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Analysis of energy saving in user equipment in LTE-A using stochastic modelling

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Abstract

Energy saving in User Equipment (UE) is one of the important issues for limited sources of power in the device. It is critical for the UE to maximize its energy efficiency. In this paper, we have presented two stochastic models, namely the Markov model and semi-Markov model, for the UE based on the states of discontinuous reception (DRX) mechanism, i.e., a power saving method in mobile communication networks. Explicit expressions are derived for transient and steady-state system size probabilities for the Markov model. For the semi-Markov model, steady-state probabilities are computed. Further, the performance measures such as mean and variance are computed for both models. Using these models, based on the states of DRX mechanism, energy saving in the UE is calculated. Finally, sensitivity analysis is performed in which the results obtained are compared for both models. Numerical results obtained in this paper ensure that energy saving can be maximized in the UE using the Markov modelling of DRX mechanism rather than semi-Markov modelling. The energy saving using the Markov model is atleast 33.19% more than the semi-Markov model. Also, for energy saving in the UE, the semi-Markov model for DRX mechanism is compared with the Markov model. The semi-Markov models for the DRX mechanism are available in the literature without considering the packet arrivals. Our analysis of DRX mechanism and conclusion on its performance can be designed and implemented to an extension for the existing DRX mechanism. We believe that, these models can also be extended to study the energy saving of hardware and other components of the system.

Keywords Markov model · Semi-Markov model · Working state · Short cycle · Long sleep · DRX · LTE-A

1 Introduction

Long-Term Evolution (LTE) is a standard for the wireless communication. The goal of LTE was to increase the capacity and speed of wireless data networks. LTE Advanced (LTE-A) is a standard for wireless communication that is one generation beyond LTE. It offers faster speeds and greater stability than LTE. LTE-A defined a new kind of cellular access network with great spectral efficiency, high peak data rates, fast round-trip times, and frequency and bandwidth flexibility. It denotes a changing level of performance as cellular hardware, software, and network technology capabilities, such as speed, latency, battery life, and cost effectiveness, are improved over time.

LTE-A is a leading technology that is expected to be widely used in wireless networking under the Third-Generation Partnership Project (3GPP). Currently in the wireless technology, important issue is energy consumption in the UE, taking into account that anyone have to deal with a limitation of large amount of energy consumption and LTE-A data transmission with massive speed. The energy gap between available and desired energy in wireless user equipment (UE) powered by batteries is widening year after year. As a result, the LTE-A standard incorporates energy saving technique like DRX mechanism. To this end, the Discontinuous Reception Scheme, DRX, which is a energy saving Scheme for UE in LTE-A networks is investigated.

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The most effective scenario to roll out an LTE-A network is by using a 900 MHz frequency. The 900 MHz frequency is used globally for voice and basic data communications. There are many models that have been well studied by various researchers in the literature, but there was no stochastic model proposed to study the energy saving for the UE at the packet level. Therefore, in this paper, the stochastic models that are unique and simple in nature are presented to evaluate the performance of the UE.

Stochastic models, i.e., two dimensional continuous time Markov chain and a semi-Markov model are proposed. The state space is same for both the models but the sojourn times in states vary. The distributions for the sojourn times of states used for the semi-Markov model are taken from literature. The motive of the paper is to present that the energy saving in the UE can be maximized using the Markov modelling of DRX mechanism rather than semi-Markov modelling. In this paper, we have developed the stochastic models for energy saving in the UE. We believe that, these proposed models can be used to save energy in the UE, where UE can be mobile,etc. Further, we believe that, these models can also be extended to study the energy saving of hardware and other components of the system.

The paper is organized as follows. The literature review is given in Sect. 2. The structure of DRX mechanism is explained in Sect. 3. The description of the proposed Markov model is explained in Sect. 4. Section 4.2 deals with the time dependability analysis of the proposed model, and Sect. 4.3 deals with steady-state analysis of the model. The semi-Markov model is presented in Sect. 5. Performance measures are analyzed in Sect. 6 for both the models. In Sect. 7, numerical results are presented for the proposed models. Finally concluding remarks are given in Sect. 8.

2 Related works

In recent years, in mobile communications, power savings in the UE and Quality of Service(QoS) provided are important issues, and there are many researchers that have been discussed in this direction [1]. For instance, there is a large literature on power-awareness in mobile and wireless networks. The stochastic processes are used for the modelling of wireless networks [2]. Also, there has been a great deal of research on reducing energy consumption in wireless networks. Jones et al. [3] provided an overview of many energy consumption reducing techniques. The energy issue using indexed semi-Markov modelling was addressed in [4]. The effects on QoS using semi-Markov modelling was presented in [5]. The results reported in the literature were obtained from system-level power measurement but not component-level power measurement, although component-level power measurement was supposed to provide more details on power consumption. Due to the much higher data rates and lower latency compared to LTE, a rise in UE power consumption was inevitable [6,7].

