



Limit Theorems for an extended inverse Hawkes process with general exciting functions

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ABSTRACT

An inverse Hawkes process is a process having constant intensity and stochastic jump size, depending on the past number of jumps, while a Hawkes process has the intensity which is stochastic. An extended inverse Hawkes process is a process obtained by combining a Hawkes process and an inverse Hawkes process.

The focus of this paper is to investigate the asymptotic behaviour of an extended inverse Hawkes process with general structure of the exciting functions. In particular, the results obtained are the generalized versions of the Law of Large Numbers and of the Central Limit Theorem.

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1. Introduction

The Hawkes processes are a class of point processes having intensities, which are stochastic, in the sense that the future evolution of the Hawkes processes are controlled by the timing of past events. First introduced by Hawkes (1971), for problems related to the earthquakes, the characteristics of taking into account the entire past history, as well as clustering effects, is the reason why the Hawkes processes have wide applications in several fields in the real world, including not only seismology, but also finance, insurance, neuroscience, and DNA modelling. Since the future developments are function of a timing of past events, these processes cannot be considered Markovian. Markov property can be obtained only if the exciting functions satisfy some particular conditions, such as, for instance, if the exciting functions are exponential, or sum of exponential functions. To characterize the Hawkes processes, we have to consider not only the exciting functions but also the intensity functions. If the intensity functions are nonlinear, then the related Hawkes processes are called nonlinear and are first introduced by Brémaud and Massoulié (1996), otherwise are linear. For the Hawkes processes the jump sizes are deterministic while the intensity functions are stochastic.

In the inverse Hawkes processes, the jump sizes are stochastic, while the intensity functions are constant. This means that the expected number of jumps remains constant, but the greater number of jumps in the past implies the larger value of the size of the jumps in the future. These type of processes have been widely applied in several fields such as queueing theory Gao and Zhu (2018a), finance and statistics Bacry et al. (2012), Dharmaraja et al. (2022) and Errais et al. (2010).

A natural generalization of the Poisson processes are the Hawkes processes, which are suitable to capture both exciting properties and clustering effects, which make it suitable for computations in financial applications Errais et al. (2010). Often in literature, for the Hawkes processes, authors consider main the linear case, see Gao and Zhu (2018b) for the

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LLN (Law of Large Numbers), [Bacry et al. \(2012\)](#) for the CLT (Central Limit Theorem) and [Bordenave and Torrisi \(2007\)](#) for the LDP (Large Deviations Principle). The nonlinear Hawkes processes are much less studied because of the difficulty in the computational tractability. However, [Brémaud and Massoulié \(1996\)](#) studies the nonlinear case, and [Zhu \(2013a,b, 2014, 2015\)](#) investigate the linear and the nonlinear case, obtaining several results related to the asymptotic behaviours. In [Seol \(2017\)](#), the author studies some limit theorems for the compensator of a Hawkes process, and, in [Seol \(2019\)](#), the same author extends these results to the inverse Hawkes processes. Some further progress related to the LLN, Invariance Principle and the CLT are in [Gao and Zhu \(2018a\)](#) and generalized in part in [Seol \(2021, 2022\)](#).

This paper investigates the extended inverse Hawkes processes, obtained by combining the Hawkes processes and the inverse Hawkes processes in a more general structure. For all these processes, we assume that the exciting functions is not only exponential, but has a general structure. For these processes we study the asymptotic behaviours by obtained the LLN and the CLT. To the best of our knowledge, this is the first time that the asymptotic behaviours are studied for extended inverse Hawkes processes with general structure of the exciting functions.

This paper is organized as follows: Section 2 provides the description of the Hawkes process, the inverse Hawkes process, and the extended inverse Hawkes process having exponential exciting function, and their asymptotic behaviours. Section 3 summarizes the main results of the paper, which are the LLN and the CLT for the extended inverse Hawkes processes with general exciting functions. The proofs of all the results are in Section 4. Section 5 concludes the paper with some final remarks.

2. Hawkes processes

2.1. General Hawkes processes

Let us assume that the uncertainty is described by $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ a filtered probability space, where filtration $\mathcal{F}_t$ satisfies the usual conditions of completeness and right continuity, and on which all the stochastic processes are defined.

Let $N = \{N_t, t \geq 0\}$ be a point process on $\mathbf{R}^+$, where $N_t := N(0, t]$ is the number of jumps of N in the interval $(0, t]$, $N_0 = 0$, and for $B \in \mathcal{B}(\mathbf{R}^+)$, $B \subset (0, t]$, $N(B)$ denotes the number of jumps of the process N in B . Let $\mathcal{F}_t = \sigma(N(B), B \in \mathcal{B}(\mathbf{R}^+), B \subset (0, t])$ be an increasing family of σ -algebras known as natural filtrations. Any non-negative $\mathcal{F}_t$ -progressively measurable process $\lambda = \{\lambda(t), t \geq 0\}$ is called $\mathcal{F}_t$ -intensity of N , provided that, for all intervals $(c, d]$, with $c > 0$,

$$\mathbf{E}[N(c, d) | \mathcal{F}_c] = \mathbf{E} \left[\int_c^d \lambda(s) ds | \mathcal{F}_c \right], \text{ a.s.}$$

