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Fair fee analysis of inflation-linked variable annuities with GLWB: exploring n -dimensional sub-account dynamic

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ABSTRACT

Variable annuities, in practice, employ single-asset models for pricing, hedging, and fee analysis, even though the guarantees usually involve numerous assets. This pragmatic expedient presents difficulties, including the possibility of basis risk. This research investigates Inflation-Linked Variable Annuities (ILVAs) with Guaranteed Lifetime Withdrawal Benefits (GLWB) through an advanced quantitative model. Specifically, a multi-dimensional sub-account framework where the fund value depends on a basket of connected assets has been considered. Additionally, Markovian health benefits have been incorporated into the analysis, adding another layer of sophistication to the modeling. A mixed discrete-continuous time model is proposed to calculate the fair fee of the product, using equilibrium conditions between premium and benefits. A quadratic utility optimization problem has been developed from the insurance company's point of view to have the maximum sub-account fund value with a specific proportion of risk. A numerical investigation and sensitivity assessment were carried out to demonstrate the potential of this financial product and explore potential enhancements. At the conclusion, a thorough comparative study was conducted, comparing an existing model that includes only one risky asset and does not account for inflation advantages. Our main result outperforms the existing literature due to the multi-investment choices and inflation considerations. The results show that increased healthcare demand impacts insurance expenditures and premiums.

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Variable annuity; guaranteed lifetime withdrawal benefit; inflation benefit; health status; multiple assets; optimal portfolio

1. Introduction

Variable annuities (VAs) embedded with guarantees have become increasingly attractive in recent times, emerging as a preferred financial instrument for investment and retirement planning. These instruments, enhanced with features like inflation linkage and Markovian health benefits, provide investors with a distinctive avenue of protection against early death, health-related concerns, unfavorable post-retirement market fluctuations, and the erosive impact of inflation on retirement income. The growing popularity of these financial products can be attributed to heightened awareness of longevity risk, advancements in medical technology, and concerns about economic downturns. Variable annuities (VAs) are equity-linked annuities that offer investment guarantees to policyholders. It has gained prominence as a significant product in the insurance market. The collective sales of variable annuities in the United States amounted to approximately 100 billion. These variable annuities consist of a variety of inherent options and assurances known as guarantees, for which policyholders pay additional fees to access. Notable among these are Guaranteed Minimum Death Benefits

(GMDB) and Guaranteed Minimum Living Benefits (GMLB) as detailed by Hardy (2003) and Ledlie et al. (2008). The GMLB category encompasses four specific types of guarantees:

- (1) Guaranteed Minimum Accumulation Benefits (GMAB)
- (2) Guaranteed Minimum Income Benefits (GMIB)
- (3) Guaranteed Minimum Withdrawal Benefits (GMWB)
- (4) Guaranteed Lifetime Withdrawal Benefits (GLWB)

These added guarantees provide a sense of stability and security for policyholders as they approach retirement. The GLWB, introduced in 2004, gained considerable attraction and became the most favored guarantee within the variable annuity market (Drinkwater et al., 2014). Moreover, economic development has led to an increased demand for healthcare services, consequently driving up healthcare expenditures and health insurance premiums. These interlinked dynamics concerning the population, economic growth, and healthcare costs are a matter of concern for both public health policymakers and private health insurance management. The evolving landscape prompts a critical assessment of health insurance and its financial implications.

Variable annuities offer investors the ability to distribute their assets across different separate account funds, which can be seen as similar to mutual funds, unlike fixed annuities. Investors have the freedom to distribute their assets in any way they like, typically based on a proportional basis. While most variable annuity guarantees are associated with multiple assets, professionals primarily utilize single-asset (univariate) stochastic investing models for the purposes of pricing and risk management. Utilizing an n -dimensional portfolio is a strategy to mitigate the risk of a variable annuity, involving investments across diverse asset classes, each with its unique risk-return profile. By diversifying the portfolio, an insurer can reduce the overall risk without sacrificing too much potential gain. This approach embodies a forward-thinking paradigm, integrating modern portfolio theory and financial engineering to empower policyholders with a more sophisticated and dynamic investment experience.

However, the research in this area is much less; Ng and Li (2011) suggests employing a multi-variate regime-switching framework to estimate the collective returns on several assets. They utilize Monte-Carlo-based methods to determine the prices of GMMB and GMDB riders in cases when the underlying fund consists of two assets. Extending their previous work Ng and Li (2013) developed a model for Pricing and Hedging of Variable Annuity Guarantees using Multiasset Stochastic Investment Models. Building upon the concept of numerous sub-account funds, as explained in Ng and Li (2011) research, Da Fonseca and Ziveyi (2017) created an analytical framework to determine the value of GMMB and GMDB riders that are based on various underlying funds. Horneff et al. (2015) provided the Optimal life cycle portfolio choice with variable annuities offering liquidity and investment downside protection. They considered GMWB without inflation and health benefits. Models have been developed for the real-time Valuation of Large Variable Annuity Portfolios efficiently and quickly (see, e.g. Lin & Yang, 2020; Liu & Seng Tan, 2021; Moenig, 2021).

A time-series approach was created by Sharma et al. (2020) to determine a reasonable price for the variable annuity embedded with GLWB. They initially take into consideration a lump sum payout. Singh et al. (2023) extended their research by looking at the price issue of a variable annuity (VA) contract embedded with a GLWB rider. They took into account a variable withdrawal strategy and surrender benefit with a non-constant rate of interest.

Moosavi and Payandeh (2021) created a framework to assess variable long-term care annuities embedded with guaranteed lifetime withdrawal and restricted hospitalization coverage advantages. They employed a Geometric Brownian Motion (GBM) model with consistent volatility for two investment funds. Furthermore, their analysis focused on a restricted range of health condition scenarios. D'Amico et al. (2023) proposed and evaluated a new insurance product, i.e. the variable annuity product, accompanied by the health status and GLWB.

Based on empirical evidence, it's evident that changes in inflation rates significantly impact financial derivatives, specifically variable annuities. Equities don't effectively hedge against inflation risk, and traditional annuity products can't ensure a satisfactory standard of living. Consequently, the market has seen an increase in the availability of inflation derivatives. These instruments, like Treasury Inflation-Protected Securities (TIPS), UK inflation-linked gilt-edged securities (ILGs), and German Bunds, help mitigate inflation risks. Jarrow and Yildirim (2003) demonstrated that pricing TIPS and other derivatives within the Heath–Jarrow–Morton (HJM) framework provides insights into TIPS data parameter estimation. Numerous studies have investigated the management of risk and valuation of GLWB riders through both static and dynamic withdrawal methods. Tiong (2013b) developed a multi-dimensional Gaussian HJM framework in a complete market, presenting closed-form pricing formulas for inflation-linked variable annuities. Similarly, Singor et al. (2013) derived efficient and approximate semi-closed-form pricing formulas for index and year-on-year inflation options, incorporating stochastic volatility and stochastic interest rate. Feng and Vecer (2017) develop a risk-based capital framework for Guaranteed Minimum Withdrawal Benefits (GMWB), whereas Feng and Volkmer (2016) present analytical findings regarding hitting times essential for the valuation of Guaranteed Lifetime Withdrawal Benefits (GLWB). Analytical and optimum control techniques are employed to determine GLWB prices. Feng and Jing (2017) and Feng et al. (2025) examine efficient strategies for fund deposits and withdrawals, taking into account the relevant costs. These frameworks provide critical insights that enhance our model and underscore the necessity of include genuine withdrawal incentives in discussions concerning inflation-adjusted variable annuities. However, there's a gap in research concerning inflation-linked variable annuities with health benefits and GLWB.

This article attempted to address the following main issues: capturing the complex interactions between market variables, inflation dynamics, and fee structures; complexity resulting from an n -dimensional scenario; the need for robust algorithms to accurately simulate the interplay of multiple factors over time; and striking a balance between computational feasibility and precision. Our contribution to the literature is

- (1) We introduced a new inflation-linked variable annuity product with GLWB rider and Markovian health benefits within a multi-dimensional sub-account framework, where a group of interlinked risky assets influences the fund's valuation.
- (2) We provided a mixed discrete-continuous time model is proposed to calculate the fair fee of the product, using equilibrium conditions between premium and benefits.
- (3) A quadratic utility optimization problem has been developed from the insurance company's point of view to have the maximum sub-account fund value with a specific proportion of risk.

We seek to analyse the fair-fee and provide the optimal portfolio that can be used to improve the financial outcomes of retirees by using an inflation-linked variable annuity embedded with GLWB rider and Markov chain. We do this by analyzing S&P 500 Composite Index in the US market, the FTSE MIB Index in the Italian market, and the MSCI World Index in the global market, alongside the Consumer Price Index for All Urban Consumers (CPI-U) units, using CTMC for modeling the policyholder's health status (see, e.g. Christiansen, 2012; Pitacco, 1995), and n -dimensional Geometric Brownian motion for the fund behavior. We calculate the anticipated present values of discounted life benefits and death benefits, formulating the wealth balance equation to evaluate the contract for the policyholder. We consider a policyholder's health state to be a continuous-time Markov chain. This decision was made because this class of stochastic processes is flexible in characterizing systems that change state randomly over time, because its main advantages are simplicity and accurate out-of-sample forecasting. Furthermore, no past activity affects the value of a health condition; only its current state does. Essentially, it can be forecast exclusively from the present conditions related to the variable.

The remainder of the paper is as follows: Section 2 describes the multidimensional continuous-discrete mixed model, quadratic utility optimization problem, and the mathematical formulas needed to determine the fair fee. Section 3 deals with calculating the expected life and death benefits. Section 4 includes data description, numerical results of quadratic utility optimization problem along with numerical results of calculating the fair fee of the product and the effect of age and risk-aversion parameter on the fair fee. Section 5 summarizes our contribution and results with some potential further work.

