

Limit theorems for Cox–Ingersoll–Ross process with externally and self-exciting jumps and application to finance

Shamiksha Pandey * and Dharmaraja Selvamuthu †

*Department of Mathematics,
Indian Institute of Technology Delhi,
Hauz Khas, New Delhi 110016, India*

**pandeyshamiksha18@gmail.com*

†dharmar@maths.iitd.ac.in

Received 19 March 2024

Revised 11 December 2024

Accepted 12 May 2025

Published 24 June 2025

In this paper, we discuss the Cox–Ingersoll–Ross (CIR) process which incorporates the external and self-exciting jumps having exponential exciting function. This is the generalization of the classical CIR process and CIR process with Poisson and Hawkes jumps and it belongs to the class of affine point processes. By employing the concept of infinitesimal generator, we obtain some limit theorems, such as the Law of Large Numbers and the Central Limit Theorem. Furthermore, we investigate the Large Deviation Principle by using the Feynman–Kac formula. The last section of this paper uses the proposed model to analyze the impact of defaults on financial risk and the corresponding loss processes.

Keywords: Cox–Ingersoll–Ross process; Poisson process; Hawkes process; Law of Large Numbers; Central Limit Theorem; Large Deviation Principle.

AMS Subject Classification: 60G55, 60J65, 60F05, 60F10, 60G44, 60H30

1. Introduction

The Cox–Ingersoll–Ross (CIR) process was first introduced by Cox *et al.* [4] and Feller [13], which is commonly used in finance to model the short-term interest rate and in credit risk to represent default intensity. However, in credit risk the primary focus is often on modeling default occurrences as they directly impact the current value of securities that are subject to default. In the literature, intensity-based models are proposed to effectively model defaults; thus, the CIR process is used. On the other hand, in finance, the CIR processes are used to describe how short-term interest rates evolve over time and to manage interest rate risk.

†Corresponding author.

Furthermore, in the literature, the study of jumps in the CIR process holds significant importance in financial modeling and risk management. The jumps in the CIR process are motivated by the realization that financial markets are subject to sudden and discontinuous changes. These unexpected events, which are frequently characterized by rapid changes in asset prices or credit conditions, result in financial losses or gains. Therefore, by including the jump component into the model, analysts may enhance the simulation and prediction of the consequences of severe occurrences which may strengthen risk assessment and decision-making processes (see [5, 9, 22]). As a result, point processes have been used to address the jumps in the CIR process.

Poisson process is the most commonly used point process because of its computational tractability. Consequently, a generalized version of the CIR process is employed, which includes jumps modeled by the Poisson process (see, for instance, [17, 18]).

However, looking into the real-world scenarios, the occurrence of jumps does not always exhibit time homogeneity. The occurrence of default contagion becomes more apparent during a financial crisis. As from the recent financial crisis, which led to the collapse of Lehman Brothers, highlighted the importance of contagion risk in the financial market. Therefore, developing a more advanced model than the Poisson process became necessary, as it does not sufficiently address the default clustering. Based on the analysis of real data, Errais *et al.* [12] found that this issue could be addressed more effectively using the self-exciting point process, i.e., Hawkes process, which was first proposed by Hawkes [15] and Hawkes and Oakes [16]. Simulations and distributional properties for the CIR process with self-exciting jumps have been studied by Dassios and Zhao [9].

In addition, Dassios and Zhao [7] introduced a model called “dynamic contagion process” which is the generalization of the Hawkes process and Cox process with Poisson shot-noise intensity. This model takes into consideration the fact that defaults can also be influenced by some external factors and consequently, is able to capture both the externally and self-exciting jumps. Dassios and Zhao ([7, 8]) by a martingale approach and infinitesimal generator analysis, also known as Dynkin’s formula, examine the distributional properties of the intensity process and the point process. Further, Yao and Xiao [23] extended the dynamic contagion process by considering the general structure of the exciting functions and examine their asymptotic properties. In Selvamuthu *et al.* [21], the authors considered the more general form of the intensity function of the dynamic contagion process and studied their limit theorems. In order to account for more general jumps in the CIR process, a study of the generalized CIR process with externally and self-exciting jumps, has been considered. The distributional properties of such model have been examined by Dassios *et al.* [5] and Pasricha and Selvamuthu [20].

Motivated by the work of Zhu [26], we investigate some asymptotic results, including the Law of Large Numbers (LLN), Central Limit Theorem (CLT) and

Large Deviation Principle (LDP) for CIR process with externally and self-exciting jumps.

This paper is organized as follows: Section 2 contains a detailed description of the Hawkes process, several types of the CIR process, and the asymptotic behaviors exhibited by these processes. Section 3 presents the model description of the generalized version of the CIR process with externally and self-exciting jumps. It also lists some of the assumptions and presents preliminary results. Section 4 focuses on the main results of the paper, i.e., the LLN, the CLT and the LDP. The detailed proof of the results are provided in Sec. 5. Section 6 explains the significance of the proposed model in relation to some financial models. Section 7 provides some remarks and open problems.

2. Hawkes and CIR Process

Consider, a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$, where the filtration $\mathcal{F}_t$ satisfies the usual conditions of completeness and right continuity. All the stochastic processes used throughout in the paper are defined under this filtered probability space.

2.1. Hawkes process (self-exciting process)

A Hawkes process is a point process $N = \{N_t, t \geq 0\}$ which admits an $\mathcal{F}_t$ intensity

$$\lambda_t = \phi \left(\int_0^t h(t-s) dN_s \right),$$

where $\phi(\cdot) : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is locally integrable, left continuous function and $h(\cdot) : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is assumed to be integrable, i.e. $\|h\|_{L^1} = \int_0^\infty h(t)dt < \infty$. Here $h(\cdot)$ and $\phi(\cdot)$ are known as the exciting function and the intensity function, respectively.

When $\phi(\cdot)$ is linear, say $\phi(z) = v + z$, $v > 0$, the Hawkes process is said to be linear, and it has a nice immigration-birth representation, see [16]. Under the Stability Assumption, i.e. $\|h\|_{L^1} < 1$, Daley and Vere-Jones [6], proved the LLN for the linear Hawkes process and allow us to write

$$\frac{N_t}{t} \rightarrow \frac{v}{1 - \|h\|_{L^1}}, \quad \text{a.s. as } t \rightarrow \infty.$$

Bacry *et al.* [1], proved a Functional CLT for the multivariate Hawkes process. In particular, as a special case, they obtained the following CLT for the linear Hawkes process,

$$\frac{N_t - \mu t}{\sqrt{t}} \rightarrow \sigma N(0, 1), \quad \text{as } t \rightarrow \infty,$$

where $\mu = \frac{v}{1 - \|h\|_{L^1}}$ and $\sigma^2 = \frac{v}{(1 - \|h\|_{L^1})^3}$. Moreover, Bordenave and Torrisi [2] proved a LDP, for $P\left(\frac{N_t}{t} \in \cdot\right)$. Moderate Deviation Principle (MDP) for the linear

Hawkes process is studied by Zhu [25]. Limit theorems of linear Hawkes process with random marks is obtained in [19].

When $\phi(\cdot)$ is nonlinear, then the Hawkes process is said to be nonlinear Hawkes process which is first introduced by Brémaud and Massoulié [3]. In nonlinear Hawkes process the usual immigration-birth representation is no longer valid thus a different approach must be taken into account to study the limit theorems. As a result, Zhu [24] proved the Functional CLT. LDP for the nonlinear case has been studied in [27, 28].

2.2. CIR process

CIR process $\{r_t, t \geq 0\}$ is a stochastic process proposed by Cox *et al.* [4] and satisfies the stochastic differential equation given by

$$dr_t = \delta(a - r_t)dt + \sigma\sqrt{r_t}dW_t,$$

where $\{W_t, t \geq 0\}$ is a standard Brownian motion and $\delta, \sigma > 0, a \geq 0$ and $2\delta a \geq \sigma^2$. The distributional properties of this process is studied by Dufresne [11]. The square root term, i.e. $\sqrt{r_t}$ in the volatility and the condition $2\delta a \geq \sigma^2$ help to ensure that the process $\{r_t, t \geq 0\}$ remains positive. This characteristic makes the CIR process to be widely used in interest rate modeling.

In order to describe the appearance of positive jumps in the model, a natural generalization of the CIR process is employed, where the jumps are modeled by the Poisson process, and it is defined as

$$dr_t = \delta(a - r_t)dt + \sigma\sqrt{r_t}dW_t + \alpha dN_t^{(1)},$$

where $\{N_t^{(1)}, t \geq 0\}$ is a homogeneous Poisson process with constant intensity $\rho > 0$ and $\alpha > 0$.

To account the issue of default clustering, Zhu [26] introduced the CIR process with Hawkes jumps as

$$dr_t = \delta(a - r_t)dt + \sigma\sqrt{r_t}dW_t + \beta dN_t,$$

where $\{N_t, t \geq 0\}$ is a Hawkes process with intensity $\lambda_t = b + cr_t$ where $b, c, \beta > 0$, and the author has proved the limit theorems such as LLN, CLT and LDP.