A Hawkes process is a point process N admitting an $\mathcal{F}_t$ intensity $\lambda(t) = \lambda \left(\int_0^t h(t-s) dN_s \right)$, where $\lambda(\cdot) : \mathbf{R}^+ \rightarrow \mathbf{R}^+$ is a locally integrable, left continuous function and $h(\cdot) : \mathbf{R}^+ \rightarrow \mathbf{R}^+$ is assumed to be integrable, $\|h\|_{L^1} = \int_0^\infty h(t) dt < \infty$. Note that, $h(\cdot)$ and $\lambda(\cdot)$ are commonly referred in literature as the exciting function and the intensity function, respectively. Moreover, if $\lambda(\cdot)$ is linear, the Hawkes process is linear; otherwise, it is nonlinear.

Generally, the Hawkes process is non-Markovian in nature since the timing of the past events determine the future growth. But, under some particular assumptions on the exciting function, such as, for instance, if the exciting function is exponential or sum of exponential functions, the couple (N, λ) , given by the Hawkes process and its intensity, is proven to be a Markov process (see for instance [Dharmaraja et al. \(2022\)](#), [Errais et al. \(2010\)](#) and [Zhu \(2015\)](#) and references therein).

Hence, if $h(t) = ae^{-bt}$, $a > 0$ and $b > 0$, the couple (N, λ) satisfies the Markov property, and we are able to define $\{Z_t, t \geq 0\}$ a Markov process which fulfils the dynamics:

$$dZ_t = -bZ_t dt + a dN_t, \quad t \geq 0.$$

The intensity of the process $\{N_t, t \geq 0\}$, for each t , is given by $\lambda(t) = v + Z_{t-}$, where the so-called baseline intensity v takes positive real values, and the process Z_t has the infinitesimal generator

$$\mathcal{L}f(z) = -bz f'(z) + (v+z)[f(z+a) - f(z)].$$

Furthermore, looking to the existing literature, [Gao and Zhu \(2018a\)](#), as a consequence of the LLN, we get that

$$\frac{1}{t} \int_0^t Z_s ds \rightarrow \frac{v}{b-a} \quad \text{a.s. as } t \rightarrow \infty.$$

While, as a consequence of the CLT, and, as usual, setting $N(0, \sigma^2)$ a random variable with normal distribution of mean 0 and variance σ^2 , we have

$$\frac{1}{\sqrt{t}} \left[\int_0^t Z_s ds - \frac{v}{b-a} t \right] \rightarrow N \left(0, \frac{a^2 v b}{(b-a)^3} \right) \quad \text{in distribution as } t \rightarrow \infty.$$

When $\lambda(\cdot)$ is linear, say $\lambda(z) = v + z$, $v > 0$, and $h(\cdot)$ is integrable, the Hawkes process has a nice immigration-birth representation, [Hawkes and Oakes \(1974\)](#). For the linear Hawkes process, the LLN, [Daley and Vere-Jones \(2003\)](#), allow us to write that,

$$\frac{N_t}{t} \rightarrow \frac{v}{1 - \|h\|_{L^1}}, \quad \text{a.s. as } t \rightarrow \infty.$$

While, under some regularity assumptions i.e., $\|h\|_{L^1} < 1$, Bacry et al. (2012) proved the CLT, namely,

$$\frac{N_t - \mu t}{\sqrt{t}} \rightarrow \sigma N(0, 1), \text{ as } t \rightarrow \infty,$$

where $\mu = \frac{v}{1-\|h\|_{L^1}}$ and $\sigma^2 = \frac{v}{(1-\|h\|_{L^1})^3}$. Note that in this case there is a weak convergence on $D[0, 1]$, the space of càdlàg functions on $[0, 1]$, equipped with Skorokhod topology. Then, Bordenave and Torrisi (2007) proved a LDP for $P(\frac{N_t}{t} \in \cdot)$.

When $\lambda(\cdot)$ is nonlinear, however, the usual immigration-birth representation no longer works, therefore, to establish limit theorems, a different approach must be considered. For nonlinear Hawkes processes, Zhu (2013a) proved the Functional Central Limit Theorem. Assuming that $h(\cdot)$ is exponential or sum of exponential functions, and consequently the Hawkes process is Markovian, Zhu (2015) developed a LDP for $P(\frac{N_t}{t} \in \cdot)$. Moreover, Zhu (2014) showed a process-level, i.e., level-3 LDP for the general $h(\cdot)$, and, hence, obtained the level-1 LDP for $P(\frac{N_t}{t} \in \cdot)$ by Contraction Principle.