2. Model description

This section describes the multi-dimensional continuous-discrete mixed model, the quadratic utility optimization problem, and the mathematical expressions essential for determining a fair fee. The section is divided into two primary parts. The initial part outlines the significant assumptions made for the development of the multi-dimensional continuous-discrete mixed model. Subsequently, the second part introduces the Stochastic Differential Equation (SDE), describing the progression of the inflation rate over time, the withdrawal strategy, the evolution of sub-account funds, and the optimal portfolio selection.

This paper uses capital letters to represent stochastic variables, and the subsequent notations are taken into account.

Notations	
y	Initial insured age, i.e. insured age at time period 0
r_r	Real spot rate
r_n	Nominal spot rate
$I(t)$	CPI-U inflation index at time t
t	Time in years
$S_i(t)$	Value of the i th stock price at time t
s_0^i	Initial value of the i th stock price
γ	Rate of the fee charged by the insurance company
W	Periodic random withdrawal % of h_0
$D_k(t)$	$e^{-r_k t}$: Discounted money market value at time t for $k \in \{r, n\}$

The main goal is to comprehend how the health-status dynamic affects the benefits, identify the optimal portfolio choice from the insurer's point of view, and calculate the fair fee of the contract. The following presumptions will be considered when formulating our product mathematically.

Assumption 2.1: A one-time lump sum investment of h_0 made by the insured at the start of the contract.

Assumption 2.2: n -risky asset $\mathbf{S}(t) = [S_1(t), S_2(t), \dots, S_n(t)]^T, t \geq 0$ such that:

$$dS_i(t) = \mu_i S_i(t) dt + \sum_{j=1}^n \sigma_{ij} S_i(t) dW_j(t), \quad (1)$$

where, $\mathbf{W}(t) = [W_1(t), W_2(t), \dots, W_n(t)]^T$ are n independent standard Brownian motions, μ_i is the drift parameter of the i th risky asset, $\sigma_{ij} = \text{Cov}(S_i(t), S_j(t))$ and s_0^i is the initial value of the i th risky asset $S_i(t)$. Using Itô's lemma, we get the general solution of Equation (1):

$$S_i(t) = s_0^i \exp \left(\left(\mu_i - \frac{1}{2} \sum_{j=1}^n \sigma_{ij}^2 \right) t + \sum_{j=1}^n \sigma_{ij} W_j(t) \right), \quad i = 1, 2, \dots, n. \quad (2)$$

Assumption 2.3: The premiums are invested in the proportion of $\omega = [\omega_1, \omega_2, \dots, \omega_n]^T$ in the risky assets, where ω_i is the portion of h_0 invested in i th risky asset according to the company such that: $\omega_i \geq 0$ and $\sum_{i=1}^n \omega_i = 1$, where $i = 1, 2, \dots, n$.

Assumption 2.4: The insurance company's fee is deducted continuously from the fund's value when fund units are canceled at a rate equal to γ i.e. total fee $(\gamma) = \sum_{i=0}^n \gamma_i$, where the fee $(\gamma_i$ of $H_i(t)$, where $H_i(t)$ is the i th fund value at time t) is deducted from the i th risky asset's fund value.

Assumption 2.5: We will assume the fee is charged continuously, and the guarantee $(\tilde{G}(t) \%$ of h_0) is deducted by the policyholder over a discrete-time set T_a defined as

$$T_a = \{t \geq 0 : t = ak \text{ with } a \in \mathbb{R}^+ \text{ and } k \in \mathbb{N}\}. \quad (3)$$

Assumption 2.6: Death benefits are paid when the insured dies.

2.1. Inflation index

Let $\{I(t), t \geq 0\}$ be the stochastic process representing the benchmark of the inflation index, i.e. CPI. $\{I(t), t \geq 0\}$ can be modeled using the following SDE (see Tiong, 2013b):

$$\frac{dI(t)}{I(t)} = \mu_I dt + \sigma_I dW_I(t), \quad (4)$$

where, μ_I and σ_I are the drift and volatility parameters respectively. $\{W_I(t), t \geq 0\}$ is the Brownian motion independent of $\{W_i(t), t \geq 0\}$, where $i = 1, 2, \dots, n$.

Let us define $\{D_r^*(t), t \geq 0\}$ be an inflation-linked discount process by:

$$D_r^*(t) = \frac{D_r(t)}{I(t)} \quad (5)$$

where, $D_r(t)$ is the discounted money market value at time t . Based on the above-defined processes, let us prove the following lemma:

Lemma 2.1: *There is a risk neutral measure $\mathbb{Q}$ being equivalent to $\mathbb{P}$, and under which we have:*

$$\frac{dI(t)}{I(t)} = (r_n - r_r)dt + \sigma_I d\bar{W}_I(t), \quad (6)$$

$$S_i(t) = s_0^i \exp \left(\left(r - \frac{1}{2} \sum_{j=1}^n \sigma_{i,j}^2 \right) t + \sum_{j=1}^n \sigma_{i,j} \bar{W}_j(t) \right), \quad i = 1, 2, \dots, n. \quad (7)$$

Where, r is the risk-free rate of interest.

Proof: For proof, refer to Appendix 1. ■

Corollary 2.2: *Under the risk-neutral measure $\mathbb{Q}$, we have:*

$$D_r^*(t) = \frac{e^{(-r_n + \frac{\sigma_I^2}{2})t - \sigma_I d\bar{W}_I(t)}}{I(0)}, \quad (8)$$

Proof: For proof refer Appendix 2. ■

2.2. Markovian health status

The health status of the policyholder is described with a continuous-time finite state space Markov process $\{X(t), t \geq 0\}$ with state-space $E = \{1, 2, \dots, D\}$. The elements $1, 2, \dots$ represent different states of the policyholder's health and D is the death of the insured. Figure 1 represents the state transition diagram of the Markovian health process with seven states.

For $t > s$, let us define $p_{ij}(t-s) = \mathbb{P}(X(t) = j \mid X(s) = i)$ be the transition probability from state i at time s to state j at time t and let $\mathbf{Q}$ be the generator matrix for $\{X(t), t \geq 0\}$ such that

$$q_{ij} = \lim_{t \rightarrow 0} \frac{p_{ij}(t)}{t} \quad \forall i \neq j \quad \text{and} \quad q_{ii} = - \sum_{j \neq i} q_{ij}. \quad (9)$$

Then, the following formula holds true

$$p_{ij}(t-s) = \mathbb{P}(X(t) = j \mid X(s) = i) = (e^{\mathbf{Q}(t-s)})_{(i,j)} = \left(\sum_{l=0}^{\infty} \frac{(t-s)^l \mathbf{Q}^l}{l!} \right)_{(i,j)}. \quad (10)$$

The Markov chain hypothesis is supported by a large literature on life insurance mathematics, (see, e.g. Milbrodt & Stracke, 1997; Pitacco, 1999).

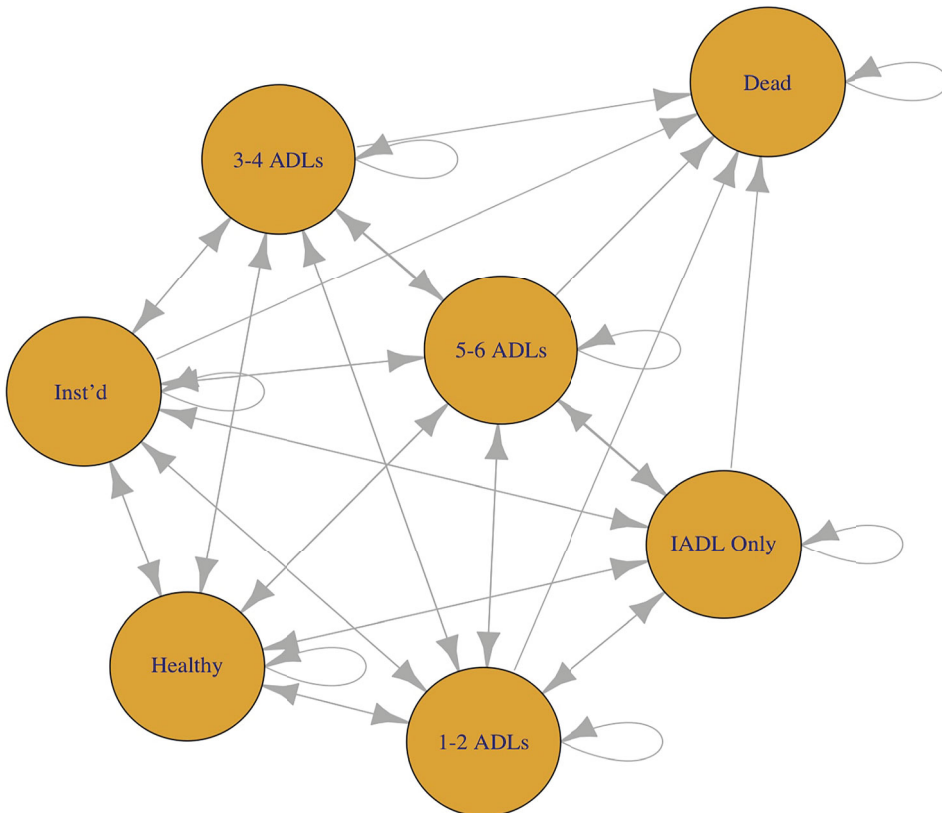


Figure 1. State transition diagram of Markovian health status.

