



Structural credit risk models with stochastic default barriers and jump clustering using Hawkes jump-diffusion processes

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Abstract

This paper derives a closed-form expression for the default probability and the default correlation of firms under a structural model of credit risk. Specifically, the underlying firms are assumed to have the value process driven by a Hawkes jump-diffusion model with the continuous parts of the trajectories being driven by correlated Brownian motions, while the jumps being driven by Hawkes processes having general structure of the exciting functions. The proposed framework takes into account the numerically observed facts about the default, i.e., clustering and unexpectedness. Furthermore, the default barriers are assumed to be stochastic in nature and modeled as stochastic processes, affected by common factors reflecting the systematic risk. A sensitivity analysis of default probability and correlation is conducted to investigate the impact of jump risk, clustering, and stochastic default barriers. These numerical studies demonstrate that jump clustering increases the default probability but reduces the correlation of defaults.

Keywords Credit risk · Default clustering · Hawkes processes · Jump-diffusion processes · Stochastic default barriers

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1 Introduction

Until recently, structural models in academic papers focused on analyzing single entities, such as individual firms with underlying assumptions that the downgrade of credit quality or default of a reference firm may be attributed to specific characteristics of that firm. However, in practice, a credit event specific to one firm may immediately have an economic or perceived impact on other firms, which is reflected in their default probability. Specifically, multiple firms may get affected due to the common macroeconomic factors, or a downgrade in credit quality or default of related firms in trade-creditor agreements. For instance, the 2008 sub-prime crisis, which saw the collapse of financial institutions like Lehman Brothers, brought attention to the importance of correlations within the financial market. Further, empirical studies (refer [4, 12, 21]) on bond prices and default data, demonstrated that there is a dependence among large jumps in credit spreads of multiple issuers. Essentially, there exists a phenomenon called clustering of default, which is the sequence of defaults of related firms, that can occur due to common macroeconomic factors affecting all the firms and contagion effects, where the default of one firm directly affects the credit quality of other firms. As a result, many researchers believe that it is reasonable to assume dependence among firms due to, either, direct business links or common economic factors affecting all the firms. Another interesting observation comes from Aït-Sahalia et al. [1], who argued that financial crises occur not due to the initial jump, but because of the subsequent amplification that occurs over hours or days, and the fact that other markets also become affected.

Motivated by the above highlighted importance of jump clustering and correlation of defaults, the interest in Hawkes processes to model the default risk has grown tremendously in academic finance and industry. This growth is attributed to the property of Hawkes processes that allow us to reproduce jump clusters, since the intensity of jump increases with the occurrence of jumps and, consequently, increases the probability of future jumps. In this direction, Ma and Xu [24] investigated a model based on a structural-based approach to derive an explicit expression for the firm values and the default correlation by using exponential Hawkes jump-diffusion processes governing the firm's values. Recently, Pasricha et al. [30] adopted the dynamics considered in Ma and Xu [24], and derived a closed-form solution for the probability of default and the default correlation using the characteristic function. The authors observed that the jump risk increases the default probability law, whereas jump clustering decreases it. On the contrary, Dassios and Zhao [6] observed that jump clustering increases the probability of default. In conclusion, several authors showed that the default correlation obtained from models driven by Hawkes jump-diffusion processes is more prominent than those obtained from models driven by Poisson jumps.

In this paper, motivated by the success of Hawkes processes, we extend the models developed in Ma and Xu [24], and Pasricha et al. [30], and propose a structural

credit risk model under a Hawkes jump-diffusion process, when the exciting function of the underlying Hawkes process is a general function.

First, our paper generalises the previous literature, and in particular Ma and Xu [24], and considers Hawkes processes not only having an exponential structure of the exciting functions, but having a general structure. In a real world scenario, this choice could allow us to better describe the speed of the decay of the effect of the jumps. Specifically, the Hawkes jump-diffusion processes are assumed to govern the dynamics of the firm values. A common Brownian motion is incorporated in the continuous part and a common Hawkes process having general exciting function is considered for the jumps to model the uncertainty resulting from the systemic risk. A specific Brownian motion and a specific Hawkes process with general exciting function are introduced to describe better the uncertainty arising out of idiosyncratic shocks. Taking into account this model, we obtain the closed-form expressions for the probability law of default of a firm and default correlation. This paper, thus, contributes to the growing applications of Hawkes process in modeling of default risk.

Second, this paper relaxes the assumption of a constant default barrier in Ma and Xu [24] and proposes to consider a stochastic default barrier, inspired by Wang [36]. Since the liabilities of many financial institutions are mostly financial assets and, hence, stochastic, we believe it is reasonable to model the default barrier as a stochastic process. To take into account that the firm and the investors have incomplete information about the default boundary, because the structure of the debt could change either due to policy changes or some exogenous events, we assume to incorporate jumps in the dynamics of the default barrier. Further, liabilities being financial assets are exposed to the common factors driving the market, i.e., the factors that affect the firm value process, and important changes in financial asset prices are a consequence of the arrival of important news¹. This motivates the fact that the stochastic dynamics of the default barrier have to be similar to the dynamics of the firm value process². Therefore, we assume that the stochastic default barriers are also affected by some common factors: the common Brownian motion and the common Hawkes process with general exciting function.

Finally, we conduct an extensive sensitive analysis to study the effect of stochastic default barriers on the probability of default and default correlation. Specifically, we study the change in default probability and default correlation with respect to the various parameters governing the stochastic barriers. Our analysis shows that the probability of default obtained by a Hawkes jump-diffusion model with a stochastic

¹ For instance, Ruf [31] explicitly modeled both firm value and default barrier by two independent jump-diffusion stochastic processes. Intuitively, the firm value and the default barrier may be affected by some common market factors, so that both processes may jump together. To address this, Dong and Wang [7] models them using two dependent jump-diffusion processes.

² It is important to highlight here that, although we are considering a default barrier, which is stochastic in nature, we still assume that the default occurs only at maturity time T , unlike the default barrier model of Black and Cox [3] which allows for early default prior to maturity T . One can call the stochastic default barrier in this paper as a stochastic default point, since default can occur only at time T , however, following Wang [36], we call it stochastic default barrier by which we mean a default barrier (point) at maturity time T , which is stochastic.

default barrier is close to that obtained by a Hawkes jump-diffusion model with a constant default barrier. However, there is a significant impact of the inclusion of a stochastic default barrier on the default correlation.