3. Model Description and Preliminaries

In order to incorporate both externally and self-exciting jumps, we consider a generalization of the CIR process $\{r_t, t \geq 0\}$ which is defined by

$$dr_t = \delta(a - r_t)dt + \sigma\sqrt{r_t}dW_t + \alpha dN_t^{(1)} + \beta dN_t, \quad (3.1)$$

where $\{W_t, t \geq 0\}$ is a standard Brownian motion, $\{N_t^{(1)}, t \geq 0\}$ is a Poisson process with parameter $\rho > 0$, $\{N_t, t \geq 0\}$ is a point process with intensity

$\lambda_t = b + cr_t$ and we require that the constants are such that $\delta, \sigma, \alpha, \beta, b, c > 0$, $a \geq 0$ and $2\delta a \geq \sigma^2$.

Remark 3.1. Using Itô lemma of jump-diffusion process, the CIR process r_t defined in Eq. (3.1) can be expressed as

$$\begin{aligned} r_t = & a + (r_0 - a)e^{-\delta t} + \sigma \int_0^t e^{-\delta(t-s)} \sqrt{r_s} dW_s \\ & + \alpha \int_0^t e^{-\delta(t-s)} dN_s^{(1)} + \beta \int_0^t e^{-\delta(t-s)} dN_s, \end{aligned} \quad (3.2)$$

where $r_0 > 0$ is the initial value of r_t at time $t = 0$.

The joint process $\{(r_t, N_t), t \geq 0\}$ is a Markov process in the state space $D = \mathbb{R}^+ \times (\mathbb{N} \cup \{0\})$ since the corresponding exciting function is exponential. The process (r_t, N_t, t) has an infinitesimal generator $\mathcal{A}$ defined on a function $f : D \times \mathbb{R}^+ \rightarrow \mathbb{R}$ which is given by

$$\begin{aligned} \mathcal{A}f(r, n, t) = & \frac{\partial f}{\partial t} - \delta(r - a) \frac{\partial f}{\partial r} + \frac{1}{2} \sigma^2 r \frac{\partial^2 f}{\partial r^2} + \rho[f(r + \alpha, n, t) - f(r, n, t)] \\ & + (b + cr)[f(r + \beta, n + 1, t) - f(r, n, t)]. \end{aligned} \quad (3.3)$$

Assumption 3.1. (i) $\{W_t, t \geq 0\}$, $\{N_t^{(1)}, t \geq 0\}$ and $\{N_t, t \geq 0\}$ are independent of each other.

(ii) $\delta > \beta c$. This condition is known as the stability condition.

(iii) $\frac{a\delta + \alpha\rho + b\beta}{\delta - \beta c} > r_0$.

Remark 3.2. Under the assumption of $\delta > \beta c$ and $2\delta a \geq \sigma^2$, there exist an unique probability measure for N_t and r_t (for instance see [3, 26] and the lecture notes by Hairer [14]).

The following lemma gives us the expression of the first moment of r_t and shows that $\mathbb{E}[r_\infty] < \infty$. These results are necessary in the proof of the main results of this paper.

Lemma 3.1. Consider the process $\{r_t, t \geq 0\}$ as defined in Eq. (3.1), for any $t \geq 0$,

$$\mathbb{E}[r_t] = \frac{\delta a + \alpha\rho + b\beta}{\delta - \beta c} + \left(r_0 - \frac{\delta a + \alpha\rho + b\beta}{\delta - \beta c}\right) e^{-(\delta - \beta c)t}. \quad (3.4)$$

In particular,

$$\mathbb{E}[r_\infty] = \frac{a\delta + \alpha\rho + b\beta}{\delta - \beta c} < \infty. \quad (3.5)$$

Proof. Using Dynkin's formula,

$$\mathbb{E}[f(r_t)] = f(r_0) + \mathbb{E}\left(\int_0^t \mathcal{A}f(r_s) ds\right). \quad (3.6)$$

In Eq. (3.3) consider, $f(r, n, t) = r$, we get $\mathcal{A}(r) = \delta(a - r) + \rho\alpha + (b + cr)\beta$. Substitute this in Eq. (3.6),

$$\mathbb{E}[r_t] = r_0 + \mathbb{E} \left(\int_0^t (\delta a + \rho\alpha + b\beta - (\delta - \beta c)r_s) ds \right). \quad (3.7)$$

Differentiating Eq. (3.7) with respect to t , we get

$$\frac{d}{dt} \mathbb{E}[r_t] = \delta a + \rho\alpha + b\beta - (\delta - \beta c) \mathbb{E}[r_t],$$

which implies that

$$\mathbb{E}[r_t] = \frac{\delta a + \alpha\rho + b\beta}{\delta - \beta c} (1 - e^{-(\delta - \beta c)t}) + r_0 e^{-(\delta - \beta c)t}.$$

By taking the limit $t \rightarrow \infty$, and by Assumption 3.1, we obtain Eq. (3.5). $\square$

Remark 3.3. From Assumption 3.1, i.e., $\frac{a\delta + \alpha\rho + b\beta}{\delta - \beta c} > r_0$, we get that $\mathbb{E}[r_t] \leq \mathbb{E}[r_\infty]$ for all $t \geq 0$.

4. Main Results

This section is devoted to the asymptotic study of CIR process with externally and self-exciting jumps. The main results are presented in Theorems 4.1–4.3.

Theorem 4.1 (Law of Large Numbers). *Under the Assumption 3.1, let $\{r_t, t \geq 0\}$ be the CIR process and $\{N_t, t \geq 0\}$ be the point process with intensity $\lambda_t = b + cr_t$ as defined in Eq. (3.1). Then*

- (i) $\frac{N_t}{t} \rightarrow \frac{b\delta + (a\delta + \alpha\rho)c}{\delta - \beta c}$ in probability as $t \rightarrow \infty$.
- (ii) $\frac{1}{t} \int_0^t r_s ds \rightarrow \frac{a\delta + \alpha\rho + b\beta}{\delta - \beta c}$ in probability as $t \rightarrow \infty$.

Theorem 4.2 (Central Limit Theorem). *Under the Assumption 3.1, let $\{r_t, t \geq 0\}$ be the CIR process and $\{N_t, t \geq 0\}$ be the point process with intensity $\lambda_t = b + cr_t$ as defined in Eq. (3.1). Then*

- (i) $\frac{N_t - (\frac{b\delta + (a\delta + \alpha\rho)c}{\delta - \beta c})t}{\sqrt{t}} \rightarrow N(0, \sigma_1^2)$ in distribution as $t \rightarrow \infty$, where

$$\sigma_1^2 = \frac{\delta^2(b\delta + (a\delta + \alpha\rho)c) + (c\alpha)^2\rho(\delta - \beta c) + (c\sigma)^2(a\delta + \alpha\rho + b\beta)}{(\delta - \beta c)^3}. \quad (4.1)$$
- (ii) $\frac{\int_0^t r_s ds - (\frac{a\delta + \alpha\rho + b\beta}{\delta - \beta c})t}{\sqrt{t}} \rightarrow N(0, \sigma_2^2)$ in distribution as $t \rightarrow \infty$, where

$$\sigma_2^2 = \frac{\alpha^2\rho(\delta - \beta c) + \beta^2(b\delta + (a\delta + \alpha\rho)c) + \sigma^2(a\delta + \alpha\rho + b\beta)}{(\delta - \beta c)^3}. \quad (4.2)$$

Remark 4.1. If in Eq. (3.1) $\alpha = 0$, then Theorems 4.1 and 4.2 correspond to the LLN and CLT of CIR process with Hawkes jumps with exponential exciting function as presented in [26].

Before moving on to the results related to the LDP for N_t and $\int_0^t r_s ds$, let us consider a sequence of probability measures $(P_n)_{n \in \mathbb{N}}$ defined on a topological space $\mathbf{X}$. This sequence satisfies the LDP with rate function $I : \mathbf{X} \rightarrow \mathbb{R}$, if I is non-negative and lower semi-continuous and for any measurable set B in $\mathbb{R}$, the following inequalities hold

$$-\inf_{x \in B^0} I(x) \leq \liminf_{n \rightarrow \infty} \frac{1}{n} \log P_n(B) \leq \limsup_{n \rightarrow \infty} \frac{1}{n} \log P_n(B) \leq -\inf_{x \in \bar{B}} I(x),$$

where B^0 represents the interior of B (also denoted as $\text{int}(B)$) and $\bar{B}$ represents its closure.

Theorem 4.3 (Large Deviation Principle). *Under Assumption 3.1,*

- (i) $P(\frac{N_t}{t} \in \cdot)$ satisfies the Large Deviation Principle with rate function $I^*(x)$ given by

$$I^*(x) = \sup_{\theta \leq \theta_m} \left\{ \theta x - \delta a y \left(\frac{\theta}{\beta} \right) - \rho(e^{\alpha y (\frac{\theta}{\beta})} - 1) - b(e^{\beta y (\frac{\theta}{\beta})} - 1) \right\} \quad (4.3)$$

where $y(\frac{\theta}{\beta})$ and θ_m , respectively, satisfy the equation

$$-\delta y \left(\frac{\theta}{\beta} \right) + \frac{1}{2} \sigma^2 y \left(\frac{\theta}{\beta} \right)^2 + c(e^{\beta y (\frac{\theta}{\beta})} - 1) + \frac{\theta \delta}{\beta} = 0, \quad (4.4)$$

and

$$\theta_m = \frac{\beta}{\delta} \left(\delta y_m - \frac{1}{2} \sigma^2 y_m^2 - c(e^{\beta y_m} - 1) \right), \quad (4.5)$$

where y_m is the unique solution of $\delta = \sigma^2 y_m + \beta c e^{\beta y_m}$.