2.2. Inverse Hawkes processes

An inverse version of the Hawkes process is developed and studied by Seol (2019). Note that, in the general Hawkes process, the jump size takes a constant value and the intensity is linear in Z_t . This structure implies that the size of the jumps remains constant, while Z_t increases with the number of the jumps, and therefore increases the expected number of jumps in the future. But in the inverse Hawkes process, the intensity takes a constant value 1 and the jump size is linear in Z_t . This means that the expected number of jumps in the future remains constant, while the expected value of the size of the jumps in the future increases if Z_t increases. The self-excitation for the Hawkes process is based on the intensity, but the self-excitation for the inverse Hawkes process is based on the jump size. An inverse Markovian Hawkes process, Seol (2019), is defined as

$$dZ_t = -bZ_t dt + (v + aZ_{t-})dN_t, \quad t \geq 0,$$

where $Z_0 = 0$ and $\{N_t, t \geq 0\}$ is a Poisson process with intensity 1. Thus, the infinitesimal generator of the process Z_t is represented as

$$\mathcal{L}f(z) = -bz f'(z) + f(z + az + v) - f(z),$$

and

$$Z_t = \int_0^t (aZ_{s-} + v)e^{-b(t-s)}dN_s. \quad (1)$$

In Seol (2019), as a consequence of the LLN and of the CLT, respectively, and under some assumptions i.e., $b > a$, $2b - 2a - a^2 > 0$, and $(1 + a)^4 < 1 + 4b$, the author gets that

$$\frac{1}{t} \int_0^t Z_s ds \rightarrow \frac{v}{b-a}, \text{ a.s., as } t \rightarrow \infty,$$

and

$$\frac{1}{\sqrt{t}} \left[\int_0^t Z_s ds - \frac{v}{b-a} t \right] \rightarrow N \left(0, \frac{v^2 + 2va \frac{v}{b-a} + a^2 \frac{v^2(a+b+2)}{(b-a)(2b-2a-a^2)}}{(b-a)^2} \right) \text{ in distribution as } t \rightarrow \infty.$$

Furthermore, Seol (2022) generalizes the previous model by assuming a general structure for the exciting function $h(\cdot) : \mathbf{R}^+ \rightarrow \mathbf{R}^+$, which is integrable and satisfy $\frac{1}{\sqrt{t}} \int_0^t \int_t^\infty h(s-u)dsdu \rightarrow 0$ as $t \rightarrow \infty$ and $\int_0^\infty h(t)\sqrt{t} dt < \infty$. This general structure of $h(\cdot)$ implies that the inverse Hawkes process cannot always satisfy the Markov property, and that Z_t given in Eq. (1) is replaced by

$$\bar{Z}_t = \int_0^t (a\bar{Z}_{s-} + v)h(t-s)dN_s.$$

In addition, Seol (2022) gets that

$$\begin{aligned} \frac{1}{t} \int_0^t \bar{Z}_s ds &\rightarrow \frac{v\|h\|_{L^1}}{1-a\|h\|_{L^1}} \text{ in probability as } t \rightarrow \infty, \\ \frac{1}{\sqrt{t}} \left[\int_0^t \bar{Z}_s ds - \frac{v\|h\|_{L^1}}{1-a\|h\|_{L^1}} t \right] &\rightarrow N(0, \sigma^2) \text{ in distribution as } t \rightarrow \infty, \end{aligned}$$

where

$$\begin{aligned} \sigma^2 &:= \frac{v^2\|h\|_{L^1}^2}{(1-a\|h\|_{L^1})^2} \left[a^2 \mathbf{E} \left[\left(\sum_{k=0}^{\infty} a^k h^{*(k+1,N)}(\infty) \right)^2 \right] + 2a \frac{\|h\|_{L^1}}{1-a\|h\|_{L^1}} + 1 \right], \\ h^{*(k+1,N)}(t) &:= \int_{0 < s_{k+1} < s_k < \dots < s_1 \leq t} h(t-s_1)h(s_1-s_2) \dots h(s_k-s_{k+1})dN_{s_{k+1}}dN_{s_k} \dots dN_{s_1}. \end{aligned}$$

2.3. Extended inverse Hawkes processes

In [Seol \(2021\)](#), the author proposes an extended inverse Markovian Hawkes process, that combines a Hawkes process and an inverse Hawkes process, both having exponential exciting functions. More precisely, given a_1 , a_2 and b positive constants, and the baseline intensities v_1 and v_2 taking positive real values, the extended inverse Markovian Hawkes process has a dynamics driven by:

$$dZ_t = -bZ_t dt + a_1 dN_t^{(1)} + (v_2 + a_2 Z_{t-}) dN_t^{(2)}, \quad t \geq 0$$

where $\{N_t^{(1)}, t \geq 0\}$ is a Hawkes process with intensity, at any time t , given by $v_1 + Z_{t-}$, and $\{N_t^{(2)}, t \geq 0\}$ is a Poisson process with intensity 1. Taking into account that $Z_0 = 0$, and that

$$d(e^{bt} Z_t) = a_1 e^{bt} dN_t^{(1)} + (v_2 + a_2 Z_{t-}) e^{bt} dN_t^{(2)},$$

$$Z_t = \int_0^t a_1 e^{-b(t-s)} dN_s^{(1)} + \int_0^t (v_2 + a_2 Z_{s-}) e^{-b(t-s)} dN_s^{(2)}.$$

Consequently, the infinitesimal generator of the process Z_t assumes the representation given by

$$\mathcal{L}f(z) = -bf'(z) + (v_1 + z)[f(z + a_1) - f(z)] + f(z + a_2 z + v_2) - f(z).$$