The remainder of the paper is organized as follows. Section 2 gives the literature review of credit risk modeling. Section 3 builds the firm value model with stochastic default barriers. Section 4 derives the analytic solution for the default probability and the default correlation. Section 5 presents the sensitivity analysis to study the effect of stochastic default barriers. Section 6 concludes the paper and discusses future directions.

2 Literature review

Credit risk analysis is a central topic in modern finance and consists of finding the likelihood of default of an obligor going into debt. Basically, in the area of credit risk, there are two categories of models: reduced-form models and firm value models, also known as structural models. The structural approach, pioneered by Merton [26] and Black and Cox [3], gives an explicit mechanism of default in terms of a relation between liabilities and assets at maturity time T , a firm defaults when the company value drops below a default barrier (liabilities). This last model has been generalized in various directions by incorporating stochastic interest rates, default at any time, correlated defaults, and so on, see for instance Longstaff and Schwartz [20], Zhou [38] and references therein. On the other hand, reduced-form models, introduced by Duffie and Singleton [8] and Jarrow and Turnbull [15], set the default times as non-negative random variables with distributions depending on the economic co-variables. Structural models are more grounded in economic fundamentals and give a practical and realistic description for the occurrence of default. Interested readers can refer to Schönbucher [32] and Lando [17], which present development of structural models starting from Merton [26] and their long-established significance within credit modeling.

One crucial step of Merton's modeling approach is the assumption about the evolution of firm value as diffusion process. But because of the continuity of diffusion processes, a sudden negative jump in the firm's value is impossible; consequently, a firm can never default unexpectedly. However, many incidences had occurred in the financial world when a firm suddenly went bankrupt, for instance, the Enron scandal in 2001. Also, diffusion models cannot explain why some corporate bonds have large credit spreads in a short maturity time.

Therefore, to abide by reality, i.e., produce unexpected defaults, many authors set models that incorporated jumps into the processes governing dynamics of firm values. For example, Zhou [39] proposes a model with jump risk to describe the unforeseen occurrence of default, Willemann [37] extends a structural jump-diffusion model to handle correlations through the diffusion of assets and common jumps, Selvamuthu et al. [5] models the credit risk by a discrete-time Markov chain subject to a catastrophe, the default, and in Tardelli [34, 35], the author studies the problem of hedging a defaultable claim, given the dynamics of the stocks driven by jump processes.

To this end, we need models that are capable of incorporating the default clustering effect to get a more realistic dependence structure, see for instance Leonidov and Rumyantsev [18] and Kim et al. [16]. In this direction, Luciano and Schoutens [22] and Moobrucker [27] propose a model based on Lévy processes that governs the dynamics of firm values using time changed Brownian motions. They introduce dependence by considering a common stochastic time change, which is assumed to be made up of idiosyncratic and systemic components. Though it provides a realistic and flexible dependence structure, it does not incorporate default contagion. However, Poisson jump-diffusion models and even Lévy models are incapable of capturing clustering of jumps due to the independent increments property.

To overcome the limitations of these models, Giesecke and Goldberg [10] and Giesecke [9] extend structural models in order to incorporate contagion effects. In Berndt et al. [2], the authors propose a family of multi-factor Heath-Jarrow and Morton (HJM) models to study default clustering, default correlation and to capture credit shocks in a framework of common jumps. To capture two important sources of default clustering, which are the exposure of firms to a systemic risk process, and contagion, Giesecke et al. [11] develop a model based on point processes and Pasricha and Goel [29] and Pasricha et al. [30] develop models based on self-exciting exponential Hawkes jump-diffusion processes. These processes are able to reproduce the jump clustering effect because the occurrence of a jump increases the likelihood of future jumps occurring. This is the major advantage of these models over the standard jump-diffusion models. Recently, Ma and Xu [24] proposes a structural-based approach and derives an explicit expression for firm values and default correlation by modeling the firm value using Hawkes jump-diffusion process, where the Hawkes processes have exponential exciting functions. As a typical example of its application in derivatives pricing, Ma et al. [25], and Liu and Zhu [19] adopt Hawkes jump-diffusion processes for pricing vulnerable options and variance swaps, respectively. Pasricha and Goel [29] considers a Hawkes jump-diffusion model to price power exchange options. Ma et al. [23] extends the previous model on exchange options, to allow for clustered jump contagion dynamics. In Sharma et al. [33], the authors study the problem of valuation of equity-indexed annuities (EIA), when the stock index follows a Hawkes jump-diffusion model. Moreover, Pasricha et al. [30] presents a closed-form solution for the probability of default and the default correlation using the characteristic function, substantially improving the computational efficiency.

For an overview of the literature related to Hawkes jump-diffusion processes and their applications in finance, see Hawkes [13] and, for more details, Hawkes [14].

3 The model description

The uncertainty in the economy is assumed to be described by $(\Omega, \mathcal{F}, P)$, a complete real world probability space, satisfying the usual conditions and on which all the processes are defined. As usual in the existing literature, to avoid too many technicalities, in the rest of the paper, we restrict our attention to two reference firms, even if this procedure can be extended to a larger number of firms.

Let the dynamics of the values of two underlying firms, say V_1 and V_2 , be described by Hawkes jump-diffusion processes. The uncertainty resulting from the systemic risk is described by a common Hawkes process, while a specific Brownian motion and a specific Hawkes process are introduced for each firm to better describe the uncertainty arising out of idiosyncratic shocks. Note that we assume that the Hawkes processes have exciting functions not only exponential but with a general structure.

3.1 Default model setup

On a fix finite time horizon $[0, T]$, let us consider three independent Hawkes processes N , N_1 and N_2 . Their intensity processes are given by

$$\lambda_t = \lambda \left(\int_0^t h(t-s) dN_s \right), \quad \text{and} \quad \lambda_{i,t} = \lambda_i \left(\int_0^t h_i(t-s) dN_{i,s} \right), \quad i = 1, 2.$$

where $\lambda : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ and $\lambda_i : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ are locally integrable, left continuous functions and $h : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ and $h_i : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ are assumed to be integrable functions, namely $\int_0^{+\infty} h(t)dt < +\infty$ and $\int_0^{+\infty} h_i(t)dt < +\infty$. The functions λ , λ_i and h , h_i are commonly referred in literature as intensity functions and exciting, or kernel, functions. The jump times of the processes N and N_i are respectively given by the sequences of random variables $\{T_k\}_{k \geq 1}$ and $\{T_{i,k}\}_{k \geq 1}$, and

$$N_t = \sum_{k=1}^{\infty} \mathbf{1}_{\{T_k \leq t\}}, \quad \text{and} \quad N_{i,t} = \sum_{k=1}^{\infty} \mathbf{1}_{\{T_{i,k} \leq t\}}, \quad i = 1, 2.$$

The Hawkes process N , is assumed to describe the common jump process, in the sense that, N represents the jumps resulting from systemic shocks that affects all the assets in the financial market, such as macroeconomic events or financial crisis. On the other hand, the process N_i is assumed to reflect idiosyncratic jumps, namely, the jumps resulting from the firm-specific events, such as investment losses or some bad news specific of the firm labelled by i . In this setting, if a systemic shock occurs, then the intensity process λ , of the common Hawkes process, increases with a jump of size $h(0)$, and then this increment decays at a speed which depends on the particular structure chosen for the exciting function h . In other words, when a systemic shock occurs, the probability law of the occurrence of future systemic shocks increases, thus resulting in jump clustering. Further, larger values of $h(0)$ implies more clustering of jumps and, even, with a smaller value of the speed of the decay of h , we have more clustering of jumps.

