is presented to verify the accuracy of the derived formula. For the Monte Carlo simulation generated options prices, 100 000 sample paths have been generated with 500 time steps. To generate the correlated Brownian motion, Cholesky decomposition is applied, and the stochastic evolution of the asset price processes, $S_1(t)$, $S_2(t)$ and $V(t)$ is modeled using the Euler–Maruyama discretization scheme. The algorithm for Monte-Carlo simulation is given in Algorithm 1. The power exchange options prices given by Eq. (12) is computed according to Algorithm 2.

Table 3 presents the comparison of vulnerable power exchange options prices with the Monte Carlo simulation prices and it can be observed that the relative percentage error is less than 0.9%. Table 4 presents the comparison of the prices of power exchange options without default risk with the Monte Carlo generated prices. It is evident from Table 4 that the relative percentage error is less than 0.71%. This verifies the accuracy of the derived formula. Tables 3 and 4 illustrate that options prices increase as the maturity time increases. This is because the asset price will have a higher probability of moving favorably. The vulnerable power exchange options carry higher risks for holders due to default risk, so their premiums should be lower compared to options without such risks. From Tables 3 and 4, a similar pattern is observed as options prices with default risk are lower than options without default risk.

Derived formula price under GBM model: The Black–Scholes model prices are obtained from the derived formula by considering $\beta_1 = \beta_2 = 0$ in Eq. (19). Fig. 4 presents a comparison of power exchange options prices from derived formula under the GBM model

Table 3

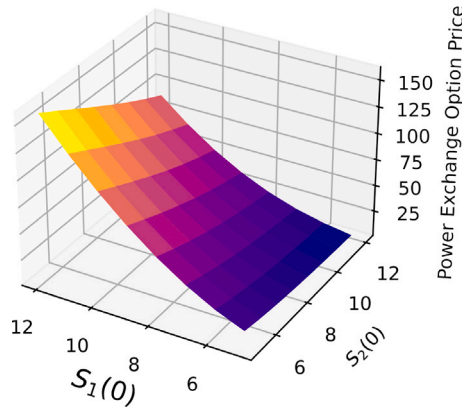
Comparison of derived vulnerable power exchange options price with Monte Carlo simulation price.

Time	Model price	MCS price	RE (%)	Time	Model price	MCS price	RE (%)
0.1	10.126	10.202	0.7523	0.9	40.625	40.630	0.0123
0.2	15.024	14.943	0.5389	1.0	44.025	44.271	0.5590
0.3	19.217	19.307	0.4712	1.1	47.432	47.022	0.8628
0.4	23.081	23.111	0.1324	1.2	50.856	50.791	0.1288
0.5	26.755	26.686	0.2561	1.3	54.309	54.392	0.1532
0.6	30.309	30.486	0.5850	1.4	57.798	57.869	0.1212
0.7	33.787	33.720	0.1957	1.5	61.333	61.020	0.5100
0.8	37.218	37.137	0.2167	1.6	64.919	64.973	0.0835

Table 4

Comparison of the derived power exchange options price with Monte Carlo simulation price.

Time	Model price	MCS price	RE (%)	Time	Model price	MCS price	RE (%)
0.1	10.127	10.155	0.2805	0.9	41.415	41.572	0.3789
0.2	15.031	15.018	0.0888	1.0	45.058	44.972	0.1895
0.3	19.244	19.244	0.0033	1.1	48.740	48.445	0.6056
0.4	23.154	23.177	0.1022	1.2	52.474	52.102	0.7086
0.5	26.904	26.865	0.1460	1.3	56.269	55.97	0.5318
0.6	30.568	30.558	0.0311	1.4	60.136	59.827	0.5144
0.7	34.189	34.045	0.4202	1.5	64.082	63.821	0.4072
0.8	37.798	37.614	0.4866	1.6	68.116	68.04	0.1112

**Fig. 5.** Comparison of power exchange options price against various values of $S_1(0)$ and $S_2(0)$.

and those from Blenman and Clark [1] closed-form formula. Fig. 4 depicts that the prices from both methods align point by point. This comparison highlights that the proposed model is an extension of the GBM approach.

Options prices against underlying assets: To demonstrate the impact of the initial spread between the prices of both assets on power exchange options prices, a comparison of power exchange options price with $S_1(0)$ and $S_2(0)$ is presented in Fig. 5. Power exchange options prices increase with an increase in $S_1(0)$, but decrease with an increase in $S_2(0)$. It can be observed from Fig. 5 that as the $S_1(0) - S_2(0)$ decreases, the options price decreases. This is because the payoff of options increases with an increase in $S_1(0) - S_2(0)$, which increases the options premium, which explains this behavior.