- (ii) $P(\frac{1}{t} \int_0^t r_s ds \in \cdot)$ satisfies the Large Deviation Principle with rate function $I^*(x)$ given by

$$I^*(x) = \sup_{\theta \leq \theta_{\max}} \{ \theta x - a \delta y(\theta) - \rho(e^{\alpha y(\theta)} - 1) - b(e^{\beta y(\theta)} - 1) \}, \quad (4.6)$$

where $y(\theta)$ and $\theta_{\max}$ satisfy the equation

$$-\delta y(\theta) + \frac{1}{2} \sigma^2 y(\theta)^2 + c(e^{\beta y(\theta)} - 1) + \theta = 0, \quad (4.7)$$

and

$$\theta_{\max} = \delta y_{\max} - \frac{1}{2} \sigma^2 y_{\max}^2 - c(e^{\beta y_{\max}} - 1), \quad (4.8)$$

respectively, where $y_{\max}$ is the unique solution of the equation $\delta = \sigma^2 y_{\max} + \beta c e^{\beta y_{\max}}$.

5. Proof of the Main Results

In this section, we present detailed proof of the results.

5.1. Proof of Theorem 4.1

(i) By the definition given in Eq. (3.2), we have

$$\begin{aligned}
 \int_0^t r_s ds &= at + \frac{(r_0 - a)}{\delta}(1 - e^{-\delta t}) + \sigma \int_0^t \int_0^s e^{-\delta(s-\tau)} \sqrt{r_\tau} dW_\tau ds \\
 &\quad + \alpha \int_0^t \int_0^s e^{-\delta(s-\tau)} dN_\tau^{(1)} ds + \beta \int_0^t \int_0^s e^{-\delta(s-\tau)} dN_\tau ds \\
 &= at + \frac{(r_0 - a)}{\delta}(1 - e^{-\delta t}) + I_1(t) + I_2(t) + I_3(t). \tag{5.1}
 \end{aligned}$$

Now, simplify $I_1(t)$, $I_2(t)$ and $I_3(t)$ by the change of variables under integration, we get

$$\begin{aligned}
 I_1(t) &= \sigma \int_0^t \int_0^s e^{-\delta(s-\tau)} \sqrt{r_\tau} dW_\tau ds \\
 &= \sigma \int_0^t \int_\tau^t e^{-\delta(s-\tau)} \sqrt{r_\tau} ds dW_\tau \\
 &= \frac{\sigma}{\delta} \int_0^t \sqrt{r_\tau} (1 - e^{-\delta(t-\tau)}) dW_\tau, \tag{5.2}
 \end{aligned}$$

$$\begin{aligned}
 I_2(t) &= \frac{\alpha}{\delta} \int_0^t (1 - e^{-\delta(t-\tau)}) dN_\tau^{(1)} \\
 &= \frac{\alpha}{\delta} N_t^{(1)} - \frac{\alpha}{\delta} \int_0^t e^{-\delta(t-\tau)} dN_\tau^{(1)} \tag{5.3}
 \end{aligned}$$

and

$$I_3(t) = \frac{\beta}{\delta} N_t - \frac{\beta}{\delta} \int_0^t e^{-\delta(t-\tau)} dN_\tau. \tag{5.4}$$

respectively. Consequently, by substituting $I_1(t)$, $I_2(t)$ and $I_3(t)$ into Eq. (5.1), we deduce that

$$\begin{aligned}
 \int_0^t r_s ds &= at + \frac{(r_0 - a)}{\delta}(1 - e^{-\delta t}) + \frac{\sigma}{\delta} \int_0^t \sqrt{r_\tau} (1 - e^{-\delta(t-\tau)}) dW_\tau \\
 &\quad + \frac{\alpha}{\delta} N_t^{(1)} + \frac{\beta}{\delta} N_t - \epsilon_t, \tag{5.5}
 \end{aligned}$$

where

$$\epsilon_t = \frac{\alpha}{\delta} \int_0^t e^{-\delta(t-\tau)} dN_\tau^{(1)} + \frac{\beta}{\delta} \int_0^t e^{-\delta(t-\tau)} dN_\tau. \tag{5.6}$$

Taking into account that $\lambda_t = b + cr_t$, we get

$$\begin{aligned} \int_0^t \lambda_s ds &= (b + ac)t + \frac{c(r_0 - a)}{\delta}(1 - e^{-\delta t}) + \frac{c\sigma}{\delta} \int_0^t \sqrt{r_\tau}(1 - e^{-\delta(t-\tau)})dW_\tau \\ &\quad + \frac{c\alpha}{\delta} N_t^{(1)} + \frac{c\beta}{\delta} N_t - c\epsilon_t, \end{aligned}$$

and consequently,

$$\begin{aligned} N_t &= \frac{N_t - \int_0^t \lambda_s ds}{(1 - \frac{\beta c}{\delta})} + \frac{(b + ac)t}{(1 - \frac{\beta c}{\delta})} + \frac{c(r_0 - a)}{\delta(1 - \frac{\beta c}{\delta})}(1 - e^{-\delta t}) \\ &\quad + \frac{c\sigma}{\delta(1 - \frac{\beta c}{\delta})} \int_0^t \sqrt{r_\tau}(1 - e^{-\delta(t-\tau)})dW_\tau + \frac{c\alpha}{\delta(1 - \frac{\beta c}{\delta})} N_t^{(1)} \\ &\quad - \frac{c}{(1 - \frac{\beta c}{\delta})} \epsilon_t. \end{aligned} \tag{5.7}$$

Since $N_t - \int_0^t \lambda_s ds$ is a martingale, by Jensen's inequality, we get $(N_t - \int_0^t \lambda_s ds)^2$ is a sub-martingale, and by Doob's Decomposition, we have

$$\mathbb{E} \left(N_t - \int_0^t \lambda_s ds \right)^2 = \mathbb{E} \left(\int_0^t \lambda_s ds \right).$$

By using Lemma 3.1, i.e. $\mathbb{E}[r_\infty] < \infty$, we obtain

$$\begin{aligned} \mathbb{E} \left[\left(\frac{N_t - \int_0^t \lambda_s ds}{t} \right)^2 \right] &= \frac{1}{t^2} \mathbb{E} \left(\int_0^t \lambda_s ds \right) \\ &= \frac{1}{t^2} \int_0^t \mathbb{E}[\lambda_s] ds \leq \frac{1}{t} \mathbb{E}[\lambda_\infty] \\ &= \frac{1}{t} (b + c\mathbb{E}[r_\infty]) \rightarrow 0 \quad \text{as } t \rightarrow \infty. \end{aligned} \tag{5.8}$$

Thus, by Markov's inequality, we conclude that

$$\frac{N_t - \int_0^t \lambda_s ds}{t} \rightarrow 0 \quad \text{in probability as } t \rightarrow \infty. \tag{5.9}$$

Now let's consider the error terms ϵ_t , as defined in Eq. (5.6), and we get that

$$\begin{aligned} \mathbb{E}[\epsilon_t] &= \frac{\alpha}{\delta} \int_0^t e^{-\delta(t-\tau)} \rho d\tau + \frac{\beta}{\delta} \int_0^t e^{-\delta(t-\tau)} \mathbb{E}[\lambda_\tau] d\tau \\ &\leq \frac{\alpha}{\delta} \int_0^t e^{-\delta(t-\tau)} \rho d\tau + \frac{\beta}{\delta} (b + c\mathbb{E}[r_\infty]) \int_0^t e^{-\delta(t-\tau)} d\tau \\ &= \left(\frac{\alpha\rho}{\delta^2} + \frac{\beta}{\delta^2} (b + c\mathbb{E}[r_\infty]) \right) (1 - e^{-\delta t}). \end{aligned} \tag{5.10}$$

Using Lemma 3.1, i.e. $\mathbb{E}[r_\infty] < \infty$, we deduce that $\frac{1}{t}\mathbb{E}[\epsilon_t] \rightarrow 0$ as $t \rightarrow \infty$, this in turn implies that

$$\frac{\epsilon_t}{t} \rightarrow 0 \quad \text{in probability as } t \rightarrow \infty. \quad (5.11)$$

In order to use the zero mean property of Itô integral, i.e.,

$$\mathbb{E} \left(\frac{c\sigma}{\delta} \int_0^t \sqrt{r_\tau} (1 - e^{-\delta(t-\tau)}) dW_\tau \right) = 0, \quad (5.12)$$

we need to show that

$$\mathbb{E} \left(\frac{c^2\sigma^2}{\delta^2} \int_0^t r_\tau (1 - e^{-\delta(t-\tau)})^2 d\tau \right) < \infty. \quad (5.13)$$

To this end, note that

$$\begin{aligned} \mathbb{E} \left(\frac{c^2\sigma^2}{\delta^2} \int_0^t r_\tau (1 - e^{-\delta(t-\tau)})^2 d\tau \right) &= \frac{c^2\sigma^2}{\delta^2} \int_0^t \left(1 - e^{-\delta(t-\tau)} \right)^2 \mathbb{E}[r_\tau] d\tau \\ &\leq \frac{c^2\sigma^2}{\delta^2} \mathbb{E}[r_\infty] \int_0^t (1 - e^{-\delta(t-\tau)})^2 d\tau \\ &< \infty \end{aligned}$$

as $\mathbb{E}[r_\infty] < \infty$. This implies that the integral in Eq. (5.13) is finite and Eq. (5.12) holds, therefore

$$\frac{1}{t} \left(\frac{c\sigma}{\delta} \int_0^t \sqrt{r_\tau} (1 - e^{-\delta(t-\tau)}) dW_\tau \right) \rightarrow 0 \quad \text{in probability as } t \rightarrow \infty. \quad (5.14)$$