2.3. Withdrawal benefits

The withdrawal mechanism in our model is designed to reflect both the guaranteed lifetime withdrawal benefits (GLWB) and the health-dependent variability of withdrawals. The policyholder makes a lump sum investment h_0 at the beginning of the contract, which is then allocated across n risky assets represented by the vector $\mathbf{S}(t) = [S_1(t), S_2(t), \dots, S_n(t)]^T$. The insurer deducts a continuous fee at rate $\gamma = [\omega_1 \cdot \gamma, \omega_2 \cdot \gamma, \dots, \omega_n \cdot \gamma]^T$ proportional to the investment weights ω_i in each asset.

Let $\mathbf{H}(t) = [H_1(t), H_2(t), \dots, H_n(t)]^T$ denote the fund values, which evolve over time according to the dynamics of the underlying risky assets, fee deductions, and discrete-time withdrawals. The withdrawals occur at predetermined time points $t \in T_a = \{ak : a \in \mathbb{R}^+, k \in \mathbb{N}\}$ and are modeled as a combination of two components: a fixed health-related expenditure and a stochastic GLWB rider payout.

Specifically, at each withdrawal time $t \in T_a$, the amount withdrawn depends on the health state $X(t)$ of the policyholder, modeled as a continuous-time Markov chain. The health-dependent withdrawal is given by two vectors:

- $\mathbf{g}(X(t)) = [g_1(X(t)), \dots, g_n(X(t))]^T$, representing the health-related withdrawal from each asset.
- $\mathbf{R}(X(t)) = [R_1(X(t)), \dots, R_n(X(t))]^T$, representing the GLWB payouts from each asset.

The total withdrawal amount vector is then:

$$\mathbf{G}(X(t)) = \mathbf{g}(X(t)) + \mathbf{R}(X(t)).$$

To standardize the total withdrawal as a proportion of the initial investment h_0 , we define:

$$\tilde{G}(t) = \left[\mathbf{G}^T(X(t)) \cdot \boldsymbol{\omega} \right] \cdot h_0.$$

This represents the health- and GLWB-adjusted withdrawal amount at time t .

Thus, the withdrawal process is explicitly modeled as a function of the evolving health state and the insurer-determined portfolio weights. The variability and timing of withdrawals reflect realistic uncertainties faced by annuity holders, which contrasts with traditional models that assume deterministic or fixed-rate withdrawals. To ensure actuarial fairness, we compute the break-even fee $\gamma^* = [\omega_1 \cdot \gamma^*, \dots, \omega_n \cdot \gamma^*]^T$ by equating the expected present value of benefits to the initial premium:

$$h_0 = \text{ELB} + \text{EDB}, \quad (11)$$

where ELB and EDB are the expected life and death benefits, respectively.

This formulation captures the key trade-offs faced by insurers: variable withdrawal timing, health evolution, and the joint impact of fees and portfolio allocation on the sustainability of the fund.

2.4. Optimal portfolio

The conservative n -dimensional GBM model assumes that the covariance matrix remains constant. Under a risk-neutral metric, the stock price dynamic for n -dimensional GBM is given as per Equation (7). Additionally, as customary, we designate the insurance premium paid as a vector lump sum amount. In addition, a series of random vectors $\{\tilde{G}(t)\}_{t \in T_a}$ indicating the time-varying withdrawal from the fund at time t are present. Please note that we do not consider a withdrawal rate but rather a succession of withdrawals of arbitrary amounts based on the policyholder's health. According to these presumptions, and following D'Amico et al. (2023), the fund's dynamic can be summarized

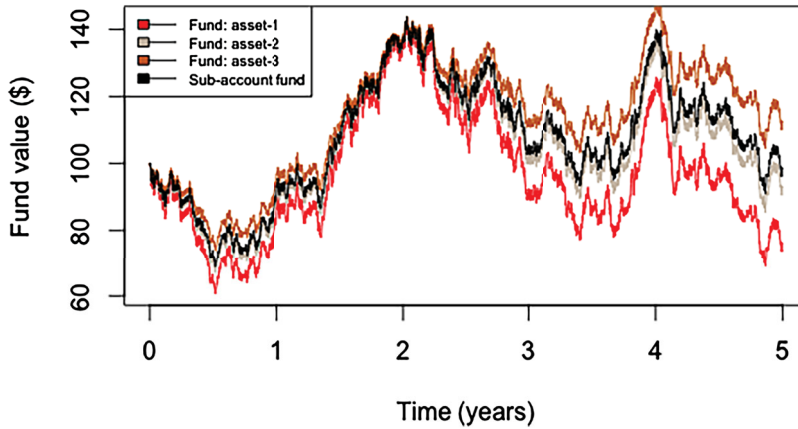


Figure 2. Fund evolution corresponding to the portfolio choice $\omega = [\frac{1}{4}, \frac{1}{4}, \frac{1}{2}]^T$.

as follows:

$$\begin{cases} dH_i(t) = (r - \gamma_i)H_i(t)dt + \sum_{j=1}^n \sigma_{ij}H_i(t)dW_j(t), & \forall t : ak < t < a(k+1), \\ H_i(t) = H_i(t^-) - G_i(X(t)) \cdot \omega_i \cdot h_0, & \forall t \in T_a. \end{cases} \quad (12)$$

Graphical visualization of the fund evolution with $\omega = [\frac{1}{4}, \frac{1}{4}, \frac{1}{2}]^T$ can be seen in Figure 2. Equation (12) represents the evolution of the underlying i th fund i.e, fund corresponding to the i th risky asset with time. The first part of the equation represents the evolution of the fund within two discrete-time epochs. The capital evolves as the risky asset $S_i(t)$ from which the fee rate γ_i is deduced instantaneously and with different initial values $H_i(t)$ given by subsequent equations. The final part describes the variation of the i th fund at the withdrawal epochs. The withdrawals are proportional to the level of the fund $H_i(t^-)$ through the function $(1 - G_i(X(t)) \cdot \omega_i \cdot h_0)$, which is dependent on the health status process at that time.

Using the above fund dynamics, let us describe our portfolio optimization problem with $\omega = [\omega_1, \omega_2, \dots, \omega_n]^T$ as decision variables which give the insurance company the best choice for the investment of the premium. The optimization problem is as follows:

$$\begin{aligned} \text{Max} \quad Z &= \sum_{i=1}^n \omega_i \cdot \mathbb{E}[H_i(\tau)] - \delta \text{Var}(\sum_{i=1}^n \omega_i H_i(\tau)) \\ \text{Constraints :} \quad &\sum_{i=1}^n \omega_i = 1 \quad \text{and} \quad \omega_i \geq 0. \end{aligned} \quad (13)$$

The aim of the above portfolio optimization problem is to maximize the sub-account fund value and minimize the risk from the insurance company's point of view at the initiation time of the withdrawal phase (i.e. τ) with $\delta \geq 0$ as the risk aversion parameter. The positive values of the weights ensure that no short selling is allowed. The above portfolio optimization problem can be rewritten in the vector form as:

$$\begin{aligned} \text{Max} \quad Z &= \omega^T * \mathbf{E} - \delta \omega^T * \mathbf{Cov} * \omega \\ \text{Constraints :} \quad &u^T * \omega = 1 \quad \text{and} \quad \omega \geq 0, \end{aligned} \quad (14)$$

where, $u = [1, 1, \dots, 1]^T \in \mathbb{R}^n$ and $\mathbf{E} = [\mathbb{E}(H_1(\tau)), \mathbb{E}(H_2(\tau)), \dots, \mathbb{E}(H_n(\tau))]^T$. Equation (14) can be solved in order to get the value of ω which gives the insurer the optimal choice for the investment of the premium.

3. Expected benefits

This section deal with the calculation of the expected life benefits (ELB) and expected death benefits (EDB) in order to calculate the Equilibrium equation given by Equation (11). Later, the Equilibrium equation can be treated as an inverse problem which gives us the value of the fair fee.

3.1. Computation of life benefits

Benefits are paid in accordance with the set of periodic periods T_a as a random amount $\mathbf{G}^T(X(t)) \cdot \omega \cdot h_0$. Over the course of the policyholder's lifetime, the accumulated benefits are provided by:

$$LB(0) = \sum_{t \in T_a} \left(\sum_{j=1}^n G_j(X(t)) \cdot \omega_j \right) h_0 \cdot D_r^*(t) \cdot \mathbb{1}_{\{X(t) \neq D\}}. \quad (15)$$

Where, D is the death of the insured. Now, take the conditional expectation of Equation (15), that is

$$ELB_i = \mathbb{E}[LB(0) \mid X(0) = i] = \mathbb{E}_i[LB(0)]. \quad (16)$$

Given the policyholder's health state i at initial time zero, it represents the conditional expected life benefits. The ELB_i can be obtained by following steps:

$$ELB_i = \sum_{t \in T_a} \left[\sum_{j=1}^n \mathbb{E}_i \left(G_j(X(t)) \cdot D_r^*(t) \cdot \mathbb{1}_{\{X(t) \neq D\}} \right) \omega_j h_0 \right], \quad (17)$$

where $ELB_D = 0$. Now since the health state of an individual is independent of the inflation rate hence $X(t)$ is independent of $I(t)$, which implies $G_j(X(t))$ is independent of $D_r^*(t) \forall j = 1, 2, \dots, n$. Hence we get the following:

$$ELB_i = \sum_{t \in T_a} \left[\sum_{j=1}^n \mathbb{E}_i \left(G_j(X(t)) \cdot \mathbb{1}_{\{X(t) \neq D\}} \right) \cdot \mathbb{E}_i \left(D_r^*(t) \right) \omega_j h_0 \right],$$

This expected value can be evaluated according to the time scale of benefit payments

$$\begin{aligned} ELB_i &= \sum_{k \geq 1} \left[\sum_{j=1}^n \mathbb{E}_i \left(G_j(X(ka)) \cdot \mathbb{1}_{\{X(ka) \neq D\}} \right) \cdot \mathbb{E}_i \left(D_r^*(ka) \right) \omega_j h_0 \right], \\ &= \sum_{k \geq 1} \left[\sum_{j=1}^n \left[\sum_{u \neq D} G_j(u) \cdot \mathbb{P}(X(ka) = u \mid X(0) = i) \cdot \omega_j h_0 \right] \frac{e^{(-r_n + \sigma_I^2)ka}}{I(0)} \right], \\ &= \sum_{k \geq 1} \sum_{j=1}^n \sum_{u \neq D} G_j(u) \cdot (e^{\mathbf{Q}^{ka}})_{(i,j)} \cdot \frac{e^{(-r_n + \sigma_I^2)ka}}{I(0)} \cdot \omega_j h_0. \end{aligned} \quad (18)$$

The formula for ELB_i requires the calculation of transition probabilities at different time intervals and involves an infinite sum. An alternative method for calculating the ELB_i is to create a set of linear equations for them and solve it using matrix representation, which is more efficient. Proposition 3.1 provides the matrix formula for calculating the expected life benefits conditioned on the initial health state i , whose proof can be found in Appendix 3.