4.2. Sensitivity analysis

In this subsection, the sensitivity of options price with various liquidity parameters and correlation between the assets is presented. For the sake of computational simplicity, power exchange options price without default risk is considered in this subsection.

- Parameters of market liquidity:** The market is perfectly liquid when $L(t) = 0$. For fixed values of β_i , high deviations in values of $|L(t)|$ imply that stock has lower market liquidity. As the liquidity decreases, the risk associated with the asset increases, hence, the options premium increases.

- Mean reversion level:** Fig. 6 depicts the sensitivity of power exchange options prices with liquidity risk to the variations in the mean reversion rate (κ) of the market liquidity process $L(t)$. It is observed that the options prices decrease with

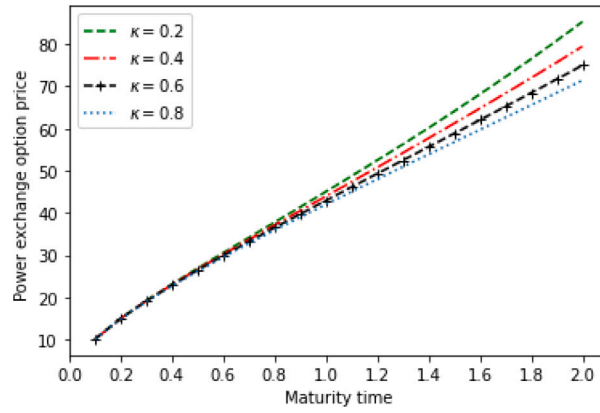


Fig. 6. Power exchange options price against maturity time for various values of κ .

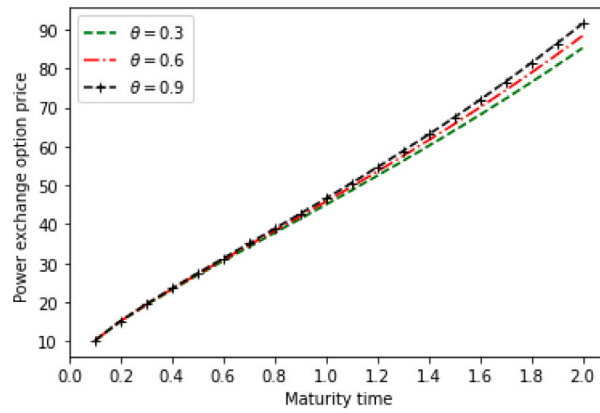


Fig. 7. Power exchange options price against maturity time for various values of θ .

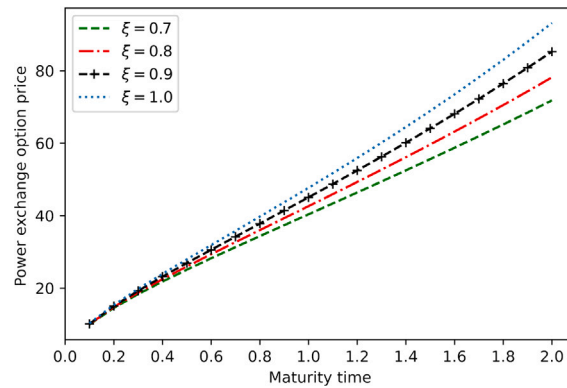


Fig. 8. Power exchange options price against maturity time for various values of ξ .

an increase in the value of κ . The higher values of κ imply that $L(t)$ will fluctuate less. Hence the options premium decreases.

(b) **Long run mean:** Fig. 7 illustrates the impact of long-term mean (θ) of market liquidity on power exchange options prices for different maturity time. Power exchange options prices increase with an increase in θ . Larger values of θ indicate larger illiquidity in the long term, which results in higher options prices.

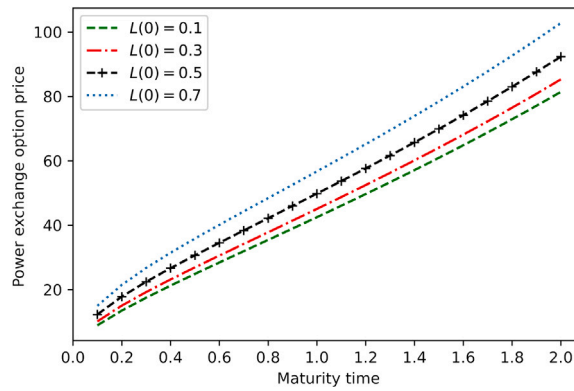


Fig. 9. Power exchange options price against maturity time for various values of $L(0)$.