Recalling that $N_t^{(1)}$ is a standard Poisson process, we have that

$$\frac{N_t^{(1)}}{t} \rightarrow \rho \quad \text{a.s. as } t \rightarrow \infty. \quad (5.15)$$

Substituting all the results obtained in Eqs. (5.9), (5.11), (5.14) and (5.15) into Eq. (5.7), we get

$$\frac{N_t}{t} \rightarrow \frac{b\delta + (a\delta + \alpha\rho)c}{\delta - \beta c} \quad \text{in probability as } t \rightarrow \infty. \quad (5.16)$$

(ii) Again, by substituting all the results obtained so far in Eqs. (5.9), (5.11), (5.14), (5.15) and (5.16) into Eq. (5.5), we get

$$\frac{1}{t} \int_0^t r_s ds \rightarrow \frac{a\delta + \alpha\rho + b\beta}{\delta - \beta c} \quad \text{in probability as } t \rightarrow \infty. \quad (5.17)$$

□

Remark 5.1. The convergence in Theorem 4.1 also holds almost surely, which can be established by using the existence of a unique invariant probability measure (as discussed in Remark 3.2) and the ergodic property. Consequently, using the ergodic

theorem and Eq. (3.5), we obtain

$$\frac{1}{t} \int_0^t r_s ds \rightarrow \mathbb{E}[r_\infty] = \frac{a\delta + \alpha\rho + b\beta}{\delta - \beta c} \quad \text{a.s. as } t \rightarrow \infty.$$

Note that, from Eq. (5.9)

$$\frac{N_t}{t} \rightarrow \frac{1}{t} \int_0^t \lambda_s ds \rightarrow \mathbb{E}[\lambda_\infty] = b + c\mathbb{E}[r_\infty] = \frac{b\delta + (a\delta + \alpha\rho)c}{\delta - \beta c} \quad \text{a.s. as } t \rightarrow \infty.$$

5.2. Proof of Theorem 4.2

(i) By Eq. (5.7), we deduce that

$$\begin{aligned} N_t - \left(\frac{b\delta + (a\delta + \alpha\rho)c}{\delta - \beta c} \right) t &= \frac{N_t - \int_0^t \lambda_s ds}{(1 - \frac{\beta c}{\delta})} + \frac{c\alpha}{\delta(1 - \frac{\beta c}{\delta})} (N_t^{(1)} - \rho t) \\ &\quad + \frac{c\sigma}{\delta(1 - \frac{\beta c}{\delta})} \int_0^t \sqrt{r_\tau} (1 - e^{-\delta(t-\tau)}) dW_\tau \\ &\quad + \frac{c(r_0 - a)}{\delta(1 - \frac{\beta c}{\delta})} (1 - e^{-\delta t}) - \frac{c}{(1 - \frac{\beta c}{\delta})} \epsilon_t. \end{aligned} \quad (5.18)$$

By following the same approach as demonstrated in Theorem 4.1, we can derive the result that

$$\frac{1}{\sqrt{t}} \left(\frac{c(r_0 - a)}{\delta(1 - \frac{\beta c}{\delta})} (1 - e^{-\delta t}) - \frac{c}{(1 - \frac{\beta c}{\delta})} \epsilon_t \right) \rightarrow 0 \quad \text{in probability as } t \rightarrow \infty. \quad (5.19)$$

Since

$$\frac{N_t - \int_0^t \lambda_s ds}{(1 - \frac{\beta c}{\delta})} + \frac{c\alpha}{\delta(1 - \frac{\beta c}{\delta})} (N_t^{(1)} - \rho t) + \frac{c\sigma}{\delta(1 - \frac{\beta c}{\delta})} \int_0^t \sqrt{r_\tau} (1 - e^{-\delta(t-\tau)}) dW_\tau$$

is a martingale and its quadratic variation is given by

$$\frac{\delta^2 N_t}{(\delta - \beta c)^2} + \frac{(c\alpha)^2 N_t^{(1)}}{(\delta - \beta c)^2} + \frac{(c\sigma)^2}{(\delta - \beta c)^2} \int_0^t r_\tau (1 - e^{-\delta(t-\tau)})^2 d\tau. \quad (5.20)$$

Note that,

$$\begin{aligned} \frac{1}{t} \mathbb{E} \left[\int_0^t r_\tau |e^{-2\delta(t-\tau)} - 2e^{-\delta(t-\tau)}| d\tau \right] &= \frac{1}{t} \int_0^t (2e^{-\delta(t-\tau)} - e^{-2\delta(t-\tau)}) r_\tau d\tau \\ &\leq \frac{1}{t} \int_0^t (2e^{-\delta(t-\tau)} - e^{-2\delta(t-\tau)}) d\tau \cdot \mathbb{E}[r_\infty] \\ &\rightarrow 0 \quad \text{as } t \rightarrow \infty, \end{aligned} \quad (5.21)$$

which implies

$$\frac{1}{t} \int_0^t r_\tau |e^{-2\delta(t-\tau)} - 2e^{-\delta(t-\tau)}| d\tau \rightarrow 0 \quad \text{in probability as } t \rightarrow \infty.$$

By the LLN for N_t and $\int_0^t r_s ds$ as derived in Eqs. (5.16) and (5.17) of Theorem 4.1, we conclude that

$$\begin{aligned} & \frac{1}{t} \frac{\delta^2 N_t}{(\delta - \beta c)^2} + \frac{1}{t} \frac{(c\alpha)^2 N_t^{(1)}}{(\delta - \beta c)^2} + \frac{1}{t} \frac{(c\sigma)^2}{(\delta - \beta c)^2} \int_0^t r_\tau (1 - e^{-\delta(t-\tau)})^2 d\tau \\ & \rightarrow \frac{\delta^2(b\delta + (a\delta + \alpha\rho)c)}{(\delta - \beta c)^3} + \frac{(c\alpha)^2 \rho}{(\delta - \beta c)^2} + \frac{(c\sigma)^2(a\delta + \alpha\rho + b\beta)}{(\delta - \beta c)^3} \end{aligned} \quad (5.22)$$

in probability as $t \rightarrow \infty$. Thus, by the Martingale Central Limit Theorem, we obtain

$$\frac{N_t - \left(\frac{b\delta + (a\delta + \alpha\rho)c}{\delta - \beta c} \right) t}{\sqrt{t}} \rightarrow N(0, \sigma_1^2) \quad (5.23)$$

in distribution as $t \rightarrow \infty$, where σ_1^2 is defined in Eq. (4.1).

(ii) By Eq. (5.5), we have

$$\begin{aligned} \int_0^t r_s ds &= at + \frac{(r_0 - a)}{\delta} (1 - e^{-\delta t}) + \frac{\sigma}{\delta} \int_0^t \sqrt{r_\tau} (1 - e^{-\delta(t-\tau)}) dW_\tau \\ &+ \frac{\alpha}{\delta} N_t^{(1)} + \frac{\beta}{\delta} N_t - \epsilon_t \\ &= \frac{\alpha}{\delta(1 - \frac{\beta c}{\delta})} (N_t^{(1)} - \rho t) + \frac{\beta}{\delta(1 - \frac{\beta c}{\delta})} (N_t - \int_0^t \lambda_s ds) \\ &+ \frac{\sigma}{\delta(1 - \frac{\beta c}{\delta})} \int_0^t \sqrt{r_\tau} (1 - e^{-\delta(t-\tau)}) dW_\tau + \frac{at}{(1 - \frac{\beta c}{\delta})} \\ &+ \frac{(r_0 - a)}{\delta(1 - \frac{\beta c}{\delta})} (1 - e^{-\delta t}) + \frac{\alpha \rho t}{\delta(1 - \frac{\beta c}{\delta})} + \frac{\beta b t}{\delta(1 - \frac{\beta c}{\delta})} - \frac{\epsilon_t}{(1 - \frac{\beta c}{\delta})}. \end{aligned}$$

Now consider,

$$\begin{aligned} & \int_0^t r_s ds - \left(\frac{a\delta + \alpha\rho + b\beta}{\delta - \beta c} \right) t \\ &= \frac{\alpha}{\delta(1 - \frac{\beta c}{\delta})} (N_t^{(1)} - \rho t) + \frac{\beta}{\delta(1 - \frac{\beta c}{\delta})} (N_t - \int_0^t \lambda_s ds) - \frac{\epsilon_t}{(1 - \frac{\beta c}{\delta})} \\ &+ \frac{\sigma}{\delta(1 - \frac{\beta c}{\delta})} \int_0^t \sqrt{r_\tau} (1 - e^{-\delta(t-\tau)}) dW_\tau + \frac{(r_0 - a)}{\delta(1 - \frac{\beta c}{\delta})} (1 - e^{-\delta t}). \end{aligned}$$

From Eq. (5.19), we have that

$$\frac{1}{\sqrt{t}} \left(\frac{c(r_0 - a)}{\delta(1 - \frac{\beta c}{\delta})} (1 - e^{-\delta t}) - \frac{c}{(1 - \frac{\beta c}{\delta})} \epsilon_t \right) \rightarrow 0 \quad \text{in probability as } t \rightarrow \infty.$$