Thus, as a consequence of the LLN and of the CLT, [Seol \(2021\)](#) gets that

$$\frac{1}{t} \int_0^t Z_s ds \rightarrow \frac{v_1 a_1 + v_2}{b - a_1 - a_2} \quad \text{a.s. as } t \rightarrow \infty, \quad (2)$$

$$\frac{1}{\sqrt{t}} \left[\int_0^t Z_s ds - \frac{v_1 a_1 + v_2}{b - a_1 - a_2} t \right] \rightarrow N(0, \sigma^2) \quad \text{in distribution as } t \rightarrow \infty, \quad (3)$$

where

$$\sigma^2 := \frac{(a_1^2 v_1 + v_2^2) M_1 M_2 + (a_1^2 v_1 + 2v_1 a_2) M_2 M_3 + a_2^2 (M_3 M_5 + M_1 M_4)}{M_1^3 M_2}, \quad (4)$$

$$M_1 = b - a_1 - a_2, \quad M_2 = 2M_1 - a_2^2, \quad M_3 = v_1 a_1 + v_2, \quad M_4 = v_1 a_1^2 + v_2^2, \quad M_5 = a_1^2 + 2v_2 a_2 + 2M_3,$$

under some regularity conditions, i.e., $M_1, M_2 > 0$.

3. Main results

This section is devoted to study the extended inverse Hawkes process having general structure of the exciting function, and its asymptotic behaviour.

To this end, let a_1 and a_2 positive constants, and v_1 and v_2 the baseline intensities taking positive real values. Assuming that $h_1(\cdot), h_2(\cdot) : \mathbf{R}^+ \rightarrow \mathbf{R}^+$, the exciting functions, are integrable, and $\|h_1(t)\|_{L^1} < 1$, the extended inverse Hawkes process has the dynamics driven by

$$Z_t = \int_0^t h_1(t-s) dN_s^{(1)} + \int_0^t (v_2 + a_2 Z_{s-}) h_2(t-s) dN_s^{(2)}, \quad (5)$$

where $\{N_t^{(1)}, t \geq 0\}$ is a Hawkes process with intensity, at any time t , given by $v_1 + Z_{t-}$, and $\{N_t^{(2)}, t \geq 0\}$ is a Poisson process with intensity 1.

Starting from the results obtained for a model having a structure driven by Eq. (5), in this paper, we are going to generalize them, proving the same results but for a model having a more general structure. More precisely, this model has the dynamics driven by

$$Z_t = \int_0^t h_1(t-s) dN_s^{(1)} + \int_0^t (h_{21}(t-s) + Z_{s-} h_{22}(t-s)) dN_s^{(2)}, \quad (6)$$

where $\{N_t^{(1)}, t \geq 0\}$ is a Hawkes process with intensity, at any time t , given by $v_1 + Z_{t-}$, and $\{N_t^{(2)}, t \geq 0\}$ is a Poisson process with intensity 1.

Assumption 3.1. For $g = h_1, h_2, h_{21}$ and h_{22} , let

- (1) $\|g(t)\|_{L^1} < \infty$ and, in particular, $\|h_1(t)\|_{L^1} < 1$.
- (2) $\frac{1}{\sqrt{t}} \int_0^t \int_t^\infty g(s-u) ds du \rightarrow 0$,
- (3) $\int_0^\infty g(t) \sqrt{t} dt < \infty$.

The main results of this paper are given by [Theorems 3.2](#), and [3.3](#), their proofs are given in [Section 4](#).

Theorem 3.2. Let Z_t be defined as in Eq. (6). Under Assumption 3.1, points (1) and (2), we have that

$$\frac{1}{t} \int_0^t Z_s ds \rightarrow \frac{v_1 \|h_1\|_{L^1} + \|h_{21}\|_{L^1}}{1 - \|h_1\|_{L^1} - \|h_{22}\|_{L^1}}, \quad \text{in probability as } t \rightarrow \infty. \quad (7)$$

In particular if $h_{21} = v_2 h_2$ and $h_{22} = a_2 h_2$ then we have the LLN for the process Z_t defined in (5) i.e.,

$$\frac{1}{t} \int_0^t Z_s ds \rightarrow \frac{v_1 \|h_1\|_{L^1} + v_2 \|h_2\|_{L^1}}{1 - \|h_1\|_{L^1} - a_2 \|h_2\|_{L^1}}, \quad \text{in probability as } t \rightarrow \infty. \quad (8)$$

Also, in the particular case in which $h_1(t) = a_1 e^{-bt}$ and $h_2(t) = e^{-bt}$, the model is Markovian, and we have that

$$\frac{1}{t} \int_0^t Z_s ds \rightarrow \frac{a_1 v_1 + v_2}{b - a_1 - a_2}, \quad \text{in probability as } t \rightarrow \infty. \quad (9)$$

Note that Eq. (9) coincides with Eq. (2), which means that Theorem 3.2 generalized the analogous result given in Seol (2021).