Proposition 3.1: The expected life benefits can be calculated according to the following matrix formula:

$$ELB = \left(I - \frac{e^{(-r_n + \sigma_l^2)a}}{I(0)} \cdot P(a) \right)^{-1} * \left(\frac{e^{(-r_n + \sigma_l^2)a}}{I(0)} \cdot P(a) * \tilde{G} \right), \quad (19)$$

where

$$ELB = [ELB_1, ELB_2, \dots, ELB_D]^T \quad \text{with} \quad ELB_D = 0,$$

$$(\tilde{G}(t) = [\tilde{G}(1), \tilde{G}(2), \dots, \tilde{G}(D)]^T \quad \text{with} \quad G(D) = 0.$$

and $P(a) = e^{Qa}$ represents the transition probability matrix at time a , with $(*)$ as the standard matrix product operator.

3.2. Computation of death benefit

Let l be the time of death of the policyholder that is $l = \inf \{t \in \mathbb{R} : X(t) = D\}$. Let $DB_\beta(\gamma)$ be the death benefit conditional on the death at time $l = \beta \in \mathbb{R}_+$ then,

$$DB_\beta(\gamma) = \max \left\{ \omega^T \cdot H(\beta), 0 \right\},$$

$$= \max \left\{ \sum_{i=1}^n \omega_i H_i(\beta), 0 \right\}.$$

It results that

$$DB_\beta(\gamma) = \max \left\{ 0, \sum_{i=1}^n \omega_i h_0 \exp \left(\left(r - \frac{1}{2} \sum_{j=1}^n \sigma_{ij}^2 \right) \beta + \sum_{j=1}^n \sigma_{ij} W_j(\beta) \right) \right. \\ \left. - \sum_{i=1}^n \sum_{v=1}^{\lfloor \beta \rfloor} \omega_i G(X(va)) h_0 \exp \left(\left(-r + \frac{1}{2} \sum_{j=1}^n \sigma_{ij}^2 \right) (\beta - va) - \sum_{j=1}^n \sigma_{ij} W_j(\beta - va) \right) \right\},$$

where $\lfloor \beta \rfloor = \max \{k \in \mathbb{N} : k \cdot a \leq \beta\}$.

Remark 3.1: It is challenging to determine $DB_\beta(\gamma)$ analytically since it is a random variable. Consequently, the value of $DB_\beta(\gamma)$ is obtained using the Monte Carlo simulation technique, as explained in Subsection 3.3.

Now, the expected death benefit is just the discounted expectation of the conditional death benefit with respect to the random time of death, i.e.

$$EDB(\gamma) = \int_0^\infty D_r^*(\beta) \mathbb{E}[DB_\beta(\gamma)] \delta(\beta) d\beta,$$

where

$$\delta(\beta) = \lim_{h \rightarrow 0} \frac{\mathbb{P}(\beta < l \leq \beta + h \mid l > \beta)}{h},$$

is the so-called failure rate function. For an n -state continuous-time Markov chain, it can be expressed according to Sadek and Limnios (2005)

$$\delta(\beta) = \begin{cases} \frac{-1}{R(\beta)} \cdot \frac{dR(\beta)}{d\beta} = -\frac{\alpha_1 e^{\beta Q_{(n-1, n-1)}} \cdot Q_{(n-1, n-1)} \mathbb{1}_{(n-1)}}{\alpha_1 e^{\beta Q_{(n-1, n-1)}} \mathbb{1}_{(n-1)}} & \text{if } R(\beta) \neq 0, \\ 0 & \text{if } R(\beta) = 0, \end{cases} \quad (20)$$

where $\mathbf{Q}$ is the generator matrix of the Markov chain, α is the initial probability distribution, $R(\beta)$ the reliability function and $\mathcal{K}_{(n-1)}$ is the unitary vector of the dimension $n-1$. We then have:

$$LB_0 + DB_0 = \sum_{i=1}^D \alpha_i \cdot ELB_i + \int_0^\infty D_r^*(\beta) \cdot \mathbb{E}[DB_\beta(\gamma)] \delta(\beta) d\beta, \quad (21)$$

hence, we get the final equation as

$$h_0 = \alpha^T * \mathbf{ELB} + EDB(\gamma). \quad (22)$$

The above equation in γ is solved numerically to obtain the break-even value for the fee (γ^*) charged by the insurer. Algorithm 1 provides a root finding algorithm to find the solution of Equation (22). In which $matrix(0, n, 2)$ represents a zero matrix with dimensions n by 2 and $value(0, grate[i], r)$ the value of the contract corresponding to withdrawal rate $grate[i]$ and rate of interest r .

Algorithm 1 Calculate Root Values

```

1: Initialize  $grate \leftarrow seq(start, end, step)$ 
2: Set parameter  $r \leftarrow (some\ value)$ 
3:  $n \leftarrow length(grate)$ 
4:  $root65 \leftarrow matrix(0, n, 2)$ 
5: for  $i \leftarrow 1$  to  $n$  do
6:   if  $value(0, grate[i], r) > 100$  and  $value(10000, grate[i], r) < 100$  then
7:      $root65[i, 1] \leftarrow grate[i]$ 
8:      $root65[i, 2] \leftarrow uniroot(x \mapsto value(x, grate[i], r) - 100, [0, 10000])\$root$ 
9:   end if
10: end for

```

3.3. Monte Carlo technique for fund process

Within this particular section, we utilize the Monte Carlo simulation technique to computationally solve the stochastic differential equation (SDE) provided by

$$dH_i(t) = (r - \gamma_i)H_i(t)dt + \sum_{j=1}^n \sigma_{ij}H_i(t)dW_j(t), \quad \forall t: ak < t < a(k+1), \quad (23)$$

$$H_i(t) = H_i(t^-) - G_i(X(t)) \cdot \omega_i \cdot h_0, \quad \forall t \in T_a.$$

3.3.1. Discretization procedure

In order to carry out the Monte Carlo simulation, we employ the Euler-Maruyama method to discretize the stochastic differential equation (SDE). The discretized form of Equation (23) is given by

$$H_i(t_{k+1}) = H_i(t_k) + (r - \gamma_i)H_i(t_k)\Delta t + \sum_{j=1}^n \sigma_{ij}H_i(t_k)\sqrt{\Delta t}Z_{ij}, \quad (24)$$

$$H_i(t_{k+1}) = H_i(t_k) - G_i(X(t_k)) \cdot \omega_i \cdot h_0 \cdot \Delta t.$$

where Z is a standard normal random variable. The time points t_k are determined by the discretization of the time interval $ak < t < a(k+1)$ and $t \in T_a$. Following Kloeden and Platen (1992), Algorithm 2 provides the Monte Carlo simulation of the fund process given by Equation (23). Figure 2 represents the fund evolution corresponding to the portfolio choice $\omega = [\frac{1}{4}, \frac{1}{4}, \frac{1}{2}]^T$.

Algorithm 2 Monte Carlo Simulation for SDE

Input: Parameters for the SDE: $r, \gamma_i, n, ak, T_a, n_paths, h_0$, and functions $G_i(X)$ and $\sigma_{i,j}$.

Output: Simulated paths of $H_i(t)$.

- (1) Set initial conditions: $t_{\text{start}} = ak, t_{\text{end}} = T_a, \Delta t = \frac{t_{\text{end}} - t_{\text{start}}}{n_paths}$.
- (2) Initialize matrix to store simulation results: H_paths .
- (3) **For** $i = 1$ to n_paths :
 - (a) Initialize H_i at t_{start} : $H_i[t_{\text{start}}] = h_0$.
 - (b) **For** $k = t_{\text{start}}$ to $t_{\text{end}} - 1$:
 - (i) Generate Wiener increments: $dW \sim \mathcal{N}(0, \Delta t)$.
 - (ii) Compute $G = G_i(X)$.
 - (iii) Update H_i using Euler-Maruyama method:

$$H_i[k + 1] = H_i[k] + (r - \gamma_i)H_i[k]\Delta t + \sum_{j=1}^n \sigma_{i,j}H_i[k]dW - G\omega_i h_0 \Delta t.$$

- (c) Store H_i in H_paths .
 - (4) Plot the simulation results.
-

4. Numerical and sensitivity analysis

In this section description of the data taken for the numerical calculation is mentioned, along with the optimal portfolio weights and numerical results for the given model.