Consequently,

$$\begin{aligned} & \frac{\alpha}{\delta(1 - \frac{\beta c}{\delta})} (N_t^{(1)} - \rho t) + \frac{\beta}{\delta(1 - \frac{\beta c}{\delta})} \left(N_t - \int_0^t \lambda_s ds \right) \\ & + \frac{\sigma}{\delta(1 - \frac{\beta c}{\delta})} \int_0^t \sqrt{r_\tau} (1 - e^{-\delta(t-\tau)}) dW_\tau \end{aligned} \quad (5.24)$$

is a martingale and its quadratic variation is given by

$$\frac{\alpha^2 N_t^{(1)}}{(\delta - \beta c)^2} + \frac{\beta^2 N_t}{(\delta - \beta c)^2} + \frac{\sigma^2}{(\delta - \beta c)^2} \int_0^t r_\tau (1 - e^{-\delta(t-\tau)})^2 d\tau. \quad (5.25)$$

From Eq. (5.21), we know that

$$\frac{1}{t} \int_0^t r_\tau |e^{-2\delta(t-\tau)} - 2e^{-\delta(t-\tau)}| d\tau \rightarrow 0 \quad \text{in probability as } t \rightarrow \infty.$$

Similarly, by the results obtain in Eqs. (5.16) and (5.17), we obtain

$$\begin{aligned} & \frac{1}{t} \frac{\alpha^2 N_t^{(1)}}{(\delta - \beta c)^2} + \frac{1}{t} \frac{\beta^2 N_t}{(\delta - \beta c)^2} + \frac{1}{t} \frac{\sigma^2}{(\delta - \beta c)^2} \int_0^t r_\tau (1 - e^{-\delta(t-\tau)})^2 d\tau \\ & \rightarrow \frac{\alpha^2 \rho}{(\delta - \beta c)^2} + \frac{\beta^2 (b\delta + (a\delta + \alpha\rho)c)}{(\delta - \beta c)^3} + \frac{\sigma^2 (a\delta + \alpha\rho + b\beta)}{(\delta - \beta c)^3} \end{aligned} \quad (5.26)$$

in probability as $t \rightarrow \infty$. Thus, by the Martingale Central Limit Theorem, we deduce that

$$\frac{\int_0^t r_s ds - (\frac{a\delta + \alpha\rho + b\beta}{\delta - \beta c})t}{\sqrt{t}} \rightarrow N(0, \sigma_2^2) \quad (5.27)$$

in distribution as $t \rightarrow \infty$, where σ_2^2 is defined in Eq. (4.2). $\square$

5.3. Proof of Theorem 4.3

(i) In order to establish the Large Deviation, let us first find the expression for the moment generating function of N_t . From Eq. (3.1), we have that

$$N_t = \frac{1}{\beta} \left(r_t - r_0 - \delta \int_0^t (a - r_s) ds - \sigma \int_0^t \sqrt{r_s} dW_s - \alpha N_t^{(1)} \right). \quad (5.28)$$

Now by Eq. (5.28), we can determine that $\mathbb{E}[e^{\theta N_t} | r_0 = r] = e^{-\frac{\theta}{\beta} r} u(\theta, t, r)$ where

$$u(\theta, t, r) = \mathbb{E}[e^{\frac{\theta}{\beta} r t - \frac{\theta \delta}{\beta} \int_0^t (a - r_s) ds - \frac{\theta \sigma}{\beta} \int_0^t \sqrt{r_s} dW_s - \frac{\theta \alpha}{\beta} N_t^{(1)}} | r_0 = r].$$

Using the Feynman–Kac formula, u satisfies the partial differential equation given by

$$\begin{aligned} \frac{\partial u}{\partial t} = & \delta a \frac{\partial u}{\partial r} - \delta r \frac{\partial u}{\partial r} + \frac{1}{2} \sigma^2 r \frac{\partial^2 u}{\partial r^2} + \rho[u(\theta, t, r + \alpha) - u(\theta, t, r)] \\ & + (b + cr)[u(\theta, t, r + \beta) - u(\theta, t, r)] + \frac{\theta \delta}{\beta} ru, \end{aligned}$$

with initial condition given as $u(\theta, 0, r) = e^{\frac{\theta}{\beta}r}$. Taking into account the affine structure (see for instance [12]), we know that $u(\theta, t, r) = e^{A(t; \frac{\theta}{\beta})r + B(t; \frac{\theta}{\beta})}$, where $A(t; \frac{\theta}{\beta})$ and $B(t; \frac{\theta}{\beta})$ satisfy the following ordinary differential equation

$$\begin{aligned} A' \left(t; \frac{\theta}{\beta} \right) &= -\delta A \left(t; \frac{\theta}{\beta} \right) + \frac{1}{2} \sigma^2 A^2 \left(t; \frac{\theta}{\beta} \right) + c(e^{\beta A(t; \frac{\theta}{\beta})} - 1) + \frac{\theta \delta}{\beta}, \\ B' \left(t; \frac{\theta}{\beta} \right) &= \delta a A \left(t; \frac{\theta}{\beta} \right) + \rho(e^{\alpha A(t; \frac{\theta}{\beta})} - 1) + b(e^{\beta A(t; \frac{\theta}{\beta})} - 1), \\ A \left(0; \frac{\theta}{\beta} \right) &= \frac{\theta}{\beta}, \quad B \left(0; \frac{\theta}{\beta} \right) = 0. \end{aligned}$$

Note that $\lim_{t \rightarrow \infty} A(t; \frac{\theta}{\beta}) = y(\frac{\theta}{\beta})$, where $y(\cdot)$ satisfies the equation given by

$$-\delta y + \frac{1}{2} \sigma^2 y^2 + c(e^{\beta y} - 1) + \frac{\theta \delta}{\beta} = 0. \quad (5.29)$$

If Eq. (5.29) admits a solution then the limit exist, otherwise, $\lim_{t \rightarrow \infty} A(t; \frac{\theta}{\beta}) = \infty$. We claim that $y(\frac{\theta}{\beta})$ is the solution of Eq. (5.29) whenever $\theta < \theta_m$, where

$$\begin{aligned} \theta_m &= \frac{\beta}{\delta} \left(\sup_{y \in \mathbb{R}^+} \left\{ \delta y - \frac{1}{2} \sigma^2 y^2 - c(e^{\beta y} - 1) \right\} \right) \\ &= \frac{\beta}{\delta} \left(\delta y_m - \frac{1}{2} \sigma^2 y_m^2 - c(e^{\beta y_m} - 1) \right) \end{aligned}$$

and y_m is the solution of $\delta = \sigma^2 y_m + \beta c e^{\beta y_m}$. Thus, we get that for $\theta \leq \theta_m$, $\lim_{t \rightarrow \infty} A(t; \frac{\theta}{\beta}) = y(\frac{\theta}{\beta})$ which further implies that for $\theta \leq \theta_m$

$$\lim_{t \rightarrow \infty} B' \left(t; \frac{\theta}{\beta} \right) = \delta a y \left(\frac{\theta}{\beta} \right) + \rho(e^{\alpha y(\frac{\theta}{\beta})} - 1) + b(e^{\beta y(\frac{\theta}{\beta})} - 1).$$

Consequently, for $\theta \leq \theta_m$ by L'Hôpital's rule,

$$\begin{aligned} I(\theta) &= \lim_{t \rightarrow \infty} \frac{1}{t} \log \mathbb{E}[e^{\theta N_t} \mid r_0 = r] \\ &= \delta a y \left(\frac{\theta}{\beta} \right) + \rho(e^{\alpha y(\frac{\theta}{\beta})} - 1) + b(e^{\beta y(\frac{\theta}{\beta})} - 1). \end{aligned} \quad (5.30)$$

Now to obtain the LDP we apply Gärtner–Ellis Theorem (see [10, Theorem 2.3.6]) and to this end, we have to check the, so-called, essential smoothness condition. Define, $D_I = \{\theta : I(\theta) < \infty\}$ and recall that for $\theta \leq \theta_m$, $I(\theta) < \infty$ which implies that the interior of D_I contains a nonempty neighborhood of the origin, i.e. $\text{int}(D_I) \neq \emptyset$. Also, from Eq. (5.29) we have that $y(\cdot)$ is differentiable at θ for $\theta < \theta_m$, which implies that $I(\theta)$ is differentiable for $\theta < \theta_m$. Now differentiate Eq. (5.29) with respect to θ , we get

$$\frac{\partial y(\frac{\theta}{\beta})}{\partial \theta} = \frac{\delta}{\delta - \sigma^2 y(\frac{\theta}{\beta}) - \beta c e^{\beta y(\frac{\theta}{\beta})}}.$$

Since, $y \uparrow y_m$ as $\theta \uparrow \theta_m$, where y_m satisfies the equation $\delta = \sigma^2 y_m + \beta c e^{\beta y_m}$, therefore, for $\theta \uparrow \theta_m$, we get

$$\frac{\partial y(\frac{\theta}{\beta})}{\partial \theta} \rightarrow +\infty. \quad (5.31)$$

Again by differentiating both the sides of Eq. (5.30) with respect to θ , we obtain

$$\frac{\partial I(\theta)}{\partial \theta} = \frac{1}{\beta} (\delta a + \rho \alpha e^{\alpha y(\frac{\theta}{\beta})} + b \beta e^{\beta y(\frac{\theta}{\beta})}) \frac{\partial y(\frac{\theta}{\beta})}{\partial \theta}.$$

Using Eq. (5.31), we get $\frac{\partial I(\theta)}{\partial \theta} \rightarrow +\infty$ as $\theta \uparrow \theta_m$. Since we have proved that I satisfies the essential smoothness condition, thus by Gärtner–Ellis Theorem, $P(\frac{N_t}{t} \in \cdot)$ satisfies the LDP with rate function $I^*(x)$ as defined in Eq. (4.3).