Theorem 3.3. Let Z_t be defined as in Eq. (6). Under Assumption 3.1, points (1), (2) and (3), we have

$$\frac{1}{\sqrt{t}} \left[\int_0^t Z_s ds - \left(\frac{v_1 \|h_1\|_{L^1} + \|h_{21}\|_{L^1}}{1 - \|h_1\|_{L^1} - \|h_{22}\|_{L^1}} \right) t \right] \rightarrow N(0, \sigma^2) \quad \text{in distribution as } t \rightarrow \infty, \quad (10)$$

where

$$\sigma^2 := \frac{(\|h_1\|_{L^1})^2 (v_1 + \frac{v_1 \|h_1\|_{L^1} + \|h_{21}\|_{L^1}}{1 - \|h_1\|_{L^1} - \|h_{22}\|_{L^1}}) + [(\|h_{22}\|_{L^1})^2 \mathbf{E}[Z_\infty^2] + 2 \|h_{21}\|_{L^1} \|h_{22}\|_{L^1} (\frac{v_1 \|h_1\|_{L^1} + \|h_{21}\|_{L^1}}{1 - \|h_1\|_{L^1} - \|h_{22}\|_{L^1}}) + (\|h_{21}\|_{L^1})^2]}{(1 - \|h_1\|_{L^1} - \|h_{22}\|_{L^1})^2}, \quad (11)$$

and $\mathbf{E}[Z_\infty^2]$ is given in Proposition 4.1, Eq. (17). In particular if $h_{21} = v_2 h_2$ and $h_{22} = a_2 h_2$ then we have the CLT for the process Z_t defined in (5) i.e.,

$$\frac{1}{\sqrt{t}} \left[\int_0^t Z_s ds - \left(\frac{v_1 \|h_1\|_{L^1} + v_2 \|h_2\|_{L^1}}{1 - \|h_1\|_{L^1} - a_2 \|h_2\|_{L^1}} \right) t \right] \rightarrow N(0, \sigma^2) \quad \text{in distribution as } t \rightarrow \infty, \quad (12)$$

where

$$\sigma^2 := \frac{(\|h_1\|_{L^1})^2 (v_1 + \frac{v_1 \|h_1\|_{L^1} + v_2 \|h_2\|_{L^1}}{1 - \|h_1\|_{L^1} - a_2 \|h_2\|_{L^1}}) + (\|h_2\|_{L^1})^2 [a_2^2 \mathbf{E}[Z_\infty^2] + 2 v_2 a_2 (\frac{v_1 \|h_1\|_{L^1} + v_2 \|h_2\|_{L^1}}{1 - \|h_1\|_{L^1} - a_2 \|h_2\|_{L^1}}) + (v_2)^2]}{(1 - \|h_1\|_{L^1} - a_2 \|h_2\|_{L^1})^2}. \quad (13)$$

Furthermore, when $h_1(t) = a_1 e^{-bt}$, $h_2(t) = e^{-bt}$, the model satisfies the Markov property and

$$\frac{1}{\sqrt{t}} \left[\int_0^t Z_s ds - \frac{v_1 a_1 + v_2}{b - a_1 - a_2} t \right] \rightarrow N(0, \sigma^2) \quad \text{in distribution as } t \rightarrow \infty, \quad (14)$$

where σ^2 is given as in Eq. (4).

Note that Eq. (14) coincides with Eq. (3), which means that Theorem 3.3 generalized the analogous result given in Seol (2021).

Remark 3.4. If we let $a_2 = 0$ in Z_t which is defined by Eq. (5) then it corresponds to the intensity of general dynamic contagion process (see for instance Yao and Xiao (2018)) whose intensity at time t is given by:

$$\lambda(t) = v_1 + \int_0^t v_2 h_2(t-s) dN_s^{(2)} + \int_0^t h_1(t-s) dN_s^{(1)}, \quad (15)$$

where $\{N_t^{(1)}, t \geq 0\}$ is a dynamic contagion process, and $\{N_t^{(2)}, t \geq 0\}$ is a Poisson process with intensity 1.

In the particular case consider h_1 and h_2 as exponential decay exciting functions then Eq. (15) corresponds to the intensity of dynamic contagion process for the Markovian case (see for instance Dassios and Zhao (2011)).

Based on the above analysis, we observe that the Limit theorems obtained in Dassios and Zhao (2011) and Yao and Xiao (2018) can also be obtained by using Eqs. (8) and (12). However our result is more general.

4. Proof of the main results

The proof of our main results, as well as all the preliminary results, are presented in this section. In particular, in Section 4.1, we get the first and the second moments of Z_t , whereas Sections 4.2 and 4.3 completed the main results.

4.1. Auxiliary results

Proposition 4.1. Let Z_t be defined as in Eq. (6). Under [Assumption 3.1](#), we have:

(i)

$$\mathbf{E}[Z_\infty] = \frac{v_1 \|h_1\|_{L^1} + \|h_{21}\|_{L^1}}{1 - \|h_1\|_{L^1} - \|h_{22}\|_{L^1}}, \quad (16)$$