Because of its limited on board energy, it was critical for the UE to maximize its energy efficiency [8]. The power consumption can be reduced by entering the UE to sleep mode [9]. To reduce the energy consumption, the LTE-A standard defines DRX mechanism [10] that allows UEs to turn off its components when no data is expected to be received. An energy efficient and QoS aware DRX scheme is presented in [11]. A new appliance is proposed to switch the DRX mechanism from the power active state to the power saving state and vice versa [12]. A simple but efficient application-aware DRX mechanism to optimize the system performance of LTE-A networks is presented in [13]. In [14], a semi-Markov model was developed to characterize the performance metrics of the DRX where the metrics were limited to an exponential packet arrival rate. Arunsundar et al. [15] introduced an analysis of the DRX mechanism, which integrates DRX long and short cycles that influence the average latency due to buffering, DRXSLEEP, and energy consumption and created a systematic model of the DRX mechanism.

A new DRX was introduced, called LTE-DRX, to improve UE battery life [16]. In [17] to analyze energy saving in the UE, a Markovian queuing model for the DRX mechanism in LTE/LTE-A networks was presented. There were many authors who modeled the DRX mechanism as semi-Markov models. For instance, the DRX mechanism was modeled as a five-state semi-Markov model, and an eight-state semi-Markov model in [18] and [19], respectively. Semi-Markov modelling was performed for the performance analysis of power saving in the DRX mechanism [20]. A semi-Markov model was presented to analyze the DRX mechanism in LTE/LTE-A networks [21,22]. These models derive the power saving factor by calculating the stationary probabilities and holding times for the active and sleeping states of the DRX mechanism.

A new Licensed-Assisted Access DRX mechanism (LAA-DRX) was analyzed using four-state semi-Markov model to show the probabilistic estimation of power saving [23]. A flexible DRX mechanism was analysed in [24]. A Markov chain with timer inactivity, short sleep, long sleep, and active service states is created, and the tradeoff between mean delay and power saving factor is examined in [25] and the profile of both parameters is plotted against the packet arrival rate. The technique adopted in [20] shows that the larger wake-up delay was needed for the efficiency of battery power saving. In [26], the gain in power consumption was achieved when a longer DRX cycle was applied. In [27], it was shown that extending the length of the DRX cycle helps in reducing the energy consumption of the UE, i.e., length of DRX cycle affects the energy consumption in the UE [28]. The performance of DRX in terms of energy consumption, using the Markov model considering both short and long cycles, was analyzed in [27]. In [29], the power consumption in LTE networks was discussed. The authors studied the energy saving in UE by applying carrier aggregation in [30], i.e., a narrow and a wide band UE power model was presented. An LTE smartphone power model was introduced where the power consumption of three different LTE smartphones was presented [31]. The sleep concept with the aim of minimizing the energy waste in case of unused buffered data was introduced [32].

3 DRX mechanism

In this Section, the DRX mechanism of the UE is explained in detail that helps to understand the proposed stochastic models. The DRX mechanism consists of three periods that are idle, working, and sleep periods [12,33]. The sleep period includes two types of sleeps that are given as short sleep and long sleep. The structure of DRX mechanism is shown in Fig. 1. The energy consumption values are different for different periods of DRX mechanism. These energy values are taken from the literature [27,34] and are given in Table 1.

The UE is in an idle period if there are no packets in the system to provide service. If there is a packet arrival, then the system starts providing service to the packets, hence moving to the working period. Once service is provided to all the packets, the system again moves to an idle period. After spending some random time in this period, if there is no packet indication, then the system moves to $1st$ short cycle, i.e., sleep period. If there is no packet indication in one short cycle, then the system keeps moving to the next short cycle. This is repeated for the predetermined number of cycles (say, N), and many short cycles together form the short sleep, i.e. short sleep consists of a predetermined number of short cycles N. Suppose there is packet arrival in any short cycle, then after spending random amount of time in that cycle, the system moves to the working period. If there is no packet arrival till the last short cycle, Nth short cycle, then the system enters the long sleep. The packets arriving during long sleep are kept for a random amount of time to provide service after the completion of long sleep. If there is no packet indication during long sleep, then after spending a random amount of time, the system enters an idle period. The possible transitions from one state to another state in the DRX mechanism is shown in Fig. 2.