In Poisson jump models, due to the stationarity and the independence of the increments, which implies also the Markov property, this self-exciting behaviour cannot be observed, in the sense that the occurrence of a jump doesn't stimulate the future jumps. Hence, the self-exciting property of the Hawkes processes is the key that make these processes appropriate to model the jump clustering phenomena. In the linear case, namely when $\lambda(u) = \lambda_0 + u$, $u \in \mathbb{R}$ and $\lambda_0 \in \mathbb{R}^+$, if the function h is always null, then

the process reduces to Poisson process with constant intensity λ_0 . An analogous observation can be made for the idiosyncratic Hawkes processes.

The dynamics of the values of the firms V_1 and V_2 are assumed to be governed by Hawkes jump-diffusion processes. More precisely, for each firm labelled by i , we consider two independent Hawkes processes in jump components, N , the common Hawkes process, to capture the systemic shocks, and N_i , for the idiosyncratic shocks. The magnitudes of the jump sizes are a sequence $\{Z_{i,k}\}_{i=1,2,k=1,2,\dots}$ of random variables which are independent and identical distributed and such that, for each k , $\mathbb{E}[Z_{i,k}] = \beta_i$. Regarding the continuous component of the model, let B and B_i be independent standard Brownian motions. The process B is common to both V_1 and V_2 and, in this way, represents the systemic risk, that affects all the firms across the market and, similarly, the process B_i is specific of the firm labelled by i , and thus represents the uncertainty resulting from idiosyncratic risk. As a result, we consider a correlation in both the continuous part and the jump part. Therefore, B and N represent the risk sources in firm values resulting from systemic risk. Specifically, assume that

$$W_i = B_i + \gamma_i B,$$

where σ_i is the volatilities of the underlying firm values, and to avoid too many technicalities, μ_i and σ_i are assumed to be positive real constant.

Summing up, setting, for $i = 1, 2$, $\eta_i := \mu_i - \frac{1}{2}\sigma_i^2(1 + \gamma_i^2)$, and, as usual, for each process ζ_t , $\Delta\zeta_t := \zeta_t - \zeta_{t-}$, for the dynamics of V_1 and V_2 we have that

$$\begin{aligned} \frac{dV_{i,t}}{V_{i,t-}} &= \eta_i dt + \sigma_i dW_{i,t} + d \left[\sum_{k=1}^{N_i+N_{i,t}} Z_{i,k} - \beta_i \int_0^t \lambda_s ds - \beta_i \int_0^t \lambda_{i,s} ds \right] \\ &= [\eta_i - \beta_i(\lambda_t + \lambda_{i,t})] dt + \sigma_i dW_{i,t} + \Delta \left(\sum_{k=1}^{N_i+N_{i,t}} Z_{i,k} \right). \end{aligned}$$

Following the existing literature, see for instance Wang [36], we assume stochastic default barriers. To this end the dynamics of the default barriers D_i , $i = 1, 2$, are affected by the common factors, B and N , in order to reflect the consideration that the liabilities consist of assets depending on the common factors which affect all the firms. We believe reasonable to study the effects of stochastic barriers on option prices, since large changes on the value of financial assets could be caused by the arrival of important news. Furthermore, in the case of the stochastic barrier, increasing the value of the counterparty's assets could increase the probability of default, while in the constant case this is not possible. Hence, a representation of the dynamics of the default barrier D_i is

$$\begin{aligned} \frac{dD_{i,t}}{D_{i,t-}} &= \left(\mu_{D,i} - \frac{1}{2}\sigma_{D,i}^2 \right) dt + \sigma_{D,i} dB_t + d \left[\sum_{k=1}^{N_i} Z_{D,i,k} - \beta_{D,i} \int_0^t \lambda_s ds \right] \\ &= \left(\mu_{D,i} - \frac{1}{2}\sigma_{D,i}^2 - \beta_{D,i}\lambda_t \right) dt + \sigma_{D,i} dB_t + \Delta \left(\sum_{k=1}^{N_i} Z_{D,i,k} \right), \end{aligned}$$

where, analogous to the previous setting, $\sigma_{D,i}$ is the volatility of the default barriers. The independent and identical distributed random variables $\{Z_{D,i,k}\}_{i=1,2,k=1,2,\dots}$ are such that $\mathbb{E}[Z_{D,i,k}] = \beta_{D,i}$, and represent the magnitudes of the jump sizes.

The initial value of the default barrier is assumed to be $D_{i,0}$. Thence, the default occurs when the firm value $V_{i,T}$ drops below the stochastic threshold $D_{i,T}$. Since the common factors B and N affect, simultaneously, the firm value and the stochastic default barrier, if the firm value decreases, due to these factors, the probability of default may not increase. However, in case of constant default barrier, a drop in the firm value may increase the probability of default.

Hence, this is the reason why we have to consider the dynamics of V_i/D_i .