(ii) Using the same approach as in the proof of part (i) of Theorem 4.3, let us first find the expression of the moment generating function of $\int_0^t r_s ds$. Let $u(\theta, t, r) = \mathbb{E}[e^{\theta \int_0^t r_s ds} | r_0 = r]$, then by the Feynman–Kac formula, u satisfies the partial differential equation

$$\begin{aligned} \frac{\partial u}{\partial t} &= \delta a \frac{\partial u}{\partial r} - \delta r \frac{\partial u}{\partial r} + \frac{1}{2} \sigma^2 r \frac{\partial^2 u}{\partial r^2} + \rho [u(\theta, t, r + \alpha) - u(\theta, t, r)] \\ &\quad + (b + cr)[u(\theta, t, r + \beta) - u(\theta, t, r)] + \theta r u, \end{aligned}$$

with the initial condition given by $u(\theta, 0, r) = 1$. Now by affine structure (see for instance [12]) we have $u(\theta, t, r) = e^{A(t; \theta)r + B(t; \theta)}$, where $A(t; \theta)$ and $B(t; \theta)$ satisfy the ordinary differential equation given by

$$\begin{aligned} A'(t; \theta) &= -\delta A(t; \theta) + \frac{1}{2} \sigma^2 A^2(t; \theta) + c(e^{\beta A(t; \theta)} - 1) + \theta, \\ B'(t; \theta) &= \delta a A(t; \theta) + \rho(e^{\alpha A(t; \theta)} - 1) + b(e^{\beta A(t; \theta)} - 1), \\ A(0; \theta) &= B(0; \theta) = 0. \end{aligned}$$

We conclude that $\lim_{t \rightarrow \infty} A(t; \theta) = y(\theta)$, where $y(\theta)$ satisfies the equation

$$-\delta y(\theta) + \frac{1}{2} \sigma^2 y^2(\theta) + c(e^{\beta y(\theta)} - 1) + \theta = 0, \quad (5.32)$$

otherwise, $\lim_{t \rightarrow \infty} A(t; \theta) = \infty$. Further, we claim that $y(\theta)$ is the solution of Eq. (5.32) for $\theta \leq \theta_{\max}$, where $\theta_{\max}$ is defined as

$$\begin{aligned} \theta_{\max} &= \sup_{y \in \mathbb{R}^+} \left\{ \delta y - \frac{1}{2} \sigma^2 y^2 - c(e^{\beta y} - 1) \right\} \\ &= \delta y_{\max} - \frac{1}{2} \sigma^2 y_{\max}^2 - c(e^{\beta y_{\max}} - 1) \end{aligned} \quad (5.33)$$

and $y_{\max}$ is the unique solution of the equation $\delta = \sigma^2 y_{\max} + \beta c e^{\beta y_{\max}}$. Since, $\delta > \beta c$, this implies that Eq. (5.33) admits an unique solution. Consequently, we obtain that Eq. (5.32) has a solution for $\theta \leq \theta_{\max}$, implying $\lim_{t \rightarrow \infty} A(t; \theta) = y(\theta)$ for $\theta \leq \theta_{\max}$. Furthermore, we have that

$$B'(t; \theta) = \delta a A(t; \theta) + \rho(e^{\alpha A(t; \theta)} - 1) + b(e^{\beta A(t; \theta)} - 1) \quad (5.34)$$

which implies $\lim_{t \rightarrow \infty} B'(t; \theta) = \delta a y(\theta) + \rho(e^{\alpha y(\theta)} - 1) + b(e^{\beta y(\theta)} - 1)$ for $\theta \leq \theta_{\max}$. Thus, by employing L'Hôpital's rule, for $\theta \leq \theta_{\max}$, we obtain

$$I(\theta) = \lim_{t \rightarrow \infty} \frac{1}{t} \log u(\theta, t, r) = \delta a y(\theta) + \rho(e^{\alpha y(\theta)} - 1) + b(e^{\beta y(\theta)} - 1). \quad (5.35)$$

To apply the Gärtner–Ellis Theorem, let us check the essential smoothness condition. Define $D_I = \{\theta : I(\theta) < \infty\}$ since, for $\theta \leq \theta_{\max}$, $I(\theta) < \infty$ which implies $\text{int}(D_I) \neq \emptyset$. From Eq. (5.32), $y(\theta)$ is differentiable at θ for $\theta < \theta_{\max}$ which implies $I(\theta)$ is also differentiable at θ for $\theta < \theta_{\max}$. Differentiate Eq. (5.32) both the sides with respect to θ , we get

$$\frac{\partial y(\theta)}{\partial \theta} = \frac{1}{\delta - \sigma^2 y(\theta) - \beta c e^{\beta y(\theta)}}$$

since, we know that for $\theta \uparrow \theta_{\max}$ we have $y \uparrow y_{\max}$ where $y_{\max}$ satisfies the equation $\delta = \sigma^2 y_{\max} + \beta c e^{\beta y_{\max}}$ which implies

$$\frac{\partial y(\theta)}{\partial \theta} \rightarrow +\infty \quad \text{as } \theta \uparrow \theta_{\max}. \quad (5.36)$$

Now differentiate Eq. (5.35), both the sides with respect to θ , we obtain

$$\frac{\partial I(\theta)}{\partial \theta} = (\delta a + \rho \alpha e^{\alpha y(\theta)} + b \beta e^{\beta y(\theta)}) \frac{\partial y(\theta)}{\partial \theta}.$$

Using Eq. (5.36), we obtain $\frac{\partial I(\theta)}{\partial \theta} \rightarrow +\infty$ as $\theta \uparrow \theta_{\max}$. Hence, I satisfies the essential smoothness condition, and by Gärtner–Ellis Theorem, $P(\frac{1}{t} \int_0^t r_s ds \in \cdot)$ satisfies a LDP with rate function $I^*(x)$ as defined in Eq. (4.6). $\square$

6. Application to Finance

We consider a generalized intensity-based model for modeling default probability. This model takes into account many adverse events that are relevant to the underlying company, as well as the external circumstances which lead to the occurrence of default or bankruptcy. Suppose that the occurrences of such severe events are

described by the point process $\{N_t, t \geq 0\}$ with intensity $\lambda_t = r_t$. Here, r_t represents the CIR process as stated in Eq. (3.1). Certainly, such intensity-based model effectively captures the internal risk factors of the company, as well as the external risk factors and market noise that consistently impact it.

Before we continue, let's first examine the probability generating function of the dynamic contagion process $\{N_t, t \geq 0\}$ with CIR intensity (or with diffusion) $\lambda_t = r_t$ where r_t as specified in Eq. (3.1). Although, the corresponding proof has been provided by Dassios and Zhao [8].

Theorem 6.1 (Probability Generating Function). *For $\delta > \beta$ and $\theta \in [\theta, 1)$, the probability generating function of N_T condition on λ_0 is given by*

$$\mathbb{E}[\theta^{N_T} | \lambda_0] = \exp \left(- \int_0^{\mathcal{G}_{0,\theta}^{-1}(T)} \frac{a\delta u + \rho(1 - e^{-\alpha u})}{1 - \delta u - \theta e^{-\beta u} - \frac{1}{2}\sigma^2 u^2} du \right) * e^{-\mathcal{G}_{0,\theta}^{-1}(T)\lambda_0}, \quad (6.1)$$

where

$$\mathcal{G}_{0,\theta}(L) = \int_0^L \frac{du}{1 - \delta u - \theta e^{-\beta u} - \frac{1}{2}\sigma^2 u^2}. \quad (6.2)$$

Remark 6.1. (i) For $\{N_t, t \geq 0\}$ to be dynamic contagion process without diffusion, take $\lambda_0 = a$ and $\sigma = 0$ in Eqs. (6.1) and (6.2) thus, we obtain probability generating function of N_T condition on λ_0 as

$$\mathbb{E}[\theta^{N_T} | \lambda_0] = \exp \left(- \int_0^{\mathcal{G}_{0,\theta}^{-1}(T)} \frac{\lambda_0 \delta u + \rho(1 - e^{-\alpha u})}{1 - \delta u - \theta e^{-\beta u}} du \right) * e^{-\mathcal{G}_{0,\theta}^{-1}(T)\lambda_0}, \quad (6.3)$$

where

$$\mathcal{G}_{0,\theta}(L) = \int_0^L \frac{du}{1 - \delta u - \theta e^{-\beta u}}.$$

(ii) For $\{N_t, t \geq 0\}$ to be Hawkes process, take $\lambda_0 = a$, $\sigma = 0$, $\alpha = 0$, $\rho = 0$ thus, we obtain

$$\mathbb{E}[\theta^{N_T} | \lambda_0] = \exp \left(- \int_0^{\mathcal{G}_{0,\theta}^{-1}(T)} \frac{\lambda_0 \delta u}{1 - \delta u - \theta e^{-\beta u}} du \right) * e^{-\mathcal{G}_{0,\theta}^{-1}(T)\lambda_0}, \quad (6.4)$$

where

$$\mathcal{G}_{0,\theta}(L) = \int_0^L \frac{du}{1 - \delta u - \theta e^{-\beta u}}.$$