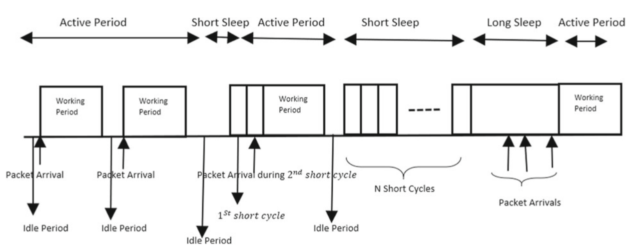

Fig. 1 Structure of DRX mechanism

Table 1 UE energy consumption values in different periods of the DRX mechanism

Period	Energy consumption (mW/ms)
Idle period	255.5
Working period	500
Short sleep	11
Long sleep	0

There are two kinds of sleep in DRX, to minimize the delay experienced and to maximize the energy saving. The energy saving in long sleep is more in comparison to that of short sleep because if there is packet arrival in any short cycle, then after spending random amount of time in that cycle, the system moves to the working period. But packets arriving during long sleep are kept for a random amount of time to provide service after the completion of long sleep. Hence, the delay experienced by the packets for service is more in the case of long sleep than short sleep [27]. Together N cycles constitute the short sleep, and this N is predetermined number that is considered to make sure that the system moves to long sleep only when there is no packet arrival for some amount of time (i.e, during short sleep). If $N = 1$ and there is no packet arrival in a short sleep, then the system moves to a long sleep. As the duration of long sleep is more than the duration of the short sleep, whenever there is packet arrival during the long sleep, the delay experienced by the packet is more. Therefore, the delay experienced by the packets increases if $N = 1$.

In this paper, the two models, namely the Markov model and the semi-Markov model, are considered based on the sojourn time distributions in sleep periods. In the Markov model, sojourn time in each state is exponentially distributed. In the semi-Markov model, sojourn time in sleep periods is different from an exponential distribution.

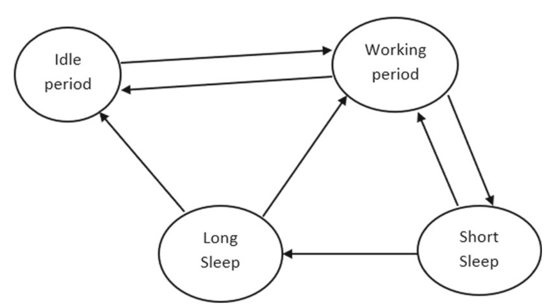

Fig. 2 State transition in DRX mechanism

4 Markov model

A stochastic model is proposed, considering the UE in different states of the DRX Mechanism. In this proposed stochastic model, sojourn time in each state follows an independent exponential distribution. Hence the underlying stochastic process is a continuous time Markov chain.

4.1 Description of the Markov model

The proposed Markov model is explained in detail in this Section. The four states of this model are explained:

- *Idle period*: It is the state in which UE is active and waiting for packets to provide service.
- *Working period*: It is the state in which the UE is active, i.e., providing service to the packets.
- *Short cycle*: It is a state in which the UE turns off most of its components. This short cycle is repeated for a predetermined number of times (say N) if no data packet arrives and ends otherwise.
- *Long sleep*: It is a state in which UE enters on the expiration of the short cycle if no data packet arrives.

Let $\{(N(t), S(t)), t \geq 0\}$ represents the continuous time Markov chain where $\{N(t), t \geq 0\}$ denote the number of packets in the system at any time t and $\{S(t), t \geq 0\}$ represents the state of the system at any time t with state space $\Omega = \{(i, j); i = 0, 1, \ldots; j = 0, 1, \ldots, N, N+1\}$.
The state (0, 0) represents that the system is in an idle period. The state $(i, 0)$, $i = 1, 2, \ldots$ represents that the system is in working period providing service to i^{th} packet and the state $(0, j)$, $j = 1, 2, \ldots, N$ represents the system is in j^{th} short cycle. The state $(i, N+1)$ represents the system is in long sleep with i number of packets. In both idle and working periods, the UE is active to provide service to the packets.
Let the time spent in the idle period before moving to the first short cycle be exponentially distributed with parameter β. Let the packet arrival in the system follow a Poisson process with parameter λ, and the service time of a packet follows an exponential distribution with parameter μ. Let the time spent in a short cycle and long sleep be exponentially distributed with parameters α and ν, respectively.
Let us assume that the system is initially in an idle period, i.e., it is waiting for data packets to provide service. If there is packet arrival in an idle period, then on the completion of an idle period, the system moves to state (1, 0) and starts providing service to packets. If before the service completion of the packet, there is packet arrival, then the system moves to $(i+1, 0)$, when i packets are already in the system. On completion of every service, the system moves to state $(i-1, 0)$ from state $(i, 0)$. If there is no packet arrival in the idle period, then the system moves to state (0, 1). It keeps