Theorem 1 For $i = 1, 2$, setting $\tilde{\eta}_i = \mu_i - \frac{1}{2}\sigma_i^2(1 + \gamma_i^2)$,

let

$$X_{i,t} = V_{i,0} \exp \left\{ \sigma_i W_{i,t} + \tilde{\eta}_i t - \beta_i \int_0^t (\lambda_s + \lambda_{i,s}) ds \right\},$$

and $J_{i,t} = \prod_{k=1}^{N_i+N_{i,t}} (1 + Z_{i,k})$, assuming that the product over a null set is one, we have that

$$V_{i,t} = X_{i,t} J_{i,t}. \quad (1)$$

Proof Taking into account that X_i are continuous stochastic processes, and that J_i are pure jump processes, setting

$$q_{i,t} = \lambda_t + \lambda_{i,t}$$

then

$$dX_{i,t} = [\eta_i - \beta_i q_{i,t}] X_{i,t} dt + \sigma_i X_{i,t} dW_{i,t} \quad (2)$$

and

$$\Delta J_{i,t} := J_{i,t} - J_{i,t-} = J_{i,t-} \Delta \left(\sum_{k=1}^{N_i+N_{i,t}} Z_{i,k} \right). \quad (3)$$

Therefore, by Ito's formula for semi-martingales, we have

$$\begin{aligned} dV_{i,t} &= d(X_{i,t} J_{i,t}) = dX_{i,t} J_{i,t-} + X_{i,t-} \Delta J_{i,t} + d[X_i, J_i](t) \\ &= dX_{i,t} J_{i,t-} + X_{i,t-} \Delta J_{i,t}. \end{aligned} \quad (4)$$

Recalling that $\eta_i = \mu_i - \frac{1}{2}\sigma_i^2(1 + \gamma_i^2)$, and substituting Eqs. (2) and (3) into Eq. (4), we get

$$dV_{i,t} = X_{i,t} J_{i,t-} [\eta_i - \beta_i q_{i,t}] dt + \sigma_i X_{i,t} J_{i,t-} dW_{i,t} + X_{i,t} J_{i,t-} \Delta \left(\sum_{k=1}^{N_i + N_{i,t}} Z_{i,k} \right),$$

which in turn implies that $V_{i,t-} = X_{i,t} J_{i,t-}$. $\square$

Remark 1 Theorem 1 provides the result for the dynamics of the values of the firms. The procedure followed is inspired by that of Proposition 2.1 in Ma and Xu [24].

Corollary 2 For $i = 1, 2$, let $K_{i,t} = \prod_{k=1}^{N_i} (1 + Z_{D,i,k})$, and

$$\tilde{X}_{i,t} = D_{i,0} \exp \left\{ \sigma_{D,i} B_t + \left(\mu_{D,i} - \frac{1}{2} \sigma_{D,i}^2 \right) t + \beta_{D,i} \int_0^t \lambda_s ds \right\},$$

assuming that the product over a null set is one, we have that

$$D_{i,t} = \tilde{X}_{i,t} K_{i,t}. \quad (5)$$

Proof Analogously to the proof of Theorem 1, $\tilde{X}_{i,t}$ are continuous stochastic processes and K_i are pure jump processes, then

$$d\tilde{X}_{i,t} = [\mu_{D,i} - \beta_{D,i} \lambda_t] \tilde{X}_{i,t} dt + \sigma_{D,i} \tilde{X}_{i,t} dB_t,$$

and $\Delta K_{i,t} := K_{i,t-} \Delta \left(\sum_{k=1}^{N_i} Z_{i,k} \right)$. Therefore, according to Ito's formula for semi-martingales, we get that

$$\begin{aligned} dD_{i,t} &= d\tilde{X}_{i,t} K_{i,t-} + \tilde{X}_{i,t} \Delta K_{i,t} \\ &= \tilde{X}_{i,t} K_{i,t-} (\mu_{D,i} - \beta_{D,i} \lambda_t) dt + \sigma_{D,i} \tilde{X}_{i,t} K_{i,t-} dB_t + \tilde{X}_{i,t} K_{i,t-} \Delta \left(\sum_{k=1}^{N_i} Z_{i,k} \right) \end{aligned}$$

which in turn implies that $D_{i,t-} = \tilde{X}_{i,t} K_{i,t-}$. $\square$

4 The probabilities of default and default correlation

From now on, as a data driven choice, the Hawkes processes are assumed to be linear, then, given the initial values of the intensities, λ_0 and $\lambda_{i,0}$, positive real constants, let the intensity functions be such that

$$\lambda_t = \lambda_0 + \int_0^t h(t-s) dN_s = \lambda_0 + \sum_{k=1}^{\infty} h(t-T_k) \mathbf{1}_{\{T_k \leq t\}},$$

and

$$\lambda_{i,t} = \lambda_{i,0} + \int_0^t h_i(t-s) dN_{i,s} = \lambda_{i,0} + \sum_{k=1}^{\infty} h_i(t - T_{i,k}) \mathbf{1}_{\{T_{i,k} \leq t\}}, \quad i = 1, 2.$$

Moreover, we use the following notations. Let T denote the maturity time of a debt contract, for $i = 1, 2$, we set $\eta_i = \mu_i - \frac{1}{2}\sigma_i^2(1 + \gamma_i^2)$ and

$$\begin{aligned} N_T &= \tilde{n}, & x_i &= \eta_i - \beta_i \lambda_0 - \beta_i \lambda_{i,0}, \\ N_{i,T} &= n_i, & x_{D,i} &= \mu_{D,i} - \frac{1}{2}\sigma_{D,i}^2 - \beta_{D,i} \lambda_0, \end{aligned}$$

$$\begin{aligned} Y_i &= -\beta_i \int_0^T \sum_{j=1}^{N_i} h(s - T_j) ds - \beta_i \int_0^T \sum_{j=1}^{N_{i,s}} h_i(s - T_{i,j}) ds \\ &= -\beta_i \sum_{j=1}^{\tilde{n}} \int_{T_j}^T h(s - T_j) ds - \beta_i \sum_{j=1}^{n_i} \int_{T_{i,j}}^T h_i(s - T_{i,j}) ds, \end{aligned}$$

and

$$Y_{D,i} = -\beta_{D,i} \int_0^T \sum_{j=1}^{N_s} h(s - T_j) ds = -\beta_{D,i} \sum_{j=1}^{\tilde{n}} \int_{T_j}^T h(s - T_j) ds.$$

On the finite time window $[0, T]$, for $i = 1, 2$, let the σ -algebra generated by the processes N and N_i be defined, as usual, by $\mathcal{F}_{i,t} = \sigma\{N_s, N_{i,s} : 0 \leq s \leq t\}$, and $\mathcal{F}_i = \mathcal{F}_{1,t} \cup \mathcal{F}_{2,t}$.