4.1. Data description

This research paper focuses on conducting a numerical analysis using the S&P 500 composite Index from the US market, the FTSE MIB index from the Italy market, and the MSCI World Index from the global market are among the datasets. This index was selected based on its popularity and data availability in the country. To obtain the S&P 500 index, FTSE MIB index, and MSCI World Index data, we accessed Thomson and Reuters Datastream. Additionally, historical monthly data for Consumer Price Index for All Urban Consumers(CPI-U) units was obtained from a reliable source (Federal Reserve Bank of St. Louis, 2025).

In evaluating an insurance health product, the health state of policyholders and the corresponding transition probabilities are of utmost importance. The classification of an individual's health status aligns with the approach proposed by D'Amico et al. (2023). Specifically based on his/her ability to perform daily activities of daily living (IADL) such as minor household chores, laundry, grocery shopping, meal preparation, navigating outdoors, managing finances, and utilizing the telephone, as well as engaging in activities of daily living (ADL) such as eating (level 1), bathing (level 2), dressing (level 3), mobility (level 4), personal hygiene (level 5), and toileting (level 6). This study utilizes two-year transition probabilities between disability states, calculated based on data from the 1982 and 1984 NLTC (National Long-Term Care) Survey. The transition probabilities, expressed as percentages, are specific to males and females, as mentioned in Pritchard (2006) work. Table 1 provides an overview of these two-year transition probabilities, where Healthy; IADL only; 1–2 ADLs; 3–4 ADLs; 5–6 ADLs; Institutionalized; and Dead represents different health states for a detailed description (see Pritchard, 2006).

Table 1. Two-year transition probabilities between disability states calculated from the 1982 and 1984 NLTCs, as a percentage, for males and females according to Pritchard (2006).

Age	1982 status	1984 status						
		Healthy	IADL only	1–2 ADLs	3–4 ADLs	5–6 ADLs	Inst'd	Dead
71	Healthy	83.73	3.60	2.41	0.95	0.93	1.11	7.28
74		76.74	5.40	4.08	1.30	1.24	1.94	9.30
77		76.74	5.40	4.08	1.30	1.24	1.94	9.30
71	IADL Only	25.27	32.53	17.48	3.64	2.77	4.93	13.38
74		15.39	30.65	18.73	5.01	5.31	7.53	17.38
77		15.39	30.65	18.73	5.01	5.31	7.53	17.38
71	1–2 ADLs	10.52	12.85	35.00	11.99	5.78	6.94	16.92
74		9.22	12.06	35.08	10.33	6.73	6.09	20.05
77		9.22	12.06	35.08	10.33	6.73	6.09	20.05
71	3–4 ADLs	8.31	4.64	18.09	29.12	18.18	3.98	17.67
74		3.73	4.37	17.57	20.78	16.35	10.97	26.23
77		3.73	4.37	17.57	20.78	16.35	10.97	26.23
71	5–6 ADLs	6.15	4.71	9.13	10.16	28.68	8.03	33.15
74		3.97	3.64	4.98	7.71	34.73	11.56	33.42
77		3.97	3.64	4.98	7.71	34.73	11.56	33.42
71	Inst'd	2.97	0.75	1.81	1.24	2.36	52.40	38.48
74		1.74	1.48	0.88	0.74	1.48	56.87	36.81
77		1.74	1.48	0.88	0.74	1.48	56.87	36.81
71	Dead	0	0	0	0	0	0	1
74		0	0	0	0	0	0	1
77		0	0	0	0	0	0	1

4.2. Estimated parameters

The model parameters are estimated using the maximum likelihood estimation (MLE) methodology, a statistical technique established by Kessler (1997). The estimation process use the *fitsde* function, which is a part of the R-package *Sim.DiffProc*. Embedded within the *Sim.DiffProc* package, this function serves as a handy tool for obtaining parameter estimates of Stochastic Differential Equation (SDE) models using empirical data.

The vector $\hat{\mu}$ and covariance matrix *Cov* are estimated using real data to facilitate the calculation of the expected death benefit (EDB) and optimal portfolio. For this research, the estimated values are:

$$\hat{\mu} = \begin{pmatrix} 3.575 \\ 4.2237 \\ 3.2398 \end{pmatrix}, \quad \text{Cov} = \begin{pmatrix} 0.9443 & 1.0451 & 0.499 \\ 1.0451 & 1.17965 & 0.6337 \\ 0.499 & 0.6737 & 1.4111 \end{pmatrix}.$$

The numerical calculations were performed using RStudio version 2021.09.0 Build 351 on a system with the following specifications: AMD Ryzen 5 3500U with Radeon Vega Mobile Gfx 2.10 GHz, a 64-bit operating system, an x64-based processor, and 8GB of RAM.

4.3. Numerical calculations

For the numerical calculation, the following assumptions are considered:

- (1) To remove the complexity of numerical calculation, the upper limit to initiate the withdrawal phase is 15 years.
- (2) Premium paid is a lump sum amount of 100, i.e. $h_0 = \$100$, and a is considered to be 24 months.
- (3) The range of break-even fee (γ^*) is considered to be 0 to 1000 basis points (bp).
- (4) The risk aversion parameter (δ) is fixed at 0.009
- (5) Initially, the State of each policyholder begins with state 1 (healthy), i.e. $\alpha = (1, 0, 0, 0, 0, 0, 0)$.

- (6) The random amount vector offered by the GLWB rider is chosen from D'Amico et al. (2023).
- (7) The withdrawal is based on the health status of the patient and the weight of that particular risky asset, such that for the health states 1, the withdrawal amount is assumed to be 1\$\$, for state 2, it is assumed to be 2\$\$, for state 3, it is assumed to be 3\$\$, for state 4, it is assumed to be 4\$\$, for state 5, it is assumed to be 5\$\$, and for 6th state it is assumed to be 6\$\$\$. Accordingly, we get the withdrawal vector as:

$$\tilde{\mathbf{G}} = [1.12\$, \quad 1.46\$, \quad 1.80\$, \quad 2.14\$, \quad 2.48\$, \quad 2.82\$]^T.$$

- (8) For numerical analysis, the risk-free rate (r) and nominal rate (r_n) are considered to be 6% per annum.
- (9) Sequence of the risky assets considered for numerical calculation is, S_1 : MSCI world index, S_2 : S&P 500 composite Index, and S_3 : FTSE MIB index.

To proceed with the application of the model, it is necessary to estimate the Markov chain's generator matrix $\mathbf{Q}$. Due to the lack of precise data, we utilize the 2-year transition probability matrices presented in Table 1. In some instances, it is possible to derive the generator matrix from these matrices.

Following D'Amico and Regnault (2018), the maximum likelihood estimator $\hat{\mathbf{Q}}$ of the generator matrix $\mathbf{Q}$ is obtained as

$$\hat{\mathbf{Q}}_{71} = \begin{pmatrix} -0.1913 & 0.0629 & 0.0265 & 0.0110 & 0.0117 & 0.0137 & 0.0676 \\ 0.4540 & -1.2719 & 0.5823 & 0.0258 & 0.0585 & 0.0754 & 0.1276 \\ 0.0940 & 0.4279 & -1.3158 & 0.4342 & 0.0408 & 0.1412 & 0.1764 \\ 0.1069 & 0.0002 & 0.5720 & -1.4850 & 0.6984 & 0.0047 & 0.1145 \\ 0.0707 & 0.0850 & 0.1921 & 0.3470 & -1.3976 & 0.1870 & 0.5019 \\ 0.0382 & 0.0074 & 0.0305 & 0.0176 & 0.0538 & -0.6540 & 0.5065 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\hat{\mathbf{Q}}_{74} = \begin{pmatrix} -0.2799 & 0.1012 & 0.0507 & 0.0144 & 0.0110 & 0.0183 & 0.0865 \\ 0.2782 & -1.3390 & 0.6011 & 0.0628 & 0.1035 & 0.1340 & 0.0872 \\ 0.1279 & 0.4041 & -1.3039 & 0.4069 & 0.0797 & 0.0643 & 0.2218 \\ 0.0185 & 0.0147 & 0.7587 & -1.8751 & 0.6317 & 0.2237 & 0.2774 \\ 0.0540 & 0.0900 & 0.0340 & 0.2974 & -1.1518 & 0.2267 & 0.4483 \\ 0.0212 & 0.0321 & 0.0054 & 0.0140 & 0.0275 & -0.5715 & 0.4718 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\hat{\mathbf{Q}}_{77} = \begin{pmatrix} -0.2799 & 0.1012 & 0.0507 & 0.0144 & 0.0110 & 0.0183 & 0.0865 \\ 0.2782 & -1.3390 & 0.6011 & 0.0628 & 0.1035 & 0.1340 & 0.0872 \\ 0.1279 & 0.4041 & -1.3039 & 0.4069 & 0.0797 & 0.0643 & 0.2218 \\ 0.0185 & 0.0147 & 0.7587 & -1.8751 & 0.6317 & 0.2237 & 0.2774 \\ 0.0540 & 0.0900 & 0.0340 & 0.2974 & -1.1518 & 0.2267 & 0.4483 \\ 0.0212 & 0.0321 & 0.0054 & 0.0140 & 0.0275 & -0.5715 & 0.4718 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

4.3.1. optimal portfolio and break even fee

In this subsection, the optimal portfolio choice along with its algorithm and the value of the fair fee obtained using our model is being discussed.