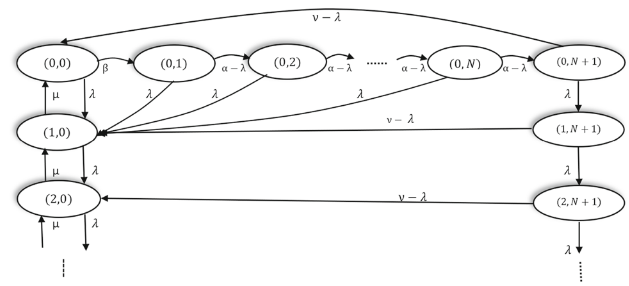

Fig. 3 State transition diagram for the proposed Markov model

on moving to state $(0, j)$, if there is no packet arrival in state $(0, j-1)$; $j = 1, 2, \ldots, N$. If there is packet arrival in any of the state $(0, j)$, $j = 1, 2, \ldots, N$, the system moves to state $(1, 0)$. If there is no packet arrival in state $(0, N)$, then the system moves to state $(0, N+1)$. If a packet arrives in the state $(i, N+1)$, then the system moves to state $(i+1, N+1)$, when i packets are already in the system. After completing the long sleep, the system moves to state $(i, 0)$ if i packets arrive during the long sleep. The state transition diagram for the proposed Markov model is presented in Fig. 3.

Let $P_{i,j}(t) = P\{N(t) = i, S(t) = j\}$, $i = 0, 1, \ldots$ and $j = 0, 1, \ldots, N+1$ be the probability that the server is in state j with i number of packets in the system at any time t. Then $P_{i,j}(t)$ satisfies the following forward Kolmogorov equations [35].

$$\begin{aligned}
P'_{0,0}(t) &= -(\beta+\lambda)P_{0,0}(t) + \mu P_{1,0}(t) \\
&\quad + (\nu-\lambda)P_{0,N+1}(t) && (1)\\
P'_{1,0}(t) &= -(\mu+\lambda)P_{1,0}(t) + \lambda P_{0,0}(t) + \mu P_{2,0}(t) \\
&\quad + \lambda(P_{0,1}(t) + P_{0,2}(t) + \ldots + P_{0,N}(t)) \\
&\quad + (\nu-\lambda)P_{1,N+1}(t) && (2)\\
P'_{i,0}(t) &= -(\mu+\lambda)P_{i,0}(t) + \lambda P_{i-1,0}(t) + \mu P_{i+1,0}(t) \\
&\quad + (\nu-\lambda)P_{i,N+1}(t),\ i = 2, 3, \ldots && (3)\\
P'_{0,1}(t) &= -\alpha P_{0,1}(t) + \beta P_{0,0}(t) && (4)\\
P'_{0,j}(t) &= -\alpha P_{0,j}(t) + (\alpha-\lambda)P_{0,j-1}(t), \\
&\quad j = 2, 3, \ldots, N && (5)\\
P'_{0,N+1}(t) &= -\nu P_{0,N+1}(t) + (\alpha-\lambda)P_{0,N}(t) && (6)\\
P'_{i,N+1}(t) &= -\nu P_{i,N+1}(t) + \lambda P_{i-1,N+1}(t), \\
&\quad i = 1, 2, \ldots && (7)
\end{aligned}$$

and assume that the system is initially empty, i.e., $P_{0,0}(0) = 1$.

4.2 Transient analysis

This Section presents the transient probabilities of different states of the proposed model, i.e., idle period, working period, short cycles and long sleep. The probability of UE being in short cycle $P_{0,j}(t)$ and long Sleep $P_{i,N+1}(t)$ are presented in terms of $P_{0,0}(t)$. Similarly, the probability of UE being in working period $P_{n,0}(t)$ is derived using generating function in terms of $P_{0,0}(t)$, where $P_{0,0}(t)$ is the transient probability for UE being in idle period.