Proposition 3 Assume that we have the information contained in $\mathcal{F}_{i,T}$, and that $N_T = \tilde{n}$, $N_{i,T} = n_i$. Given $\{Z_{i,k}\}_{k=1,2,\dots}$, the sequence of jump sizes governing the firm values, and $\{Z_{D,i,k}\}_{k=1,2,\dots}$, the sequence of jump sizes governing the default barriers, we assume that

$$\log(1 + Z_{i,k}) \sim N(a_i, b_i) \quad \text{and} \quad \log(1 + Z_{D,i,k}) \sim N(a_{D,i}, b_{D,i}).$$

Setting, for each t , $\xi_{i,t} := \ln\left(\frac{V_{i,t}}{D_{i,t}}\right)$,

$$\beta_i := \exp\left\{a_i + \frac{1}{2}b_i^2\right\} - 1 \quad \text{and} \quad \beta_{D,i} := \exp\left\{a_{D,i} + \frac{1}{2}b_{D,i}^2\right\} - 1,$$

then, $\xi_{i,T}$ follows a bivariate normal distribution with the following properties:

$$\begin{aligned} M_i &= E[\xi_{i,T}] &= \xi_{i,0} + (x_i - x_{D,i})T + (Y_i - Y_{D,i}) + (\tilde{n} + n_i)a_i - \tilde{n}a_{D,i}, \\ V_i^2 &= \text{Var}(\xi_{i,T}) &= \sigma_i^2 T + (\sigma_i \gamma_i - \sigma_{D,i})^2 T + (\tilde{n} + n_i)b_i - \tilde{n}b_{D,i}, \end{aligned}$$

and

$$\rho_{1,2} = \frac{\text{Cov}(\xi_{1,T}, \xi_{2,T})}{V_1 V_2} = \frac{(\sigma_1 \gamma_1 - \sigma_{D,1})(\sigma_2 \gamma_2 - \sigma_{D,2})T}{V_1 V_2}. \quad (6)$$

Proof Taking logarithm of Eqs. (1) and (5), we have

$$\ln(V_{i,T}) = \ln(V_{i,0}) + x_i T + \sigma_i W_{i,T} + Y_i + \sum_{k=1}^{\tilde{n}+n_i} \ln(1 + Z_{i,k}) \quad (7)$$

$$\ln(D_{i,T}) = \ln(D_{i,0}) + x_{D,i} T + \sigma_{D,i} B_T + Y_{D,i} + \sum_{k=1}^{\tilde{n}} \ln(1 + Z_{D,i,k}). \quad (8)$$

Further, subtracting Eq. (8) from Eq. (7), we have for $i = 1, 2$,

$$\begin{aligned} & \xi_{i,T} - \xi_{i,0} - (x_i - x_{D,i})T \\ &= (\sigma_i W_{i,T} - \sigma_{D,i} B_T) + Y_i - Y_{D,i} + \sum_{k=1}^{\tilde{n}+n_i} \ln(1 + Z_{i,k}) - \sum_{k=1}^{\tilde{n}} \ln(1 + Z_{D,i,k}) \\ &= [\sigma_i B_{i,T} - (\sigma_{D,i} - \gamma_i \sigma_i) B_T] + Y_i - Y_{D,i} + \sum_{k=1}^{\tilde{n}+n_i} \ln(1 + Z_{i,k}) - \sum_{k=1}^{\tilde{n}} \ln(1 + Z_{D,i,k}). \end{aligned}$$

This implies that $\xi_{1,T}$ and $\xi_{2,T}$ follow normal distributions. $\square$

Proposition 4 Given T , the maturity time of a debt contract, under the assumptions of Proposition 3, the probabilities of default and default correlation are given as follows:

$$p_i(T) = E \left[\Phi \left(-\frac{M_i}{V_i} \right) \right], \quad (9)$$

where $\Phi(\cdot)$ is the Cumulative Distribution Function (CDF) of the standard normal distribution. Setting $\Phi_2(\cdot, \cdot, \rho)$, as the CDF of the bivariate normal distribution with zero mean and covariance matrix, given by $\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$, and

$$p_{1,2}(T) = E \left[\Phi_2 \left(-\frac{M_1}{V_1}, -\frac{M_2}{V_2}, \rho \right) \right], \quad (10)$$

where E is the expectation over the paths of the Hawkes processes involved. Further, the default correlation is

$$\rho_{1,2}(T) = \frac{p_{1,2}(T) - p_1(T)p_2(T)}{\sqrt{p_1(T)(1 - p_1(T))}\sqrt{p_2(T)(1 - p_2(T))}}. \quad (11)$$

Proof Under the assumptions of Proposition 3, the conditional probability of default is

$$\begin{aligned}
P(V_{i,T} \leq D_{i,T} \mid \mathcal{F}_{i,T}) &= P\left(\ln\left(\frac{V_{i,T}}{D_{i,T}}\right) \leq 0 \mid \mathcal{F}_{i,T}\right) \\
&= P\left(\frac{\xi_{i,T} - E[\xi_{i,T}]}{\sqrt{\text{Var}(\xi_{i,T})}} \leq -\frac{E[\xi_{i,T}]}{\sqrt{\text{Var}(\xi_{i,T})}} \mid \mathcal{F}_{i,T}\right) \quad (12) \\
&= P\left(\frac{\xi_{i,T} - M_i}{V_i} \leq -\frac{M_i}{V_i} \mid \mathcal{F}_{i,T}\right) = \Phi\left(-\frac{M_i}{V_i}\right).
\end{aligned}$$

Therefore, taking the mean value of the conditional probability of default (12), we get the unconditional probability of default for i th firm as given in (9). Similarly, given $\mathcal{F}_T$, and ρ , as given in Eq. (6), the conditional joint probability of default is

$$\begin{aligned}
P(V_{1,T} \leq D_{1,T}, V_{2,T} \leq D_{2,T} \mid \mathcal{F}_T) &= P(\xi_{1,T} \leq 0, \xi_{2,T} \leq 0 \mid \mathcal{F}_T) \\
&= P\left(\frac{\xi_{1,T} - M_1}{V_1} \leq -\frac{M_1}{V_1}, \frac{\xi_{2,T} - M_2}{V_2} \leq -\frac{M_2}{V_2} \mid \mathcal{F}_T\right) \quad (13) \\
&= \Phi_2\left(-\frac{M_1}{V_1}, -\frac{M_2}{V_2}, \rho\right).
\end{aligned}$$

As before, taking the mean value of the conditional joint probability of default as given in Eq. (13), we get the unconditional joint probability of default as given in Eq. (10). Consequently, the default correlation is obtained as given by Eq. (11). $\square$

By the results obtained in Propositions 3 and 4, we are not able to get the analytical solutions to determine completely the price dynamics, the default probability and the default correlation. Thus, we have to rely on the simulations to estimate these quantities. Firstly, the paths of Hawkes processes are simulating using thinning algorithm from Ogata [28]. Then, the required measures are estimated by taking sample means over the simulated paths. Thinning algorithm for N_t is shown below, and the same algorithm is applied to obtain the paths of the processes $N_{i,t}$, $i = 1, 2$.