Fixing the risk aversion parameter at $\delta = 0.009$, the derived optimal portfolio weights are denoted by $\boldsymbol{\omega} = [0.2826, 0.4007, 0.3167]^T$, representing the weights assigned to the MSCI index, S&P 500,

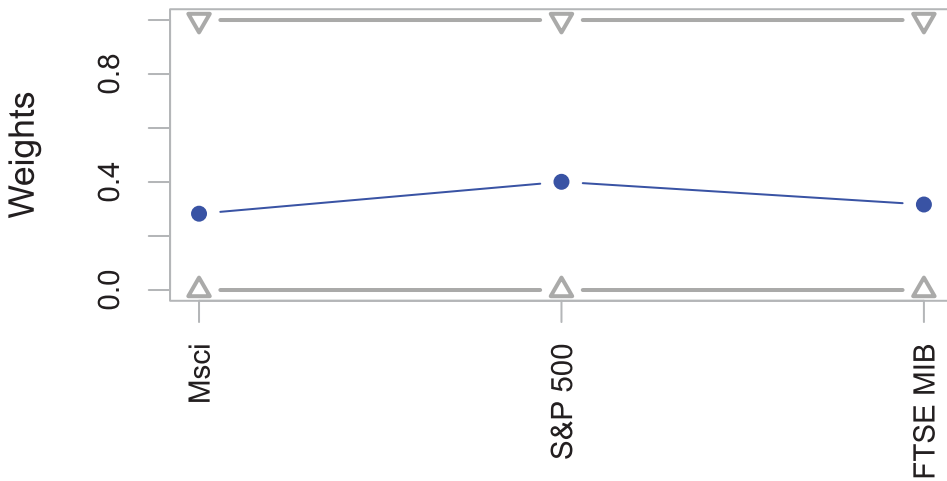


Figure 3. Optimal weights corresponding to the risk aversion(δ) = 0.009.

and FTSE MIB as $\omega_1 = 0.2826$, $\omega_2 = 0.4007$, and $\omega_3 = 0.3167$ respectively (see Figure 3). The subsequent R code which illustrates the procedure employed to ascertain this optimal portfolio is shown in Algorithm 3.

According to Equation (22), the break-even fee is the value of γ for which the expected discounted benefits equal the premium h_0 . The break-even fee corresponding to different guarantee amounts for ages 65, 71 and 76 is shown in Table 2. In the table's first column, we consider a constant variation ϵ , which is added to all the elements of the vector $\mathbf{G}$. Thus, for example when ϵ is 2 the $\mathbf{G}$ modifies to

$$\tilde{\mathbf{G}} = [3.12\$, \quad 3.46\$, \quad 3.80\$, \quad 4.14\$, \quad 4.48\$, \quad 4.82\$]^T.$$

Results from Table 2 illustrate that with the increase in ϵ , the fee associated with the contract also increases. Also, the existence of a break-even fee corresponding to every guarantee percentage is not necessary. A very low guarantee (For instance, 1% for a person aged 60) cannot make the present value of the contract equal to the premium even if the fee charged is 0 bp. Similarly, a very high guarantee (For instance, 25% for a person aged 60) cannot make the present value of the contract equal to the premium even if the fee charged is 2500 bp. Therefore, many of the blocks in Table 2 are empty as no break-even fee in the range 0–1000 bp exists for those cases. From Figure 4 The relationship between the break-even fee (γ^*), ϵ , and age appears to be generally linear. For a given age, γ^* increases as ϵ rises, indicating a positive correlation between ϵ and the break-even fee. There is a threshold value of ϵ beyond which γ^* starts to increase significantly. The value of ϵ has been selected such that the value of the break-even fee exists between 0–1000 basis points. Additionally, in Figure 5, plots are drawn of the expected present value vs. value of shift parameter (ϵ) for different risk-aversion parameters to support our results graphically. Increases in ϵ enhance the guaranteed amount, which leads to an increase in the overall expected present value and, hence, an increment in the fee.

4.3.2. Sensitivity analysis

In this subsection, we will study the change in the expected present value and fair fee of the contract with respect to the different parameters.

- (1) **Age:** From Table 2, it is observed that when a policyholder starts an insurance policy at a younger age, it affects how much he/she has to pay. As he/she gets older when starting the policy, the amount that the policyholder can withdraw over your lifetime (GLWB value) goes up, and that means the fair payment, denoted as γ^* , also goes up. This increase in the

Algorithm 3 Optimization Problem for Quadratic Utility Function in R

Input: Mean vector $\hat{\mu}$, Covariance matrix $\mathbf{Cov}$, Risk aversion parameter δ .

Output: Optimal weight vector w^* .

(1) Formulate Quadratic Utility Function:

```
U <- function(w, mu, Cov, delta) {  
  return(t(w)
```

(2) Set Up the Optimization Problem:

```
library(optim)  
constraints <- ... # Specify constraints on w  
optimization_problem <- function(w) -U(w, mu, Cov, delta)
```

(3) Choose Optimization Algorithm:

```
optimization_algorithm <- optim(par =  
  initial_w, fn = optimization_problem,  
  method = "L-BFGS-B", control = list(...))
```

(4) Optimize the Utility Function:

```
optimal_w <- optimization_algorithm$par
```

(5) Display Results:

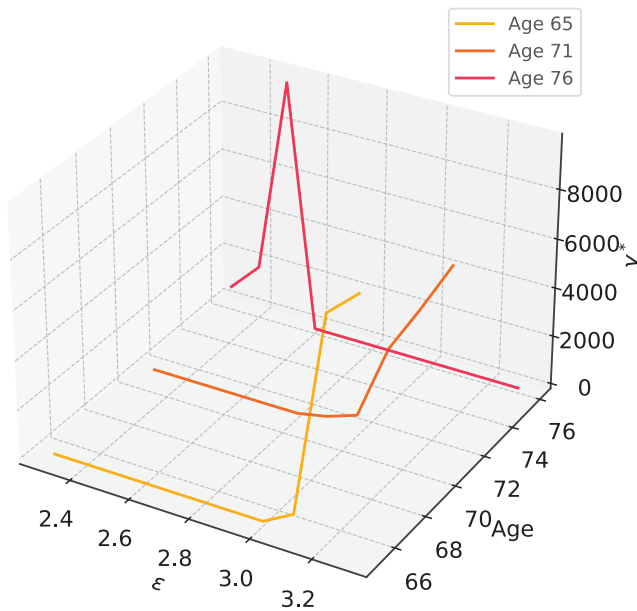
```
cat("Optimal weight vector (w*):", optimal_w, "\n")
```

fair payment is higher for younger policyholders compared to older ones because younger policyholders have longer contracts.

Our study looks at how the GLWB value changes when the age at the start of the insurance, represented by y , changes. When a policyholder starts the policy at an older age, the time he/she is covered decreases, but the level of the health status also increases. This leads to fewer withdrawals, larger withdrawal amounts, and a smaller living benefit. Moreover, from Figure 4 we can see as the age at which insurance coverage starts decreases, the duration for discounting the death benefit lessens, resulting in a significantly larger death benefit. Conversely, starting at an early age increases the overall Guaranteed Lifetime Withdrawal Benefit (GLWB) value because the reduction in living benefit is less than the reduction in the death benefit for younger policyholders. Nevertheless, this difference between the death and living benefits diminishes for older policyholders.

Table 2. Break even fee (γ^*) in basis points for age 65, 71 and 76.

ϵ	γ^* for Age 65	γ^* for Age 71	γ^* for Age 76
2.3	0	0	774.64
2.4	0	0	1961.66
2.5	0	0	9950.60
2.6	0	0	0
2.7	0	0	0
2.8	0	1.35	0
2.9	0	254.63	0
3	0	681.05	0
3.1	689.09	3776.67	0
3.2	8918.09	5725.09	0
3.3	9998	7794.95	0

**Figure 4.** Break even fee for different ages and ϵ values.

- (2) **Risk-aversion :** In Figure 5, the plot of EPV vs. ϵ for different values of δ is drawn to show the change in the behavior of the GLWB value with the change in the risk-aversion parameter δ for different value of shift parameter ϵ . The discontinuity observed in the graph is attributed to two primary factors: the change in the shift parameter and the inherent randomness of market behavior. An increase in δ increases the overall expected returns of the portfolio while increasing the risk value. The increase in the risk value overcomes the increment in the value of the expected returns of the portfolio. As a result, as the value of δ rises, the value of the GLWB as a whole increases with it. The effect is greater for the senior policyholders as the overall contract period decreases. Younger policyholders might thus have some benefit from high-risk aversion with a larger guarantee amount.

The model finds a fair fee for ILVAs with GLWB by balancing premiums and benefits, and diversifying portfolios to reduce risk. Including inflation and health benefits ensures better protection against economic uncertainties and longevity risk. Increased healthcare demand impacts expenditures and premiums, affecting insurance sectors.

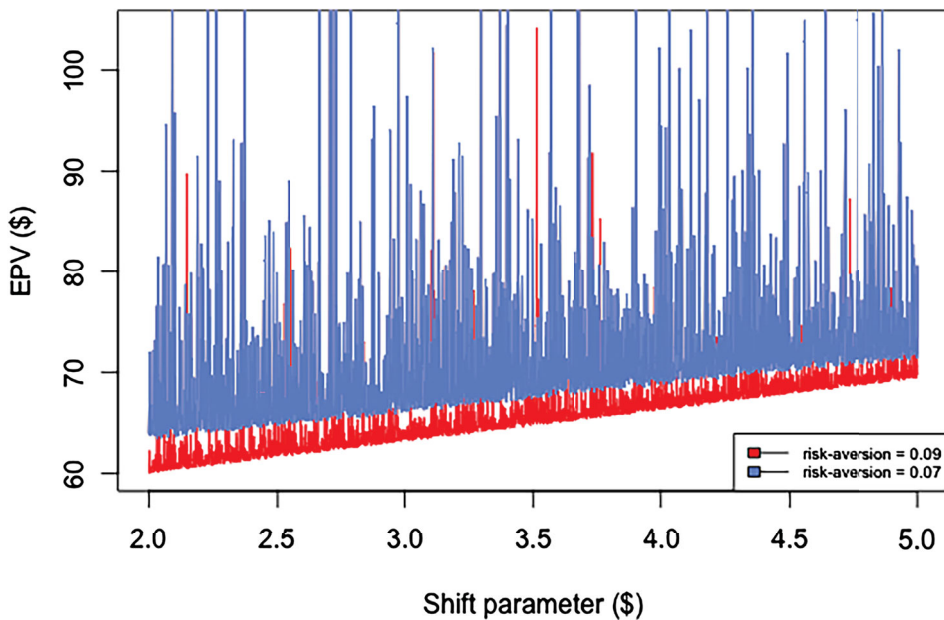


Figure 5. Fair fee for different age groups.