Theorem 1 *Time-dependent probability of UE being in short cycle* $(0, j)$ *and long sleep* $(i, N+1)$ *where* $1 \leq j \leq N; i \geq 0$ *is given by*

$$P_{0,j}(t) = \beta(\alpha-\lambda)^{j-1} e^{-\alpha t} \frac{t^{j-1}}{(j-1)!} * P_{0,0}(t),$$
$$j = 1, 2, \ldots, N.$$
$$P_{i,N+1}(t) = \beta\lambda^i(\alpha-\lambda)^N e^{-\nu t}\frac{t^i}{i!} * e^{-\alpha t}\frac{t^{N-1}}{(N-1)!} * P_{0,0}(t), \quad i = 0, 1, \ldots.$$

Proof Details of the proof are given in Appendix 1. □

Theorem 2 *Time-dependent probability of UE being in working state* $(n, 0)$ *where* $n \geq 1$ *is given by*

$$\begin{aligned}P_{n,0}(t) &= \mu b^{n+1} P_{0,0}(t) * e^{-(\mu+\lambda)(t)} \frac{2n}{\alpha t} I_n(\alpha t)\\ &+(\nu-\lambda)\sum_{i=0}^{n} b^{n-i} P_{i,N+1}(t) * e^{-(\mu+\lambda)(t)}[I_{n-i}(\alpha t) - I_{n+i}(\alpha t)]\\ &+(\nu-\lambda)\sum_{i=1}^{\infty} b^{-i} P_{n+i,N+1}(t) * e^{-(\mu+\lambda)(t)}[I_i(\alpha t) - I_{2n+i}(\alpha t)]\\ &+\lambda b^{n-1}\sum_{i=1}^{N} P_{0,i}(t) * e^{-(\mu+\lambda)(t)}\frac{2n}{\alpha t} I_n(\alpha t).\end{aligned}$$

Proof Details of the proof are given in Appendix 2. □

Theorem 3 *Time-dependent probability of UE being in idle state* $(0, 0)$ *is given by the inverse Laplace transform of equation*

$$\begin{aligned}&f_{0,0}(s)\Bigg[1 + \frac{2\mu b}{\alpha}\frac{\frac{b(d-\sqrt{d^2-\alpha^2})}{\alpha}}{1-\frac{b(d-\sqrt{d^2-\alpha^2})}{\alpha}}\\ &+\frac{\beta(\nu-\lambda)(\alpha-\lambda)^N}{(s+\nu)(s+\alpha)^N\sqrt{d^2-\alpha^2}}\Bigg[\frac{1}{1-\frac{\lambda\alpha}{b(s+\nu)(d-\sqrt{d^2-\alpha^2})}}\\ &\left(\frac{\frac{b(d-\sqrt{d^2-\alpha^2})}{\alpha}}{1-\frac{b(d-\sqrt{d^2-\alpha^2})}{\alpha}} - \frac{\lambda\alpha}{b(s+\nu-\lambda)(d-\sqrt{d^2-\alpha^2})}\right)\\ &-\frac{1}{1-\frac{\lambda(d-\sqrt{d^2-\alpha^2})}{\alpha b(s+\nu)}}\left(\frac{\frac{b(d-\sqrt{d^2-\alpha^2})}{\alpha}}{1-\frac{b(d-\sqrt{d^2-\alpha^2})}{\alpha}}\right.\\ &\left.-\frac{\frac{\lambda^2(d-\sqrt{d^2-\alpha^2})^3}{b\alpha^3(s+\nu)^2}}{1-\frac{\lambda(d-\sqrt{d^2-\alpha^2})^2}{\alpha^2(s+\nu)}}\right)\Bigg]\\ &+\frac{\beta(\nu-\lambda)(\alpha-\lambda)^N}{(s+\nu)(s+\alpha)^N}\frac{1}{\sqrt{d^2-\alpha^2}}\left(\frac{\frac{\lambda(d-\sqrt{d^2-\alpha^2})}{\alpha b(s+\nu)}}{1-\frac{\lambda(d-\sqrt{d^2-\alpha^2})}{\alpha b(s+\nu)}}\right)\\ &\times\left[\frac{\frac{\lambda}{s+\nu}}{1-\frac{\lambda}{s+\nu}} - \frac{\frac{\lambda(d-\sqrt{d^2-\alpha^2})^2}{\alpha^2(s+\nu)}}{1-\frac{\lambda(d-\sqrt{d^2-\alpha^2})^2}{\alpha^2(s+\nu)}}\right] + \frac{2\lambda\beta}{\alpha b(s+\alpha)}\\ &\times\left(\frac{\frac{b(d-\sqrt{d^2-\alpha^2})}{\alpha}}{1-\frac{b(d-\sqrt{d^2-\alpha^2})}{\alpha}}\right)\left[\frac{1-(\frac{\alpha-\lambda}{s+\alpha})^N}{1-(\frac{\alpha-\lambda}{s+\alpha})}\right]\\ &+\frac{\beta(\alpha-\lambda)^N}{(s+\nu)(s+\alpha)^N}\left(\frac{1}{1-\frac{\lambda}{s+\nu}}\right)\\ &+\frac{\beta}{s+\alpha}\left[\frac{1-(\frac{\alpha-\lambda}{s+\alpha})^N}{1-(\frac{\alpha-\lambda}{s+\alpha})}\right]\Bigg] = \frac{1}{s}.\end{aligned}$$