(ii)

$$\mathbf{E}[Z_\infty^2] = \mathbf{E} \left[\left(\sum_{k=0}^{\infty} h_{22}^{*(k, N^{(2)})} * h_{21}^{(1, N^{(2)})}(\infty) + \sum_{k=0}^{\infty} h_{22}^{*(k, N^{(2)})} * h_1^{(1, N^{(1)})}(\infty) \right)^2 \right], \quad (17)$$

provided it is finite, where

$$\begin{aligned} & h_{22}^{*(k, N^{(2)})} * h_{21}^{(1, N^{(2)})}(t) \\ &:= \int_{0 < s_{k+1} < s_k < \dots < s_1 \leq t} h_{22}(t-s_1) h_{22}(s_1-s_2) \cdots h_{22}(s_{k-1}-s_k) h_{21}(s_k-s_{k+1}) dN_{s_{k+1}}^{(2)} dN_{s_k}^{(2)} \cdots dN_{s_1}^{(2)}, \end{aligned} \quad (18)$$

and

$$\begin{aligned} & h_{22}^{*(k, N^{(2)})} * h_1^{(1, N^{(1)})}(t) \\ &:= \int_{0 < s_{k+1} < s_k < \dots < s_1 \leq t} h_{22}(t-s_1) h_{22}(s_1-s_2) \cdots h_{22}(s_{k-1}-s_k) h_1(s_k-s_{k+1}) dN_{s_{k+1}}^{(1)} dN_{s_k}^{(2)} \cdots dN_{s_1}^{(2)}. \end{aligned} \quad (19)$$

Proof. Taking into account that

$$dM_t^{(1)} = dN_t^{(1)} - (v_1 + Z_{t-})dt \quad \text{and} \quad dM_t^{(2)} = dN_t^{(2)} - dt \quad (20)$$

are martingales, by Eq. (6), we have that

$$\begin{aligned} Z_t &= \int_0^t h_1(t-s) dM_s^{(1)} + \int_0^t (v_1 + Z_{s-}) h_1(t-s) ds \\ &\quad + \int_0^t (h_{21}(t-s) + Z_{s-} h_{22}(t-s)) dM_s^{(2)} + \int_0^t (h_{21}(t-s) + Z_{s-} h_{22}(t-s)) ds. \end{aligned}$$

Hence,

$$\mathbf{E}[Z_t] = \int_0^t [(v_1 + \mathbf{E}[Z_s]) h_1(t-s) + (h_{21}(t-s) + \mathbf{E}[Z_s] h_{22}(t-s))] ds,$$

and by taking $t \rightarrow \infty$ on both sides, under [Assumption 3.1](#), we get Eq. (16).

Starting from Eq. (6), a renewal type equation, we write, successively,

$$\begin{aligned} Z_t &= \int_0^t h_1(t-s) dN_s^{(1)} \\ &\quad + \int_0^t \left(h_{21}(t-s) + \left(\int_0^s h_1(s-s_1) dN_{s_1}^{(1)} + \int_0^s (h_{21}(s-s_1) + h_{22}(s-s_1) Z_{s_1-}) dN_{s_1}^{(2)} \right) h_{22}(t-s) \right) dN_s^{(2)} \\ &= \int_0^t h_1(t-s) dN_s^{(1)} + \int_0^t \int_0^s h_{22}(t-s) h_1(s-s_1) dN_{s_1}^{(1)} dN_s^{(2)} \\ &\quad + \left(\int_0^t h_{21}(t-s) dN_s^{(2)} + \int_0^t \int_0^s h_{22}(t-s) h_{21}(s-s_1) dN_{s_1}^{(2)} dN_s^{(2)} \right) \\ &\quad + \int_0^t \int_0^s Z_{s_1-} h_{22}(t-s) h_{22}(s-s_1) dN_{s_1}^{(2)} dN_s^{(2)} \\ &= \int_0^t h_1(t-s) dN_s^{(1)} + \int_0^t \int_0^s h_{22}(t-s) h_1(s-s_1) dN_{s_1}^{(1)} dN_s^{(2)} \\ &\quad + \int_0^t \int_0^s \int_0^{s_1} h_{22}(t-s) h_{22}(s-s_1) h_1(s_1-s_2) dN_{s_2}^{(1)} dN_{s_1}^{(2)} dN_s^{(2)} \\ &\quad + \left(\int_0^t h_{21}(t-s) dN_s^{(2)} + \int_0^t \int_0^s h_{22}(t-s) h_{21}(s-s_1) dN_{s_1}^{(2)} dN_s^{(2)} \right) \end{aligned}$$

$$\begin{aligned}
& + \int_0^t \int_0^s \int_0^{s_1} h_{22}(t-s)h_{22}(s-s_1)h_{21}(s_1-s_2)dN_{s_2}^{(2)}dN_{s_1}^{(2)}dN_s^{(2)} \\
& + \int_0^t \int_0^s \int_0^{s_1} Z_{s_2-}h_{22}(t-s)h_{22}(s-s_1)h_{22}(s_1-s_2)dN_{s_2}^{(2)}dN_{s_1}^{(2)}dN_s^{(2)}.
\end{aligned}$$

Summing up, by solving Eq. (6), we get

$$Z_t = \sum_{k=0}^{\infty} h_{22}^{*(k, N^{(2)})} * h_{21}^{(1, N^{(2)})}(t) + \sum_{k=0}^{\infty} h_{22}^{*(k, N^{(2)})} * h_1^{(1, N^{(1)})}(t)$$

provided it is finite, where $h_{22}^{*(k, N^{(2)})} * h_{21}^{(1, N^{(2)})}(t)$ is given in Eq. (18) and $h_{22}^{*(k, N^{(2)})} * h_1^{(1, N^{(1)})}(t)$ is given in Eq. (19), which in turn implies Eq. (17). $\square$