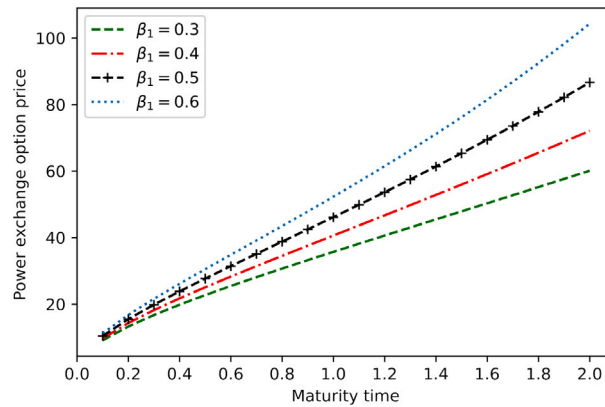


Fig. 10. Power exchange options price against maturity time for various values of β_1 where $\beta_1 = \beta_2 + 0.1$.

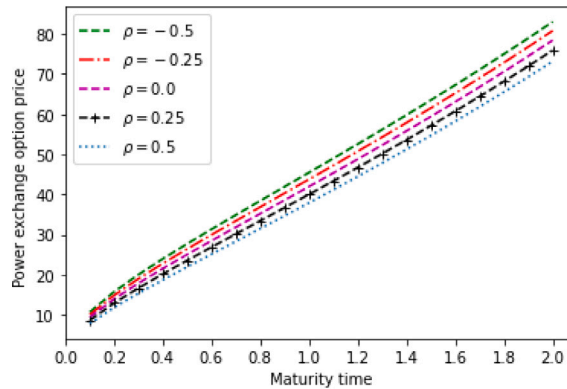


Fig. 11. Power exchange options price against maturity time for various values of ρ .

- (c) **Volatility:** The power exchange options prices are illustrated in Fig. 8 with respect to the changes in market liquidity volatility (ξ). It is observed that as the ξ value increases, option prices also increases. The higher volatility of $L(t)$ increases the fluctuations, thus increasing the options premium.
- (d) **Initial liquidity:** To illustrate the impact of the initial level of market liquidity on the pricing formula, power exchange options prices are plotted for different $L(0)$ values at various times to maturity. Fig. 9 depicts that the options prices increases with increase of $L(0)$. This behavior is logical, as increasing the initial value of market liquidity results in increasing market liquidity.

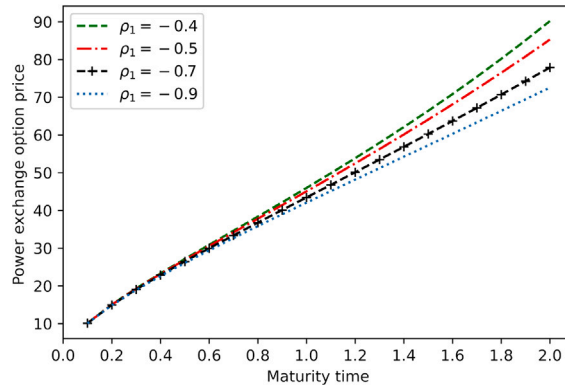


Fig. 12. Power exchange options price for different maturity time and ρ_1 .

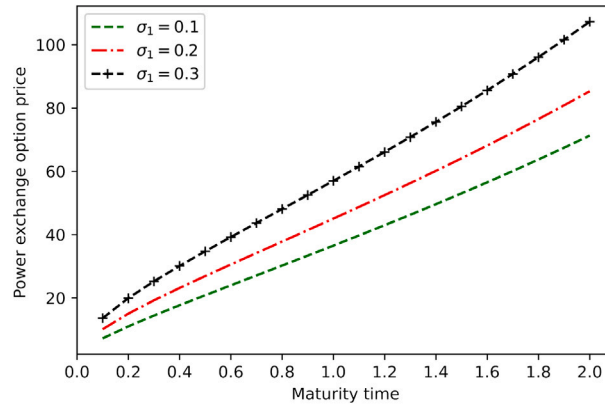


Fig. 13. Power exchange options price for different maturity time and σ_1 .