Proof Details of the proof are given in Appendix 3. □

In this Section, we have expressed the probabilities $P_{0,j}(t)$, $P_{i,N+1}(t)$ and $P_{n,0}(t)$ for $n = 1, 2, \ldots; 1 \leq j \leq N; i \geq 0$ in terms of $P_{0,0}(t)$. The probability $P_{0,0}(t)$ can be obtained taking the inverse Laplace transform of equation given in Theorem 3.

4.3 Steady-state probabilities

In this Section, the steady state probabilities of the system are obtained from the time-dependent system size probabilities which are given in Sect. 4.2. It is known that, $\lim_{t\to\infty} P_{i,j}(t) = \pi_{i,j}$ and $\lim_{t\to\infty} P'_{i,j}(t) = 0$. Hence, equations (1)-(7) becomes

$$0 = -(\beta+\lambda)\pi_{0,0} + \mu\pi_{1,0} + (\nu-\lambda)\pi_{0,N+1}, \tag{8}$$

$$\begin{aligned}0 &= -(\mu+\lambda)\pi_{1,0} + \lambda\pi_{0,0}\\ &\quad+\mu\pi_{2,0} + \lambda(\pi_{0,1} + \pi_{0,2} + \ldots + \pi_{0,N})\\ &\quad+(\nu-\lambda)\pi_{1,N+1},\end{aligned} \tag{9}$$

$$\begin{aligned}0 &= -(\mu+\lambda)\pi_{i,0} + \lambda\pi_{i-1,0}\\ &\quad+\mu\pi_{i+1,0} + (\nu-\lambda)\pi_{i,N+1}, i = 2, 3, \ldots\end{aligned} \tag{10}$$

$$0 = -\alpha\pi_{0,1} + \beta\pi_{0,0}, \tag{11}$$

$$0 = -\alpha\pi_{0,j} + (\alpha-\lambda)\pi_{0,j-1}, j = 2, 3, \ldots, N \tag{12}$$

$$0 = -\nu\pi_{0,N+1} + (\alpha-\lambda)\pi_{0,N}, \tag{13}$$

$$0 = -\nu\pi_{i,N+1} + \lambda\pi_{i-1,N+1}, i = 1, 2, \ldots. \tag{14}$$

The stability conditions for the existence of these steady state probabilities are $\lambda \leq \alpha$, $\lambda < \mu$ and $\lambda < \nu$.

Theorem 4 *Steady-state probabilities of UE being in different states, i.e.,* $(0, 0)$, $(0, j)$, $(i, N+1)$ *and* $(n, 0)$ *for* $n = 1, 2, \ldots; 1 \leq j \leq N; i \geq 0$ *are given by*

$$\pi_{0,j} = \frac{\beta(\alpha-\lambda)^{j-1}}{\alpha^j}\pi_{0,0}, j = 1, 2, \ldots, N.$$

$$\pi_{i,N+1} = \frac{\beta\lambda^i(\alpha-\lambda)^N}{\nu^{i+1}\alpha^N}\pi_{0,0},\, i = 0, 1, \ldots$$
$$\pi_{n,0} = A_n\pi_{0,0},\, n \geq 1$$

where

$$A_n = \left[\frac{\mu b^{n+1}}{(\mu-\lambda)} + \frac{\beta b^{n-1}((\alpha-\lambda)^N - \alpha^N)}{\alpha^N(\mu-\lambda)}\right]\left[b^{n+1} - b^{n-1}\right] + \frac{\beta b^n(\alpha-\lambda)^N}{(\mu-\lambda)\alpha^N}\left[\frac{(\nu-\lambda)b^{n+2}(1-(\frac{\lambda}{\nu b^2})^{n+1})}{(\nu b^2-\lambda)} - b^n\left(1-\left(\frac{\lambda}{\nu}\right)^{n+1}\right) + \left(\frac{\lambda}{\nu}\right)^{n+1}(b^{-n} - b^n)\right].$$

$$\pi_{0,0} = \frac{1}{1 + \sum_{i=1}^{\infty} A_i + \sum_{j=1}^{N}\frac{\beta(\alpha-\lambda)^{j-1}}{\alpha^j} + \sum_{i=0}^{\infty}\frac{\beta\lambda^i(\alpha-\lambda)^N}{\nu^{i+1}\alpha^N}}.$$

Proof Details of the proof are given in Appendix 4. □

Using the probability $\pi_{0,0}$, we can compute all the probabilities $\pi_{0,j}$, $\pi_{i,N+1}$; $1 \leq j \leq N$; $i \geq 0$ and $\pi_{n,0}$ for $n = 1, 2, \ldots$.