4.2. Proof of Theorem 3.2

By the definition of Z_t given in Eq. (6), we have that

$$\int_0^t Z_s ds = \int_0^t \int_0^s h_1(s-s_1)dN_{s_1}^{(1)}ds + \int_0^t \int_0^s (h_{21}(s-s_1) + Z_{s_1-}h_{22}(s-s_1))dN_{s_1}^{(2)}ds.$$

By a change of variable and taking into account the definition of $M^{(i)}$, $i = 1, 2$, given in Eq. (20), we get

$$\begin{aligned}
\int_0^t Z_s ds &= \int_0^t \int_{s_1}^t h_1(s-s_1)dsdN_{s_1}^{(1)} + \int_0^t \int_{s_1}^t (h_{21}(s-s_1) + Z_{s_1-}h_{22}(s-s_1))dsdN_{s_1}^{(2)} \\
&= \int_0^t \int_{s_1}^{\infty} h_1(s-s_1)dsdN_{s_1}^{(1)} + \int_0^t \int_{s_1}^{\infty} (h_{21}(s-s_1) + Z_{s_1-}h_{22}(s-s_1))dsdN_{s_1}^{(2)} \\
&\quad - \int_0^t \int_t^{\infty} h_1(s-s_1)dsdN_{s_1}^{(1)} - \int_0^t \int_t^{\infty} (h_{21}(s-s_1) + Z_{s_1-}h_{22}(s-s_1))dsdN_{s_1}^{(2)} \\
&= \|h_1\|_{L^1} \int_0^t (v_1 + Z_{s_1-})ds_1 + \|h_1\|_{L^1} \int_0^t dM_{s_1}^{(1)} + \int_0^t (\|h_{21}\|_{L^1} + Z_{s_1-}\|h_{22}\|_{L^1})ds_1 \\
&\quad + \int_0^t (\|h_{21}\|_{L^1} + Z_{s_1-}\|h_{22}\|_{L^1})dM_{s_1}^{(2)} - \varepsilon_t,
\end{aligned}$$

where ε_t is defined by

$$\varepsilon_t = \int_0^t \int_t^{\infty} h_1(s-s_1)dsdN_{s_1}^{(1)} + \int_0^t \int_t^{\infty} (h_{21}(s-s_1) + Z_{s_1-}h_{22}(s-s_1))dsdN_{s_1}^{(2)}. \quad (21)$$

Consequently,

$$\mathbf{E}[\varepsilon_t] = \int_0^t \int_t^{\infty} (v_1 + \mathbf{E}[Z_{s_1-}])h_1(s-s_1)dsds_1 + \int_0^t \int_t^{\infty} (h_{21}(s-s_1) + \mathbf{E}[Z_{s_1-}]h_{22}(s-s_1))dsds_1. \quad (22)$$

Since, Z_{s_1} is uniform integrable

$$\mathbf{E}[\varepsilon_t] \leq \int_0^t \int_t^{\infty} h_1(s-s_1)dsds_1 \cdot (v_1 + \mathbf{E}[Z_{\infty}]) + \int_0^t \int_t^{\infty} h_{21}(s-s_1)dsds_1 + \int_0^t \int_t^{\infty} h_{22}(s-s_1)dsds_1 \cdot (\mathbf{E}[Z_{\infty}]),$$

and, by Assumption 3.1, we get that $\frac{1}{t}\mathbf{E}[\varepsilon_t] \rightarrow 0$, as $t \rightarrow \infty$, which implies the convergences in probability of $\frac{\varepsilon_t}{t} \rightarrow 0$, as $t \rightarrow \infty$. Let,

$$G_t^{(1)} := \frac{1}{t} \int_0^t dM_{s_1}^{(1)} \quad \text{and} \quad G_t^{(2)} := \frac{1}{t} \int_0^t (\|h_{21}\|_{L^1} + Z_{s_1-}\|h_{22}\|_{L^1})dM_{s_1}^{(2)},$$

then, as $t \rightarrow \infty$,

$$\begin{aligned}
\mathbf{E}[(G_t^{(1)})^2] &= \frac{1}{t^2} \mathbf{E} \left[\left(\int_0^t dM_{s_1}^{(1)} \right)^2 \right] = \frac{1}{t^2} \mathbf{E} \left[\int_0^t (v_1 + Z_{s_1-})ds_1 \right] \leq \frac{1}{t} \mathbf{E}[v_1 + Z_{\infty}] \rightarrow 0, \\
\mathbf{E}[(G_t^{(2)})^2] &= \frac{1}{t^2} \int_0^t \mathbf{E} \left[(\|h_{21}\|_{L^1} + Z_{s_1-}\|h_{22}\|_{L^1})^2 \right] ds_1 \leq \frac{1}{t} \mathbf{E} \left[(\|h_{21}\|_{L^1} + Z_{\infty}\|h_{22}\|_{L^1})^2 \right] \rightarrow 0.
\end{aligned}$$

Since Z^2 is uniform integrable, for $i = 1, 2$, by Jensen's Inequality, we have that $\mathbf{E}[G_t^{(i)}] \leq \mathbf{E}[(G_t^{(i)})^2]^{\frac{1}{2}}$, which implies that $\mathbf{E}[G_t^{(i)}] \rightarrow 0$, as $t \rightarrow \infty$. Hence, we get the convergence in probability of $G_t^{(i)} \rightarrow 0$ as $t \rightarrow \infty$. Substituting all the results obtained so far, we get Eq. (7).