2. **Asset specific liquidity:** Fig. 10 demonstrates the power exchange options prices for different values of the assets liquidity sensitivity parameter β_1 with $\beta_2 = \beta_1 - 0.1$. Fig. 10 illustrates that the prices of power exchange options are positively correlated with the sensitivity of the underlying asset values to market liquidity. Increased sensitivity results in higher options prices as, when sensitivity is higher, the stock prices increase more for the same levels of illiquidity, thereby leading to a rise in the options price.
3. **Correlation coefficient:** Fig. 11 depicts the power exchange options price with respect to time to maturity for different values of the correlation coefficient between the two assets (ρ). The impact of the correlation coefficient ρ on the prices of the assets can be observed. The price of the power exchange options is higher when the assets have a negative correlation than when the assets have a positive correlation. When the assets are positively correlated, then the risk of relative movement between these prices is less, which results in lesser risk and hence, lesser power exchange options prices.
4. **Correlation coefficient between assets and market liquidity:** The impact of correlation between the asset price and market liquidity on power exchange options price is demonstrated in Fig. 12. Fig. 12 is plotted for various values of ρ_1 with the assumption that $\rho_1 = \rho_2$. The figure illustrates that as the correlation becomes increasingly negative, the options price decreases. This is because as the market illiquidity increases, it is difficult to trade the asset.
5. **Asset volatility:** Fig. 13 presents the power exchange options prices against maturity time for different values of asset volatility σ_1 and σ_2 with the assumption that $\sigma_1 = \sigma_2$. It is observed that the options price increases with an increase in the volatilities of assets 1 and 2. As the volatilities increase, the probability of change in asset value increases, which increases the options price.

4.3. Further discussions

Liquidity risk plays a crucial role in option pricing, often resulting in significantly higher prices compared to traditional models that assume perfect liquidity. To examine the impact of illiquidity on power exchange options prices, the comparison of the options prices under the liquidity risk model and the Black–Scholes model is plotted in Fig. 14. In both cases, the power exchange options

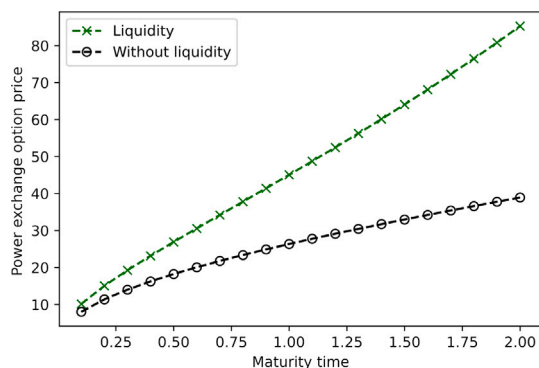


Fig. 14. Comparison of power exchange options price with liquidity and without liquidity for various values of maturity time.

prices increase as the time to maturity increases. This comparative analysis demonstrates that the options prices considering the liquidity risk are significantly higher than those without liquidity risk, highlighting its influence on valuation.

This model can help in the pricing and risk management of vulnerable power exchange options under liquidity risk. The assumption of frictionless markets may underestimate the price of options, thus resulting in mispricing and poor risk assessment. Additionally, understanding liquidity risk can help in the development of better hedging strategies and risk management techniques.

5. Conclusion and future work

The pricing of vulnerable power exchange options while incorporating the liquidity risk is presented in this article. The pricing formula is derived considering the Fourier transform of the payoff function. Numerical illustrations validate the pricing model and sensitivity analysis depicts the impact of considering liquidity risk on options prices. As the illiquidity of an asset increases, it is difficult to trade in it, which results in higher premium demand by the writer of the option. It has been illustrated graphically that liquidity increases the options prices significantly, and the difference in price increases as the time to maturity increases.

The proposed vulnerable power exchange options pricing model can be further extended by incorporating regime-switching stochastic volatility, stochastic interest rates, and jump processes to account for sudden market shocks (see [26] for a related model).

The Hurst exponent for market liquidity indicates the roughness. Hence, incorporating fractional Brownian motion with a Hurst parameter less than 0.5 can provide a more accurate representation of this behavior.

CRedit authorship contribution statement

Priya Mittal: Writing – review & editing, Writing – original draft, Visualization, Validation, Methodology, Investigation, Formal analysis, Conceptualization. **Dharmaraja Selvamuthu:** Writing – review & editing, Validation, Methodology, Formal analysis, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability

The data of this study is available on Thomson Reuters, but restrictions apply to the availability of the data. The data utilized for this article was under license and, therefore, not publicly accessible.

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