In the reminder of the paper, we assume that the exciting functions have an exponential structure. To this end, given the positive constants $\theta, \delta, \theta_1, \delta_1, \theta_2$, and, δ_2 , let

$$h(t) = \theta e^{-\delta t} \quad \text{and} \quad h_i(t) = \theta_i e^{-\delta_i t}, \quad i = 1, 2.$$

Note that, with this choice, supposing that $0 < \theta < \delta$ and $0 < \theta_i < \delta_i$, the expected number of jumps in $[0, t]$ are

$$\mathbb{E}(N_t) = \lambda_0 t \left(1 - \frac{\theta}{\delta}\right)^{-1} \quad \text{and} \quad \mathbb{E}(N_{i,t}) = \lambda_{i,0} t \left(1 - \frac{\theta_i}{\delta_i}\right)^{-1}.$$

Moreover,

$$Y_i = \sum_{j=1}^{\tilde{n}} \frac{\beta_i \theta}{\delta} (e^{-\delta(T-T_j)} - 1) + \sum_{j=1}^{n_i} \frac{\beta_i \theta_i}{\delta_i} (e^{-\delta_i(T-T_{i,j})} - 1),$$

and

$$Y_{D,i} = \sum_{j=1}^{\tilde{n}} \frac{\beta_{D,i} \theta}{\delta} (e^{-\delta(T-T_j)} - 1).$$

4.1 Thinning algorithm

The thinning algorithm used to simulate the paths of a Hawkes process is detailed in Ogata's work [28], as outlined below.

- (i) Set $\lambda \leftarrow \lambda_0$ and $k \leftarrow 1$.
- (ii) Generate a random variable U with standard uniform distribution, $U \sim U(0, 1)$, and set $s \leftarrow -\ln \frac{U}{\lambda}$.
If $s \leq T$, then $t_1 \leftarrow s$; otherwise go to Step (vi).
- (iii) Set $k \leftarrow k + 1$ and $\lambda \leftarrow \lambda_{t_{k-1}} + \theta$.
If $s \leq T$, then go to the Step (iv); otherwise go to Step (vi).
- (vi) Generate $U \sim U(0, 1)$, and let $s \leftarrow s - \ln \frac{U}{\lambda}$.
- (v) Generate $V \sim U(0, 1)$.
If $V \leq \lambda_s / \lambda$, a new event occurs at time s , i.e., $t_k \leftarrow s$, and then go to Step (iii); otherwise go to Step (iv).
- (vi) Stop. The times of the occurrence of the events are $t_1, t_2, \dots, t_k$.

5 Numerical results

First, we have a remark with some comments on the observations made in the numerical section of Ma and Xu [24]. These comments are justified by the numerical studies performed in this section through the sensitivity analysis for the proposed model.

Remark 2 In Section 3.1 in Ma and Xu [24], Fig. 1, the authors claimed that jump risk increases the default probability whereas jump clustering decreases it.

Taking into account the notations given in Sect. 4 of the present paper, and setting, for $i = 1, 2$, $R_i := e^{Y_i}$, $K_i := \tilde{n} + n_i$, the threshold level $C_{i,t}$, having an exponential model $C_{i,t} = e^{k_i t} D_i$, where k_i is regarded as the growth rate of firm i 's liabilities and

$$f_i(R_i, K_i) = - \frac{\log(R_i V_{i,0}/D_i) + (\mu_i - k_i - \beta_i \lambda_0 - \beta_i \lambda_{i,0} - \frac{1}{2} \sigma_i^2)T + a_i K_i}{\sqrt{\sigma_i^2 T + b_i^2 K_i}},$$

the expression for the probability of default of Ma and Xu [24] is given by

$$P(V_{i,T} \leq C_{i,T} \mid \mathcal{F}_{i,T}) = \Phi(f_i(R_i, K_i)).$$

We observe that $\ln(R_i)$ is a negative value (since δ and δ_i are positive values) and it decreases with increases values of θ and θ_i . Also, f_i decreases when $\ln(R_i)$ increases, which implies that f_i increases with the increases values of θ and θ_i . Therefore, the probability of default increases with the values of θ or θ_i even if all other parameters are kept equal. Since the jump clustering increases with increased values of θ_i , we have that the probability of default increases with the jump clustering, this is in contradiction with the observation made in Ma and Xu [24].

The second observation made in Ma and Xu [24] is about default correlation. In Figures 7 and 8, the authors claimed that the default correlation becomes stronger with the jump clustering. But, on the contrary, in Figure 6 of their article, we observe that default correlation decreases with increasing values of λ_0 , since the jump clustering increases with λ_0 .

In this section, we focus on stochastic default barriers. We perform a sensitivity analysis to explore the impact of the jump clustering, and the various parameters governing stochastic default barriers on the probability of default and default correlation. A comparative analysis between the models proposed is given to illustrate the impact of the jump clustering, and of the stochastic default barriers.

To get that half of the jumps results from the systemic factors and the other half grows out of the idiosyncratic factors, let $\lambda_0 = \lambda_{1,0} = \lambda_{2,0} = 1$, $\theta = \theta_1 = \theta_2 = 1$ and $\delta = \delta_1 = \delta_2 = 2$. For comparison purpose, the expected number of jumps per unit time, for the systemic jumps and for the idiosyncratic jumps (i.e. the intensities of the corresponding Poisson processes) are

$$\lambda_0 \left(1 - \frac{\theta}{\delta}\right)^{-1} \quad \text{and} \quad \lambda_0 \left(1 - \frac{\theta_i}{\delta_i}\right)^{-1}.$$

The parameter values in the base case, listed in Table 1, are chosen just for experimental purposes. As usual, for sensitivity analysis and comparison purposes, the value of only one parameter is changed at one time, while the other parameters' values are kept the same, as in the base case.