4.3. Proof of Theorem 3.3

In order to achieve the result given by Eq. (10), with the same procedure used in the proof of Theorem 3.2, we get that

$$\frac{1}{\sqrt{t}} \int_0^t Z_s ds = \frac{1}{\sqrt{t}} \left[\|h_1\|_{L^1} \int_0^t (dM_s^{(1)} + (v_1 + Z_{s-}) ds) + \int_0^t (\|h_{21}\|_{L^1} + Z_{s1-} \|h_{22}\|_{L^1}) (dM_s^{(2)} + ds) \right] - \frac{\varepsilon_t}{\sqrt{t}}.$$

Note that, the error term ε_t , which is defined in Eq. (21), is such that $\mathbf{E}[\varepsilon_t]$ is given by Eq. (22). Therefore, by Assumption 3.1, we have that $\frac{1}{\sqrt{t}} \mathbf{E}[\varepsilon_t] \rightarrow 0$, as $t \rightarrow \infty$, which implies the convergence in probability of $\frac{\varepsilon_t}{\sqrt{t}} \rightarrow 0$ as $t \rightarrow \infty$. By applying the Central Limit Theorem for martingales, we get that

$$\frac{1}{\sqrt{t}} \left[\|h_1\|_{L^1} \int_0^t dM_s^{(1)} + \int_0^t (\|h_{21}\|_{L^1} + Z_{s1-} \|h_{22}\|_{L^1}) dM_s^{(2)} \right]$$

converges in distribution, as $t \rightarrow \infty$, to

$$N \left(0, (\|h_1\|_{L^1})^2 (v_1 + \mathbf{E}[Z_\infty]) + \left(\mathbf{E} \left[(\|h_{21}\|_{L^1} + Z_\infty \|h_{22}\|_{L^1})^2 \right] \right) \right).$$

Hence, by using the results of Proposition 4.1, we get

$$\frac{1}{\sqrt{t}} \left[\int_0^t Z_s ds - \left(\frac{v_1 \|h_1\|_{L^1} + \|h_{21}\|_{L^1}}{1 - \|h_1\|_{L^1} - \|h_{22}\|_{L^1}} \right) t \right] \rightarrow N(0, \sigma^2),$$

in distribution as $t \rightarrow \infty$, where

$$\begin{aligned} \sigma^2 &:= \frac{(\|h_1\|_{L^1})^2 (v_1 + \mathbf{E}[Z_\infty]) + \mathbf{E} \left[(\|h_{21}\|_{L^1} + Z_\infty \|h_{22}\|_{L^1})^2 \right]}{(1 - \|h_1\|_{L^1} - \|h_{22}\|_{L^1})^2} \\ &= \frac{(\|h_1\|_{L^1})^2 (v_1 + \mathbf{E}[Z_\infty]) + [(\|h_{22}\|_{L^1})^2 \mathbf{E}[Z_\infty^2] + 2\|h_{21}\|_{L^1} \|h_{22}\|_{L^1} \mathbf{E}[Z_\infty] + (\|h_{21}\|_{L^1})^2]}{(1 - \|h_1\|_{L^1} - \|h_{22}\|_{L^1})^2}, \end{aligned}$$

which implies Eq. (11), where $\mathbf{E}[Z_\infty]$ is given in Eq. (16) and $\mathbf{E}[Z_\infty^2]$ is given in Eq. (17).

5. Conclusions and remarks

In real world, the Hawkes processes have wide applications in many fields due to their characteristics such as clustering, and self-excitation. Despite their significance for applications, many important theoretical results in this topic remain unknown. Several authors, in literature, specifically examine the theoretical aspects of Hawkes processes and their asymptotic behaviours. Often, they consider Hawkes processes having exponential exciting functions, since the models can be proved to be Markovian, and the results related to their asymptotic behaviours can be achieved by avoiding some computational technicalities.

This paper is devoted to the study of an extended inverse Hawkes process, obtained by combining a Hawkes process and an inverse Hawkes process, and having the structure of the exciting functions that is not only exponential but is general. In this case, the Markov property does not always hold and to achieve results relating to the asymptotic behaviours, we had to assume some regularity conditions for the exciting functions. Hence, under these assumptions, we get a generalized version of the LLN and of the CLT.

An open problem in this area for future work is to study the LDP for an extended inverse Hawkes processes having a general structure of the exciting functions. Furthermore, since we have considered a self-exciting model, a natural generalization is to investigate the asymptotic behaviours for mutually exciting processes having linear and non-linear intensity functions. Another possible extension is to study the asymptotic behaviours by including random marks, i.e., when the jump sizes can be random.

Data availability

No data was used for the research described in the article.

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