We assume that each bad event or each jump in the point process $\{N_t, t \geq 0\}$ result in default with a constant probability d , where $0 < d \leq 1$. Therefore, survival probability conditional on λ_0 is given by

$$P_S(T) = \mathbb{E}[(1 - d)^{N_T} | \lambda_0]. \quad (6.5)$$

By considering $\theta = 1 - d$ in Eqs. (6.1), (6.3), (6.4), we present a comparison graph illustrating the survival probabilities $P_S(t)$ of three processes: dynamic contagion

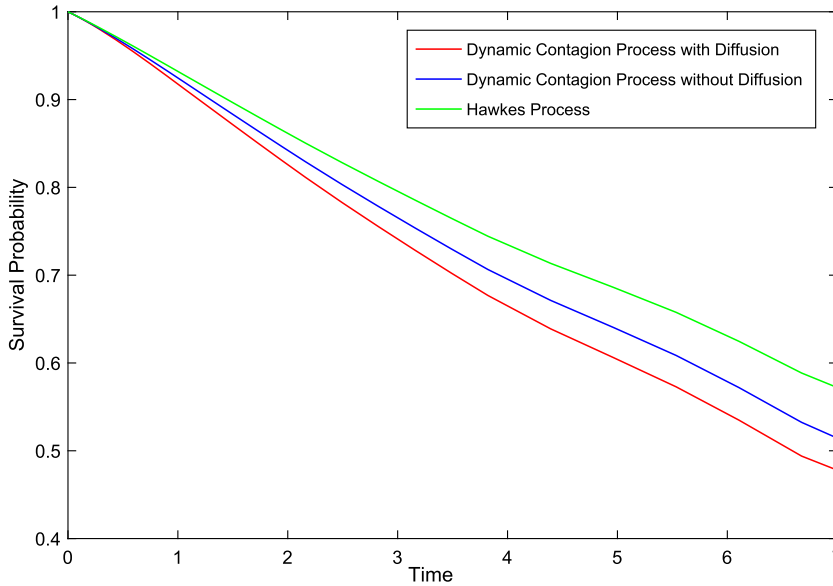


Fig. 1. Comparison of the survival probability for the dynamic contagion process with diffusion, dynamic contagion process without diffusion and Hawkes process with parameters: $d = 10\%$ and $(a; \rho; \delta; \alpha; \beta; \sigma; \lambda_0) = (0.7; 0.5; 2.0; 0.5; 0.6; 0.5; 0.6)$.

process with diffusion, dynamic contagion process without diffusion, and Hawkes process under a fixed parameter setup.

The given parameters involved can be estimated from real data. However, in this paper, we have chosen to use fixed parameter values for the sake of graphical illustration. These values are based on the assumptions stated in Assumptions 3.1. The parameter values are as follows: $d = 10\%$ and $(a; \rho; \delta; \alpha; \beta; \sigma; \lambda_0) = (0.7; 0.5; 2.0; 0.5; 0.6; 0.5; 0.6)$. The corresponding results are shown in Fig. 1.

Remark 6.2. In the comparison graph, certain parameter such as $d = 10\%$, remain same for all three processes. However, other parameters are as follows:

- For dynamic contagion process with diffusion, parameters are: $(a; \rho; \delta; \alpha; \beta; \sigma; \lambda_0) = (0.7; 0.5; 2.0; 0.5; 0.6; 0.5; 0.6)$.
- For dynamic contagion process without diffusion, we have $\lambda_0 = a$ and $\sigma = 0$. Thus, parameters are: $(a; \rho; \delta; \alpha; \beta; \sigma; \lambda_0) = (0.6; 0.5; 2.0; 0.5; 0.6; 0; 0.6)$.
- For Hawkes process, we have $\lambda_0 = a$, $\sigma = 0$, $\alpha = 0$, $\rho = 0$. Thus, parameters are: $(a; \rho; \delta; \alpha; \beta; \sigma; \lambda_0) = (0.6; 0; 2.0; 0; 0.6; 0; 0.6)$.

Remark 6.3. Note that, the dynamic contagion process with diffusion (or CIR intensity) yields the lowest survival probability compared to the other two processes. This is because it accounts for both endogenous and exogenous factors, as well as random fluctuations of the market, or market noises. Therefore, this process has the

capacity to capture a broader range of risk factors, which is particularly useful for modeling risks during financial crises, as it incorporates clusters of adverse events.

In this context, the loss process, which represents the total financial loss caused by the defaults in a portfolio, plays a critical role in risk analysis. Remark 6.3 highlights that the underlying process encompasses a broader range of risk factors. Therefore, we focus on a portfolio of investments where losses from defaults or adverse events are modeled using the point process $\{N_t, t \geq 0\}$ with intensity $\lambda_t = b + cr_t$ where r_t is defined in Eq. (3.2).

Consider the loss process $\{L_t, t \geq 0\}$ which is defined as

$$L_t = \sum_{i=1}^{N_t} \mathbf{L}_i, \quad (6.6)$$

where $\{\mathbf{L}_i\}_{i \geq 1}$ is the individual loss, which is an i.i.d. sequence of positive random variables and are assumed to be independent of $\{N_t, t \geq 0\}$ and $\{N_t^{(1)}, t \geq 0\}$.

Understanding the dynamics of this loss process is essential for assessing financial stability and informing risk management strategies. Consequently, we study the limit theorems of the loss process, which helps to determine the valuable insights into its behavior over time. The LLN helps to determine the long-term expected losses, which are crucial for setting pricing policies and maintaining stability. The CLT enables the approximation of aggregate losses by a normal distribution when the number of claims is large, which helps us to estimate the probability of exceeding critical loss thresholds. Additionally, the LDP provides tools to quantify the probability of extreme losses that significantly deviate from the mean, offering a framework for understanding rare but severe financial risks.

Theorem 6.2 (LLN of L_t). *Under the Assumption 3.1, let $\{L_t, t \geq 0\}$ be the Loss process as defined in Eq. (6.6). Then*

$$\frac{L_t}{t} \rightarrow \left(\frac{b\delta + (a\delta + \alpha\rho)c}{\delta - \beta c} \right) \mathbb{E}[\mathbf{L}_1] \quad \text{in probability as } t \rightarrow \infty. \quad (6.7)$$

Proof. Proof is the direct consequence of LLN of N_t as defined in Theorem 4.1. $\square$

Theorem 6.3 (CLT of L_t). *Under Assumption 3.1, let $\{L_t, t \geq 0\}$ be the Loss process as defined in Eq. (6.6). Then*

$$\frac{L_t - \mathbb{E}[\mathbf{L}_1] \left(\frac{b\delta + (a\delta + \alpha\rho)c}{\delta - \beta c} \right) t}{\sqrt{t}} \rightarrow N(0, \sigma_L^2) \quad \text{in distribution as } t \rightarrow \infty, \quad (6.8)$$

where

$$\sigma_L^2 = (\mathbb{E}[\mathbf{L}_1])^2 \sigma_1^2 + \left(\frac{b\delta + (a\delta + \alpha\rho)c}{\delta - \beta c} \right) \text{Var}[\mathbf{L}_1] \quad (6.9)$$

and σ_1^2 is defined in Eq. (4.1).

Proof. Note that,

$$\begin{aligned}
 L_t - \mathbb{E}[\mathbf{L}_1] \left(\frac{b\delta + (a\delta + \alpha\rho)c}{\delta - \beta c} \right) t \\
 &= \sum_{i=1}^{N_t} \mathbf{L}_i - \mathbb{E}[\mathbf{L}_1] \left(\frac{b\delta + (a\delta + \alpha\rho)c}{\delta - \beta c} \right) t \\
 &= \mathbb{E}[\mathbf{L}_1] \left[N_t - \left(\frac{b\delta + (a\delta + \alpha\rho)c}{\delta - \beta c} \right) t \right] + \sum_{i=1}^{N_t} \mathbf{L}_i - \mathbb{E}[\mathbf{L}_1] N_t. \quad (6.10)
 \end{aligned}$$

From Eqs. (5.18) and (5.19) of Theorem 4.2, Eq. (6.10) can be expressed in terms of the sum of martingales whose quadratic variation is given by

$$\begin{aligned}
 (\mathbb{E}[\mathbf{L}_1])^2 \left(\frac{\delta^2 N_t}{(\delta - \beta c)^2} + \frac{(c\alpha)^2 N_t^{(1)}}{(\delta - \beta c)^2} + \frac{(c\sigma)^2}{(\delta - \beta c)^2} \int_0^t r_\tau (1 - e^{-\delta(t-\tau)})^2 d\tau \right) \\
 + \sum_{i=1}^{N_t} (L_i - \mathbb{E}[\mathbf{L}_1])^2. \quad (6.11)
 \end{aligned}$$