4.4. Model comparison

This subsection entails a thorough comparison investigation of two separate models: the model by D'Amico et al. (2023) and our own generated model. These models are used to evaluate the fair value of variable annuity contracts that include Guaranteed Lifetime Withdrawal Benefit (GLWB) riders and other additional perks. The main goal of this comparison is to evaluate the performance of different models and ascertain which one better matches our research aims.

D'Amico et al. (2023) presented and evaluated a new type of insurance policy, specifically a variable annuity contract, enhanced with Markovian health benefits and GLWB. A mixed discrete-continuous time model was developed to calculate the fair fee for a single risky asset. This study examines a comprehensive framework in which the value of the fund is determined by a collection of interconnected assets that offer advantages tied to inflation. We introduce a model that combines discrete and continuous time to assess the fair value of this product accurately.

An essential distinction between the two models is in the number of risky assets taken into account for the pricing framework. In order to compare these models, we analyze the ratio between the expected present value (EPV) and the fair charge of the contract. The ratio plot for both models is depicted in Figure 6, assuming a starting policyholder age of 71 years. The discrepancy in the ratio can be attributed to the increase in EPV resulting from the added advantage of the inflation and multi-dimensional scenario. The provided comparison plot demonstrates superior outcomes for the multi-dimensional scenario (3-Dimension) in contrast to the single-asset scenario.

4.5. Model reduction and numerical verification

We confirm the robustness of our numerical implementation and its alignment with previous research by demonstrating that our proposed multi-asset, inflation-linked, health-dependent GLWB framework simplifies to various established models as special instances. We subsequently replicate essential numerical findings documented in the benchmark literature under these specific settings to validate the precision of our methodology.

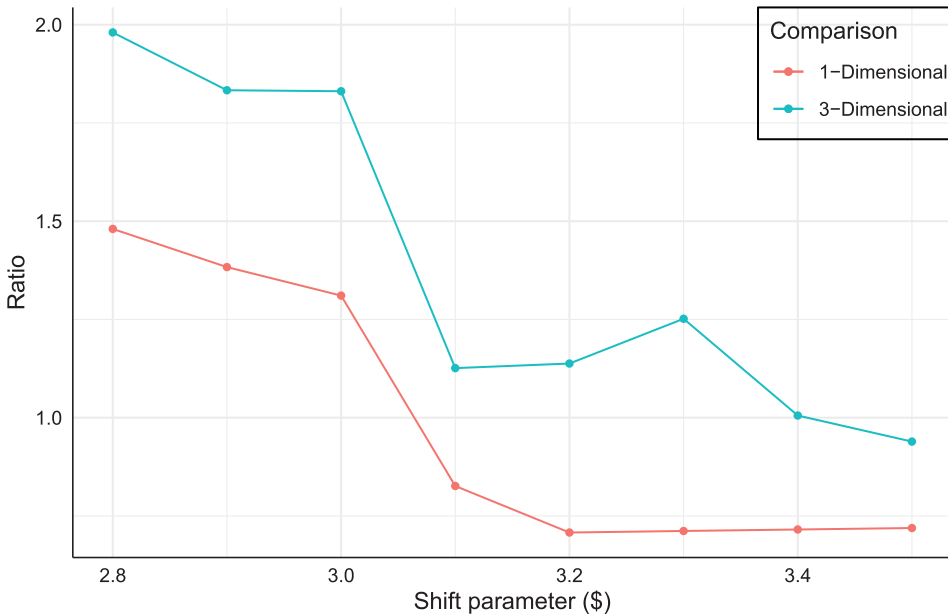


Figure 6. EPV and fair fee ratio plot.

4.5.1. Reduction to benchmark models

The following major models from the literature are special examples of our general model framework. By carefully limiting the model's features, like the amount of risky assets, the inclusion of inflation indexing, or the dynamics of health states, we may get back to the main assumptions and mathematical structures of each benchmark model:

- (1) D'Amico et al. (2023): Single risky asset, health-linked GLWB without inflation.
Special case: Set number of assets $n = 1$, eliminate inflation volatility ($\sigma_I = 0$), retain Markovian health states and withdrawal rules.
- (2) Tiong (2013b, 2013b): Single risky asset, inflation-linked GLWB without health states.
Special case: $n = 1$, health modeled as degenerate Markov chain with single alive state until death, inflation process included.
- (3) Moosavi and Payandeh (2021): Two risky assets, constant volatility, health-linked GLWB without inflation.
Special case: $n = 2$, constant volatility assumption, no inflation, health-linked withdrawals preserved.

In each case, the stochastic dynamics, withdrawal structures, and fee equilibrium equations in our framework coincide exactly with those of the referenced models when restricted as above.

4.5.2. Numerical reproduction of literature results

We used the parameter settings and model assumptions from each benchmark publication to run numerical trials and find the break-even fee γ^* . Table 3 shows the results of the comparison, which show that our solution is correct because the results are very similar.

These results confirm that our framework reliably replicates benchmark results within a margin of less than 1%, thereby establishing the numerical correctness and robustness of our approach.

Table 3. Comparison of break-even fees γ^* (in basis points) with benchmark literature results.

Case	Parameters	Benchmark γ^*	Our γ^*	Difference (%)
D'Amico et al. (2023)	65 yr, 4%	680.10	689.10	+1.3%
	71 yr, 6%	3805.70	3776.70	−0.76%
Tiong (2013b, 2013b)	20 yr, 4%	95.40	96.10	+0.73%
	20 yr, 6%	144.60	143.90	−0.48%
Moosavi and Payandeh (2021)	25 yr, 5%	112.30	113.00	+0.62%
	25 yr, 7%	156.70	155.80	−0.57%

5. Conclusion and future work

This study is focused on two main objectives; one is to analyze the fair fee of the inflation-linked variable annuity product with GLWB rider and Markovian health benefits with a multi-dimensional sub-account framework where the fund value depends on a basket of connected assets have been considered. The second objective of the paper is to provide the insurance company with the best portfolio choice to maximize the sub-account fund value with a certain amount of risk. A mixed discrete-continuous time model is proposed to calculate the fair fee of the product, and a quadratic utility optimization problem has been developed from the insurance company's point of view.

Individuals nearing retirement should secure insurance to safeguard against the depletion of their financial resources and mitigate aging challenges. The product being discussed seems to have the potential to provide a complete solution for these important needs. In this paper, we're evaluating the expected benefits in terms of life and death and also figuring out a fair cost for the service. Making these products more appealing could be achieved by improving how we calculate the health benefits. This would lead to better coverage for medical bills and hospitalization costs for those who have this insurance. By weighing benefits and premiums and diversifying portfolios to lower risk, the model determines a reasonable price for ILVAs with GLWB. Inflation and health benefits are included to provide more protection against risk associated with aging and the economy. Demand for healthcare is rising, which affects insurance companies' costs and premiums.

As outlined in the sensitivity analysis section, alterations in age and the risk aversion parameter significantly influence the product's fair fee (γ^*). Future works involve implementing more advanced health evolution models, such as semi-Markov processes, and considering health-related rewards that depend on the individual's health history. Future research may encompass broadening the actuarial factors, like evaluating surrender benefits through an appropriate penalty method. Additionally, exploring the elimination of specific financial assumptions, applying stochastic interest rates in a manner done by Bufalo et al. (2022) for distinct financial challenges, is worth investigating. An interesting challenge is to consider the risk aversion parameter as a random variable. In this case, the risk aversion parameter (δ) can modeled using a suitable stochastic process.

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Appendices

Appendix 1. Proof of Lemma 1

Proof: For the proof of Equation (6), see Liang and Sheng (2016). For Equation (7), the proof is as follows:

$S_i(t)$, be the price of i th risky asset given by the SDE:

$$dS_i(t) = \mu_i S_i(t) dt + \sum_{j=1}^n \sigma_{ij} S_i(t) dW_j(t),$$

for the risk-free rate of interest $r \geq 0$ and the exponential martingale

$$Z_t = \exp \left(\frac{-t}{2} \left(\frac{r - \mu_i}{n \cdot \sigma_{ij}} \right)^2 + \left(\frac{r - \mu_i}{n \cdot \sigma_{ij}} \right) W_j(t) \right),$$

the process $\tilde{W}_j(t) = \left(\frac{r - \mu_i}{n \cdot \sigma_{ij}} \right) t + W_j(t)$ is a $\mathbb{Q}$ -Brownian motion according to the Girsanov Theorem, and the price satisfies,

$$\begin{aligned} dS_i(t) &= \mu_i S_i(t) dt + \sum_{j=1}^n \sigma_{ij} S_i(t) d(\tilde{W}_j(t) - \left(\frac{r - \mu_i}{n \cdot \sigma_{ij}} \right) t), \\ &= \mu_i S_i(t) dt + \sum_{j=1}^n \sigma_{ij} S_i(t) (d\tilde{W}_j(t) - \left(\frac{r - \mu_i}{n \cdot \sigma_{ij}} \right) dt), \\ &= \mu_i S_i(t) dt + \sum_{j=1}^n \sigma_{ij} S_i(t) d\tilde{W}_j(t) - \mu_i S_i(t) dt + r S_i(t) dt, \\ &= r S_i(t) dt + \sum_{j=1}^n \sigma_{ij} S_i(t) d\tilde{W}_j(t). \end{aligned}$$