5 Semi-Markov model

The state space for this proposed semi-Markov model is same as that of the proposed Markov model. In contrast, the sojourn time in the state corresponds to the sleep period is following non-exponential distribution.

5.1 Description of the model

Let I_0 denotes the system is in an idle period with zero packets, i.e., the system waiting for packets, W_k denotes the UE is in working period with k number of packets, i.e., $(k, 0)^{th}$ state, S_j denotes the UE is in j^{th} short cycle, i.e., $(0, j)^{th}$ state and L_i denotes the UE is in long sleep with i number of packets to serve, i.e., $(i, N+1)^{th}$ state, where $k = 1, 2, \ldots$, $i = 0, 1, \ldots$ and $j = 1, 2, \ldots, N$. Let $F_{m,n}(t)$ denotes the CDF associated with transition from state m to n, where $m, n \in \Omega$, $\Omega = \{I_0, W_k, S_j, L_i, k = 1, 2, \ldots; i = 0, 1, \ldots; j = 1, 2, \ldots, N\}$. There are papers that talk about the sojourn time distribution for different states of DRX mechanism [22,27]. Hence, the sojourn time distribution for different states of the semi-Markov model is shown in Table 2.

The transition from one state to another is influenced by many factors and hence exhibits random behavior. In this process, the distribution of transition from one state to another is influenced by the real time scenario of DRX mechanisms. Hence, the sojourn time of some states follows the deterministic distribution, and time epochs possess the Markov property. Therefore, this stochastic process can be visualized as an SMP (Semi-Markov Process) [35]. The state transition diagram of the proposed semi-Markov model is shown in Fig. 4. The state space of this model is given as $\Omega = \{I_0, W_k, S_j, L_i, k = 1, 2, \ldots; i = 0, 1, \ldots; j = 1, 2, \ldots, N\}$.

Table 2 Distributions of sojourn times in semi-Markov model

CDF	Distribution	Parameter
$F_{I_0,S_1}(t)$	Deterministic	$\frac{1}{\beta}$
$F_{S_j,S_{j+1}}(t)$	Deterministic	$\frac{1}{\alpha}$
$F_{S_N,L_0}(t)$	Deterministic	$\frac{1}{\alpha}$
$F_{W_k,W_{k+1}}(t)$	Exponential	λ
$F_{W_1,I_0}(t)$	Exponential	μ
$F_{W_k,W_{k-1}}(t)$	Exponential	μ
$F_{S_j,W_1}(t)$	Exponential	λ
$F_{L_i,L_{i+1}}(t)$	Exponential	λ
$F_{L_0,I_0}(t)$	Deterministic	$\frac{1}{\nu}$
$F_{L_i,W_i}(t)$	Deterministic	$\frac{1}{\nu}$

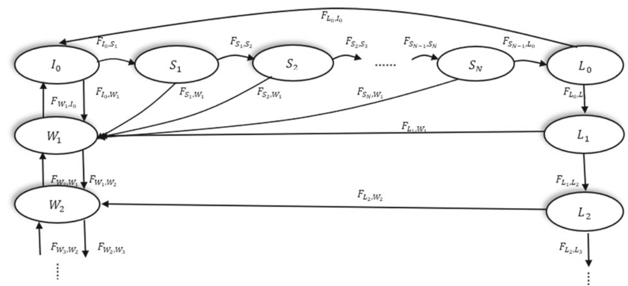

Fig. 4 State transition diagram for the proposed semi-Markov model

The steady-state probabilities $\pi_m \forall\, m \in S$ exist under the stability conditions, $\lambda < \mu$, $\lambda \leq \alpha$ and $\lambda < \nu$.