Moreover, by using Eqs. (5.21) and (5.22) of Theorem 4.2,

$$\begin{aligned}
 (\mathbb{E}[\mathbf{L}_1])^2 \left[\frac{1}{t} \frac{\delta^2 N_t}{(\delta - \beta c)^2} + \frac{1}{t} \frac{(c\alpha)^2 N_t^{(1)}}{(\delta - \beta c)^2} + \frac{1}{t} \frac{(c\sigma)^2}{(\delta - \beta c)^2} \int_0^t r_\tau (1 - e^{-\delta(t-\tau)})^2 d\tau \right] \\
 + \frac{1}{t} \sum_{i=1}^{N_t} (L_i - \mathbb{E}[\mathbf{L}_1])^2 \\
 = (\mathbb{E}[\mathbf{L}_1])^2 \left[\frac{1}{t} \frac{\delta^2 N_t}{(\delta - \beta c)^2} + \frac{1}{t} \frac{(c\alpha)^2 N_t^{(1)}}{(\delta - \beta c)^2} + \frac{1}{t} \frac{(c\sigma)^2}{(\delta - \beta c)^2} \int_0^t r_\tau (1 - e^{-\delta(t-\tau)})^2 d\tau \right] \\
 + \frac{N_t}{t} \cdot \frac{1}{N_t} \sum_{i=1}^{N_t} (L_i - \mathbb{E}[\mathbf{L}_1])^2 \\
 \rightarrow (\mathbb{E}[\mathbf{L}_1])^2 \sigma_1^2 + \left(\frac{b\delta + (a\delta + \alpha\rho)c}{\delta - \beta c} \right) \text{Var}[\mathbf{L}_1]
 \end{aligned}$$

in probability as $t \rightarrow \infty$. Thus, by the Martingale Central Limit Theorem, we deduce that

$$\frac{L_t - \mathbb{E}[\mathbf{L}_1] \left(\frac{b\delta + (a\delta + \alpha\rho)c}{\delta - \beta c} \right) t}{\sqrt{t}} \rightarrow N(0, \sigma_L^2) \quad \text{in distribution as } t \rightarrow \infty$$

where σ_L^2 as defined in Eq. (6.9). □

Theorem 6.4 (LDP of L_t). *Under Assumption 3.1 and assume that $\mathbb{E}[e^{\theta \mathbf{L}_1}] < \infty$. Then for any $r_0 = r \in \mathbb{R}^+$, $P(\frac{L_t}{t} \in \cdot)$ satisfies the LDP with the rate*

function

$$I_L^*(x) = \sup_{\theta \in \mathbb{R}} \{\theta x - I_L(\theta)\}, \quad (6.12)$$

where $I_L(\theta) = I(\log \mathbb{E}[e^{\theta \mathbf{L}_1}])$ and $I(\theta)$ is defined in Eq. (5.30) of Theorem 4.3.

Proof. For $\theta \in \mathbb{R}$ such that $\mathbb{E}[e^{\theta \sum_{i=1}^{N_t} \mathbf{L}_i}] < \infty$, we have

$$\begin{aligned} \mathbb{E}[e^{\theta \sum_{i=1}^{N_t} \mathbf{L}_i}] &= \mathbb{E}[\mathbb{E}[e^{\theta \sum_{i=1}^{N_t} \mathbf{L}_i} \mid N_t]] \\ &= \sum_{n=0}^{\infty} \mathbb{E}[e^{\theta \sum_{i=1}^{N_t} \mathbf{L}_i} \mid N_t = n] P(N_t = n) \\ &= \sum_{n=0}^{\infty} \mathbb{E}[e^{\theta \sum_{i=1}^n \mathbf{L}_i}] P(N_t = n) \\ &= \sum_{n=0}^{\infty} (\mathbb{E}[e^{\theta \mathbf{L}_1}])^n P(N_t = n) \\ &= \mathbb{E}[e^{(\log \mathbb{E}[e^{\theta \mathbf{L}_1}]) N_t}]. \end{aligned}$$

Recalling the definition of $I(\cdot)$ given in Eq. (5.30), we get

$$\begin{aligned} I_L(\theta) &= \lim_{t \rightarrow \infty} \frac{1}{t} \log \mathbb{E}[e^{\theta \sum_{i=1}^{N_t} \mathbf{L}_i}] \\ &= \lim_{t \rightarrow \infty} \frac{1}{t} \log \mathbb{E}[e^{(\log \mathbb{E}[e^{\theta \mathbf{L}_1}]) N_t}] = I(\log \mathbb{E}[e^{\theta \mathbf{L}_1}]). \end{aligned}$$

Hence, it implies that $P(\frac{L_t}{t} \in \cdot)$ satisfies the LDP with the rate function $I_L^*(x)$ as defined in Eq. (6.12). $\square$

Hence, by studying the loss process with these asymptotic behaviors, we establish a comprehensive framework for understanding and analyzing losses in portfolios, which provides critical insights for effective decision-making and risk management.

7. Conclusion and Future Work

The CIR process is a useful tool in financial modeling and in the dynamics of interest rate and credit risk. Previous models of CIR processes need to be more efficient in capturing the default clustering effect. In this paper, we have studied the CIR process with externally and self-exciting jumps with exponential exciting functions. This modification enhances the model's ability to capture the default clustering effect impacted by both endogenous and exogenous factors of the underlying system. Consequently, we have investigated the asymptotic properties of the generalized CIR process, specifically focusing on the LLN, the CLT and the LDP. Moreover, we analyze the importance of the presented model in the field of finance with the

help of default events and by studying the limiting behavior of the loss process. An open problem in this area for future work is to study the MDP.

Acknowledgment

The authors would like to thanks all reviewers for their insightful and constructive comments and suggestions.

ORCID

Shamiksha Pandey  <https://orcid.org/0009-0007-5552-1566>

Dharmaraja Selvamuthu  <https://orcid.org/0000-0003-2892-0864>

References

1. E. Bacry, S. Delattre, M. Hoffmann and J. F. Muzy, Some limit theorems for Hawkes processes and application to financial statistics, *Stoch. Process. Appl.* **123** (2013) 2475–2499.
2. C. Bordenave and G. L. Torrisi, Large deviations of Poisson cluster processes, *Stoch. Models* **23** (2007) 593–625.
3. P. Brémaud and L. Massoulié, Stability of nonlinear Hawkes processes, *Ann. Prob.* **24** (1996) 1563–1588.
4. J. C. Cox, J. E. Ingersoll Jr. and S. A. Ross, A theory of the term structure of interest rates, *Econometrica* **53** (1985) 385–407.
5. A. Dassios, J. Jang and H. Zhao, A generalised CIR process with externally-exciting and self-exciting jumps and its applications in insurance and finance, *Risks* **7** (2019) 103.
6. D. J. Daley and D. Vere-Jones, *An Introduction to the Theory of Point Processes. Vol. I and II, Elementary Theory and Methods*, 2nd edn. (Springer, 2003).
7. A. Dassios and H. Zhao, A dynamic contagion process, *Adv. Appl. Prob.* **43** (2011) 814–846.
8. A. Dassios and H. Zhao, A generalised contagion process with an application to credit risk, *Int. J. Theor. Appl. Financ.* **20** (2017) 1–33.
9. A. Dassios and H. Zhao, Efficient simulation of clustering jumps with CIR intensity, *Oper. Res.* **65** (2017) 1494–1515.
10. A. Dembo and O. Zeitouni, *Large Deviations Techniques and Applications*, 2nd edn. (Springer, 1998).
11. D. Dufresne, The integrated square-root process, Working Paper No. 90, Centre for Actuarial Studies, University of Melbourne (2001).
12. E. Errais, K. Giesecke and L. R. Goldberg, Affine point processes and portfolio credit risk, *SIAM J. Finan. Math.* **1** (2010) 642–665.
13. W. Feller, Two singular diffusion problems, *Ann. Math.* **54** (1951) 173–182.
14. M. Hairer, Convergence of Markov processes, Lecture Notes, Imperial College London (2021).
15. A. G. Hawkes, Spectra of some self-exciting and mutually exciting point processes, *Biometrika* **58** (1971) 83–90.
16. A. G. Hawkes and D. Oakes, A cluster process representation of a self-exciting process, *J. Appl. Prob.* **11** (1974) 493–503.
17. J. Jang, Jump diffusion processes and their applications in insurance and finance, *Insur.: Math. Econom.* **41** (2007) 62–70.

18. R. A. Jarrow and S. M. Turnbull, Pricing derivatives on financial securities subject to credit risk. *J. Finan.* **50** (1995) 53–85.
19. D. Karabash and L. Zhu, Limit theorems for marked Hawkes processes with application to a risk model, *Stoch. Models* **31** (2015) 433–451.
20. P. Pasricha and D. Selvamuthu, A Markov modulated dynamic contagion process with application to credit risk, *J. Statist. Phys.* **175** (2019) 495–511.
21. D. Selvamuthu, S. Pandey and P. Tardelli, Limit theorems for an extended inverse Hawkes process with general exciting functions. *Statist. Prob. Lett.* **197** (2023) 109817.
22. F. Wu, X. Mao and K. Chen, Strong convergence of Monte Carlo simulations of the mean-reverting square root process with jump, *Appl. Math. Comput.* **206** (2008) 494–505.
23. N. Yao and M. Xiao, Limit theorems for non-Markovian marked dynamic contagion processes, *J. Math. Ann. Appl.* **464** (2018) 693–706.
24. L. Zhu, Central limit theorem for nonlinear Hawkes processes, *J. Appl. Prob.* **50** (2013) 760–771.
25. L. Zhu, Moderate deviations for Hawkes processes, *Statist. Prob. Lett.* **83** (2013) 885–890.
26. L. Zhu, Limit theorems for a Cox–Ingersoll–Ross process with Hawkes jumps, *J. Appl. Prob.* **51** (2014) 699–712.
27. L. Zhu, Process-level large deviations for nonlinear Hawkes point processes, *Ann. Inst. H. Poincaré Prob. Statist.* **50** (2014) 845–871.
28. L. Zhu, Large deviations for Markovian nonlinear Hawkes processes, *Ann. Appl. Prob.* **25** (2015) 548–581.