Hence, the price of the i th risky asset is:

$$S_i(t) = s_0^i \exp \left(\left(r - \frac{1}{2} \sum_{j=1}^n \sigma_{ij}^2 \right) t + \sum_{j=1}^n \sigma_{ij} \tilde{W}_j(t) \right), \quad i = 1, 2, \dots, n.$$

Appendix 2. Proof of Corollary 1.1

Proof: From Lemma 2.1 we have, under a risk-neutral measure $\mathbb{Q}$

$$\frac{dI(t)}{I(t)} = (r_n - r_r) dt + \sigma_I d\tilde{W}_I(t), \quad (\text{A1})$$

and we have $D_r^*(t) = \frac{D_r(t)}{I(t)}$. Since the process $I(t)$, $t \geq 0$ is a Geometric Brownian motion with $I(0) \neq 0$, hence applying Ito's Lemma, we get:

$$I(t) = I(0) \cdot e^{(r_n - r_r - \frac{\sigma_I^2}{2})t + \sigma_I d\bar{W}_I(t)}.$$

Putting the value of $I(t)$ in $D_r^*(t)$ we get:

$$\begin{aligned} D_r^*(t) &= \frac{e^{-r_r t}}{I(0)} e^{(-r_n + r_r + \frac{\sigma_I^2}{2})t - \sigma_I d\bar{W}_I(t)}, \\ &= \frac{e^{(-r_n - r_r + r_r + \frac{\sigma_I^2}{2})t - \sigma_I d\bar{W}_I(t)}}{I(0)}. \end{aligned}$$

Hence, we get the desired result as:

$$D_r^*(t) = \frac{e^{(-r_n + \frac{\sigma_I^2}{2})t - \sigma_I d\bar{W}_I(t)}}{I(0)}.$$

Appendix 3. Proof of Proposition 1

Proof: According to Equation (17) we have that

$$\begin{aligned} ELB_i &= \sum_{t \in T_a} \left[\sum_{j=1}^n \mathbb{E}_i (G_j(X(t)) \cdot D_r^*(t) \cdot \mathbb{1}_{\{X(t) \neq D\}}) \omega_j h_0 \right], \\ &= \mathbb{E}_i \left(\sum_{k \geq 1} \left[\sum_{j=1}^n G_j(X(ka)) \cdot D_r^*(ka) \cdot \mathbb{1}_{\{X(ka) \neq D\}} \omega_j h_0 \right] \right). \end{aligned}$$

The application of the tower property of conditional expectation gives

$$\begin{aligned} &= \mathbb{E}_i \left[\mathbb{E} \left[\sum_{k \geq 1} \sum_{j=1}^n G_j(X(ka)) \cdot D_r^*(ka) \cdot \mathbb{1}_{\{X(ka) \neq D\}} \omega_j h_0 / X(a) \right] \right], \\ &= \mathbb{E}_i \left[\sum_{j=1}^n G_j(X(a)) \cdot \frac{e^{(-r_n + \sigma_I^2)a}}{I(0)} \cdot \mathbb{1}_{\{X(a) \neq D\}} \omega_j h_0 \right. \\ &\quad \left. + \mathbb{E} \left[\sum_{k \geq 2} \sum_{j=1}^n G_j(X(ka)) \cdot D_r^*(ka) \cdot \mathbb{1}_{\{X(ka) \neq D\}} \omega_j h_0 / X(a) \right] \right]. \end{aligned}$$

Now, we change the variable by setting $s = k-1$ in the former expectation to get

$$ELB_i = \mathbb{E}_i \left[\sum_{j=1}^n \left(G_j(X(a)) \cdot \frac{e^{(-r_n + \sigma_I^2)a}}{I(0)} \cdot \mathbb{1}_{\{X(a) \neq D\}} \right) \omega_j h_0 \right] \quad (A2)$$

$$+ \mathbb{E} \left[\sum_{s \geq 1} \sum_{j=1}^n G_j(X(sa + a)) \cdot D_r^*(sa + a) \cdot \mathbb{1}_{\{X(sa + a) \neq D\}} \omega_j h_0 / X(a) \right]. \quad (A3)$$

Observe now that

$$\begin{aligned} &\mathbb{E} \left[\sum_{s \geq 1} \sum_{j=1}^n G_j(X(sa + a)) \cdot D_r^*(sa + a) \cdot \mathbb{1}_{\{X(sa + a) \neq D\}} \omega_j h_0 / X(a) \right], \\ &= \sum_{s \geq 1} \left[\sum_{j=1}^n \left[\sum_{u \neq D} G_j(u) \cdot P(X(sa + a) = j / X(a)) \cdot \mathbb{E}(D_r^*(sa + a)) \right] \omega_j h_0 \right], \\ &= \sum_{s \geq 1} \sum_{j=1}^n \sum_{u \neq D} G_j(u) \cdot (e^{Q(sa)})_{(X(a), u)} \cdot \frac{e^{(-r_n + \sigma^2)(sa + a)}}{I(0)} \cdot \omega_j \cdot h_0, \end{aligned}$$

$$= \frac{e^{(-r_n + \sigma_I^2)a}}{I(0)} \cdot ELB_{X(a)}.$$

Then,

$$\begin{aligned} ELB_i &= \mathbb{E}_i \left[\sum_{j=1}^n \left(G_j(X(a)) \cdot \frac{e^{(-r_n + \sigma_I^2)a}}{I(0)} \cdot \mathbb{1}_{\{X(a) \neq D\}} \right) \omega_j h_0 + \frac{e^{(-r_n + \sigma_I^2)a}}{I(0)} \cdot ELB_{X(a)} \right], \\ &= \sum_{u \in E} (e^{(Qa)})_{(i,u)} \cdot \frac{e^{(-r_n + \sigma_I^2)a}}{I(0)} \left[\sum_{j=1}^n (G_j(u) \omega_j h_0) + ELB_u \right], \\ &= \sum_{u \in E} (e^{(Qa)})_{(i,u)} \cdot \frac{e^{(-r_n + \sigma_I^2)a}}{I(0)} \left[(\mathbf{G}^T(u) \cdot \boldsymbol{\omega}) h_0 + ELB_u \right], \\ &= \sum_{u \in E} (e^{(Qa)})_{(i,u)} \cdot \frac{e^{(-r_n + \sigma_I^2)a}}{I(0)} [\tilde{\mathbf{G}}(u) + ELB_u], \end{aligned}$$

The previous equation can be written in matrix form by using the column-vectors $\mathbf{ELB}$ and $\mathbf{G}$ as follows

$$\mathbf{ELB} = \frac{e^{(-r_n + \sigma_I^2)a}}{I(0)} \cdot \mathbf{P}(a) * (\tilde{\mathbf{G}} + \mathbf{ELB}).$$

Algebraic manipulations give

$$\left(I - \frac{e^{(-r_n + \sigma_I^2)a}}{I(0)} \cdot \mathbf{P}(a) \right) * \mathbf{ELB} = \frac{e^{(-r_n + \sigma_I^2)a}}{I(0)} \cdot \mathbf{P}(a) * \tilde{\mathbf{G}}, \quad (\text{A4})$$

which in turn provides

$$\mathbf{ELB} = \left(I - \frac{e^{(-r_n + \sigma_I^2)a}}{I(0)} \cdot \mathbf{P}(a) \right)^{-1} * \frac{e^{(-r_n + \sigma_I^2)a}}{I(0)} \cdot \mathbf{P}(a) * \tilde{\mathbf{G}}, \quad (\text{A5})$$

provided that $\left(I - \frac{e^{(-r_n + \sigma_I^2)a}}{I(0)} \cdot \mathbf{P}(a) \right)^{-1}$ exists.

In any case, $\mathbf{P}(a)$ is a stochastic matrix, thus its spectral radius $\rho(\mathbf{P}(a)) = 1$ as a consequence of Perron-Frobenius theorem.

Moreover, observe that the matrix $I - \frac{e^{(-r_n + \sigma_I^2)a}}{I(0)} \cdot \mathbf{P}(a)$ can be rewritten as follows:

$$I - \frac{e^{(-r_n + \sigma_I^2)a}}{I(0)} \cdot \mathbf{P}(a) = D_r^*(a) \left(\frac{e^{(-r_n + \sigma_I^2)a}}{I(0)} \cdot I - \mathbf{P}(a) \right). \quad (\text{A6})$$

The matrix $\left(\frac{e^{(-r_n + \sigma_I^2)a}}{I(0)} \cdot I - \mathbf{P}(a) \right)$ belongs to the class of M-matrix (Le & Tsatsomeros, 2021) and is non-singular if and only if $\frac{e^{(-r_n + \sigma_I^2)a}}{I(0)} \geq \rho(\mathbf{P}(a)) = 1$. ■

Appendix 4. Abbreviation

Abbreviation	Description
VAs	Variable annuities
ILVAs	Inflation-Linked Variable Annuities
GBM	Geometric Brownian Motion
TIPS	Treasury Inflation-Protected Securities
ILGs	Inflation-linked gilt-edged securities
HJM	Heath–Jarrow–Morton
CPI	Consumer Price Index
CTMC	Continuous-time Markov chain
GLWB	Guaranteed Lifelong Withdrawal Benefit
GMAB	Guaranteed Minimum Accumulation Benefits
GMIB	Guaranteed Minimum Income Benefits
SDE	Stochastic Differential Equation
GMWB	Guaranteed Minimum Withdrawal Benefits
GMDB	Guaranteed Minimum Death Benefits
ELB	Expected life benefits
EDB	Expected death benefits
NLTC	National Long-Term Care