Figure 1 presents the comparison of the probability of default obtained by the models under study, namely, the proposed model, the Poisson jump-diffusion model, and the Geometric Brownian motion. For each model, constant and stochastic default barriers are considered. The figure shows us that the probability of default is an increasing function of time for all the models. For each model (Hawkes jump-diffusion, Poisson jump-diffusion, and Geometric Brownian

Table 1 Values of the Parameters in the Base Case

Parameters	Values	Parameters	Values
$V_{1,0}$	18	$V_{2,0}$	18
$D_{1,0}$	10	$D_{2,0}$	10
μ_1	0.02	μ_2	0.02
$\mu_{D,1}$	0.02	$\mu_{D,2}$	0.02
σ_1	0.3	σ_2	0.15
$\sigma_{D,1}$	0.3	$\sigma_{D,2}$	0.15
$\lambda_{1,0}$	1	$\lambda_{2,0}$	1
θ_1	1	θ_2	1
δ_1	2	δ_2	2
λ_0	1	θ	1
δ	2		
γ_1	-0.6	γ_2	-0.6
a_1	0	a_2	0
b_1	0.1	b_2	0.1
$a_{D,1}$	0	$a_{D,2}$	0
$b_{D,1}$	0.05	$b_{D,2}$	0.05

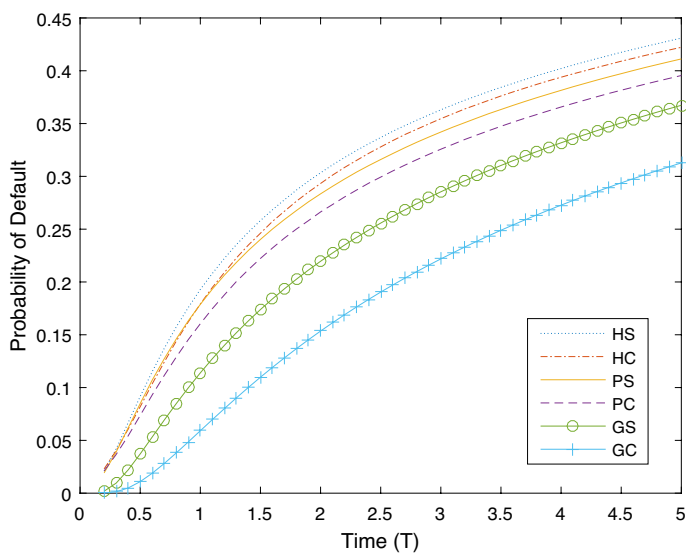


Fig. 1 Probability of default against time for six different models. *HS* Hawkes jump-diffusion stochastic default barrier, *HC* Hawkes jump-diffusion constant default barrier, *PS* Poisson jump-diffusion stochastic default barrier, *PC* Poisson jump-diffusion constant default barrier, *GS* Geometric Brownian motion stochastic default barrier, *GC* Geometric Brownian motion constant default barrier

motion), the probability of default is higher for the case with stochastic default barrier compared to the case of the constant default barrier.

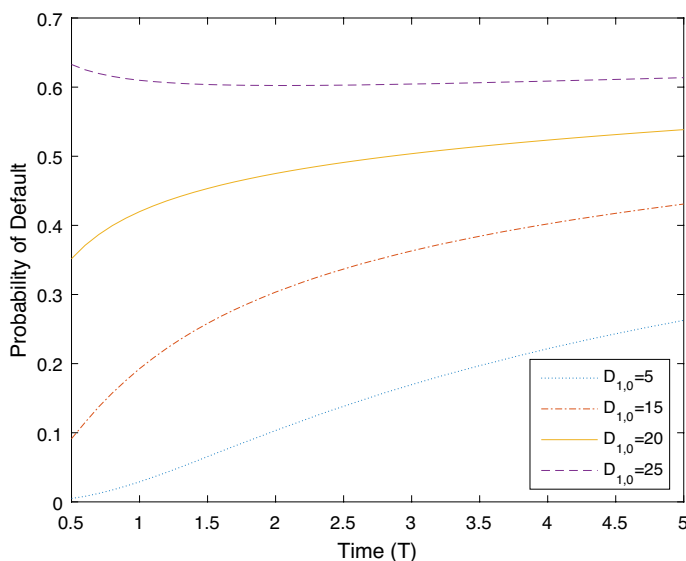


Fig. 2 Probability of default against time for different values of initial value $D_{1,0}$ of stochastic default barrier

Figure 2 shows the probability of default against the initial value of the default barrier $D_{1,0}$. In particular, the probability of default increases when the initial values of default barriers increase. Intuitively, we note that the large default barrier makes credit default more likely. In addition, for a given initial default barrier, the probability of default increases with time, and this is a confirmation of the observation made by Fig. 1.

In Fig. 3, we present the comparison of the default correlation obtained by the models under study. Note that for each model (Hawkes jump-diffusion, Poisson jump-diffusion, and Geometric Brownian motion), the default correlation is lower for the case with a constant default barrier compared to the stochastic default barrier.

Figure 4 gives the value of default correlation against time for different values of the intensity of the common Hawkes process and the firm-specific Hawkes process. As already observed from Fig. 3, the jump clustering decreases the default correlation, and the same observation is obtained from Fig. 4. The default correlation decreases with the increase in the initial intensities of the common Hawkes process and the firm-specific Hawkes processes. Figure 5 presents the value of default correlation against the jump size and the decay rate of the common Hawkes process that governs the stochastic default barriers. The observations in these figures agree with the observations in the first picture of Fig. 4 that jump clustering decreases the default correlation. Note that the sensitivity analysis in this section mainly focuses on the parameters governing stochastic default barriers. The results for δ_1 and θ_1 can be interpreted by the behavior of default correlation against $\lambda_{1,0}$. Also, due to

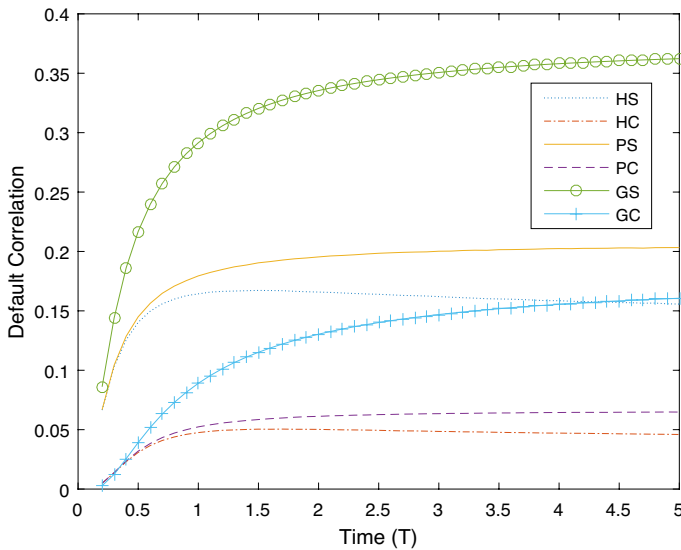


Fig. 3 Default correlation against time for six different models. *HS* Hawkes jump-diffusion stochastic default barrier, *HC* Hawkes jump-diffusion constant default barrier, *PS* Poisson jump-diffusion stochastic default barrier, *PC* Poisson jump-diffusion constant default barrier, *GS* Geometric Brownian motion stochastic default barrier, *GC* Geometric Brownian motion constant default barrier

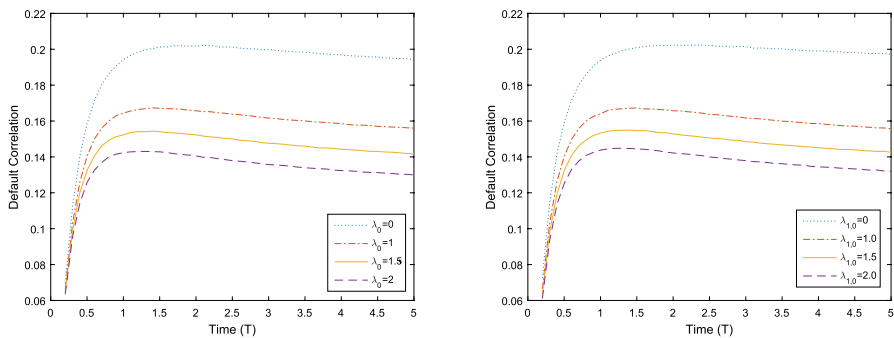


Fig. 4 Default correlation against time for different values of initial intensity λ_0 and $\lambda_{1,0}$ of the common and of the asset specific Hawkes process

symmetry, the same results holds for λ_{10} , δ_2 , θ_2 . This confirms the fact that θ and δ has an opposite effect on jump clustering.

We assumed that the firm value and the default barriers are governed by a common Hawkes process representing the systemic risk. Hence, we expected that part of the risk in the firm's value could be offset by the risk of default barriers, which makes us consider the impacts of the volatility of the default barrier. Figure 6 gives the effect of the volatility of the default barrier on the default correlation. Note that the default correlation increases with the increase in the volatility of the default

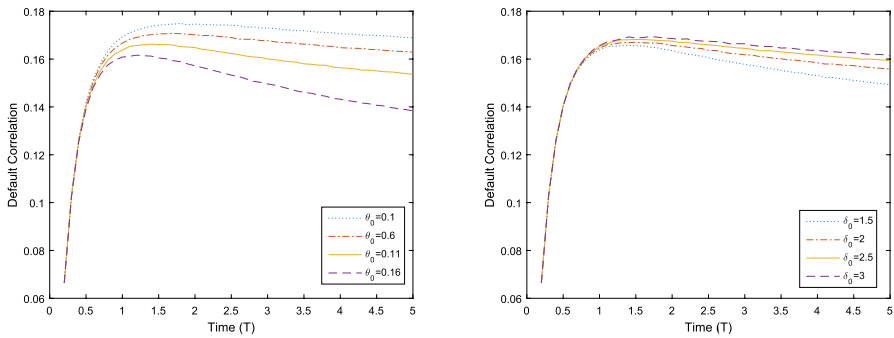


Fig. 5 The effects of the parameters θ and δ of the common Hawkes process governing stochastic default barrier on the default correlation

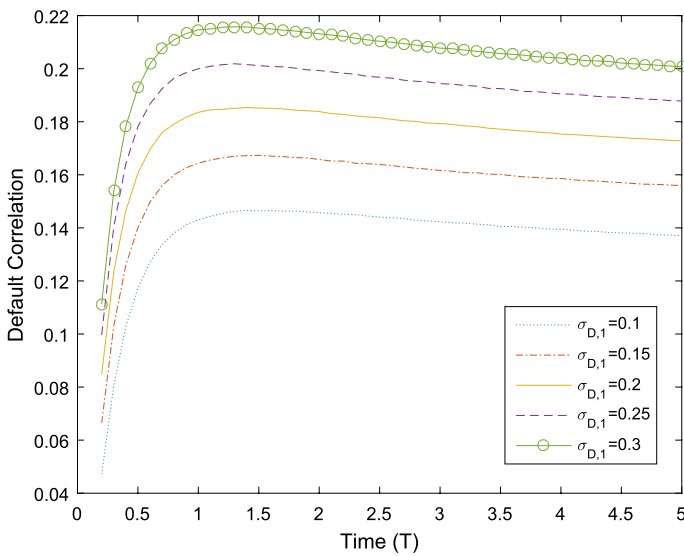


Fig. 6 The effect of volatility of stochastic default barrier on default correlation

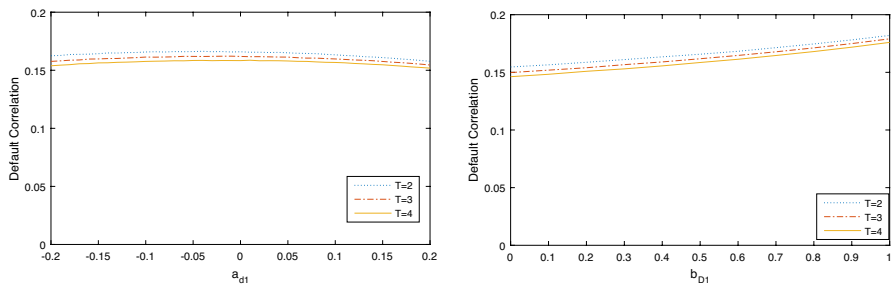


Fig. 7 Effects of the parameters $a_{D,1}$ and $b_{D,1}$ related to the jump size distribution for different risk horizon T

barrier, and that Fig. 7 show the robustness of the default correlation with respect to the mean and variance of the jump amplitude in the stochastic default barrier.

6 Conclusion and future work

The stochastic default barriers, and a Hawkes jump-diffusion model for the dynamics of a firm's value is considered. The dependence between the firms is investigated in both the continuous part and the jump part. We incorporated a common Brownian motion and a common Hawkes process with a general structure of the exciting function to model the uncertainty resulting from the systemic risk. Further, the stochastic default barriers are assumed to be affected by common factors. A closed-form expression is given for the probability of default and the default correlation. Finally, the sensitivity analysis is presented to illustrate the effect of various parameters governing stochastic default barriers. We observed that jump clustering increases the probability of default but, on the contrary, it decreases the default correlation.

We assumed that default could occur only at maturity. In general, the default can happen when the firm value drops below the default barrier, which is the first passage time model. Future research will consider the first passage time structural model with jump clustering.

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