Theorem 5 *The steady-state probabilities for the semi-Markov model are given as:*

$$\pi_{I_0} = \frac{1 - e^{-\frac{\lambda}{\beta}}}{\lambda M},$$
$$\pi_{W_1} = \frac{1 - e^{-\left(\frac{\lambda}{\beta}+\frac{\lambda N}{\alpha}+\frac{\lambda}{\nu}\right)}}{\mu M},$$
$$\pi_{W_k} = \frac{1}{\mu M}\left[\left(\frac{\lambda}{\mu}\right)^{k-1}\left(1 - e^{-\left(\frac{\lambda}{\beta}+\frac{\lambda N}{\alpha}+\frac{\lambda}{\nu}\right)}\right) + \frac{\left(\frac{\lambda}{\mu}\right)\left(1-e^{-\frac{\lambda}{\nu}}\right)^2}{\left(\frac{\lambda}{\mu} - 1 + e^{-\frac{\lambda}{\nu}}\right)}e^{-\left(\frac{\lambda}{\beta}+\frac{\lambda N}{\alpha}\right)}\left[\left(\frac{\lambda}{\mu}\right)^{k-2} - \left(1-e^{-\frac{\lambda}{\nu}}\right)^{k-2}\right]\right],\ k = 2, 3, \ldots$$

$$\pi_{S_j} = \frac{e^{-\frac{\lambda}{\beta}}\left(e^{-\frac{\lambda}{\alpha}}\right)^{j-1}\left(1-e^{-\frac{\lambda}{\alpha}}\right)}{\lambda M}, \ j = 1, 2, \ldots, N$$
$$\pi_{L_i} = \frac{e^{-\left(\frac{\lambda}{\beta}+\frac{\lambda N}{\alpha}\right)}\left(1-e^{-\frac{\lambda}{\nu}}\right)^{i+1}}{\lambda M}, \ i = 0, 1, \ldots$$

where M is given in Appendix 5.

Proof Details of the proof are given in Appendix 5. □

6 Performance analysis

Performance measures for both the proposed models are presented in this Section. Firstly, we present the measures for the Markov model in Sect. 6.1. Section 6.2 deals with the performance measures for the semi-Markov model.

6.1 Markov model

This section computes various time-dependent and long-run performance measures for the Markov model and analyzes the same. We illustrate the evaluation of attributes for the different states (i.e., active, short cycle, and long sleep) of the model.

6.1.1 Time-dependent measures

As $\{(N(t), S(t)), t \geq 0\}$ represents the continuous time Markov chain where $N(t)$ denotes the number of packets in the system at time t and $\{S(t), t \geq 0\}$ represents the state of the system at time t with state space $S = \{(i, j);\ i = 0, 1, \ldots;\ j = 0, 1, \ldots, N+1\}$.
Let $W(t), C(t), L(t)$ represent that the system is in active state, short sleep and long sleep respectively, at any time $t, t > 0$. Let $P\{(N(t), S(t)) = W(t)\}$, $P\{(N(t), S(t)) = C(t)\}$ and $P\{(N(t), S(t)) = L(t)\}$ be the probabilities for being in active state, short sleep and long sleep respectively, at any time $t, t > 0$. Then

$$P\{(N(t), S(t)) = W(t)\} = \sum_{n=0}^{\infty} P_{n,0}(t).$$

$$P\{(N(t), S(t)) = C(t)\} = \sum_{j=1}^{N} P_{0,j}(t)$$
$$= \beta e^{-\alpha t}\left\{\sum_{j=1}^{N}\frac{(t(\alpha-\lambda))^{j-1}}{(j-1)!}\right\} * P_{0,0}(t).$$

$$P\{(N(t), S(t)) = L(t)\} = \sum_{i=0}^{\infty} P_{i,N+1}(t)$$
$$= \beta(\alpha-\lambda)^N e^{(\lambda-\nu)t} * e^{-\alpha t}\frac{t^{N-1}}{(N-1)!} * P_{0,0}(t).$$

6.1.2 Steady-state measures

Let A_M, S_M and L_M represents the steady-state probabilities for system being in the active state, short sleep and long sleep respectively, for the Markov model. Then,

$$A_M = \sum_{n=0}^{\infty}\pi_{n,0} = \left\{1 - \left[\frac{\lambda}{(\mu-\lambda)} + \frac{\beta((\alpha-\lambda)^N - \alpha^N)}{\alpha^N(\mu-\lambda)}\right]\right.$$
$$+\frac{\beta(\alpha-\lambda)^N}{(\mu-\lambda)\alpha^N}\left[\frac{(\nu-\lambda)b^4}{(\nu b^2-\lambda)(1-b^2)} - \frac{\lambda^2}{\nu(\nu b^2-\lambda)}\right.$$
$$\left.\left.-\frac{b^2}{1-b^2} + \frac{\lambda^2}{\nu(\nu-\lambda)}\right]\right\}\pi_{0,0},$$

$$S_M = \sum_{j=1}^{N}\pi_{0,j} = \frac{\beta(\alpha^N - (\alpha-\lambda)^N)}{\lambda\alpha^N}\pi_{0,0},$$

$$L_M = \sum_{n=0}^{\infty}\pi_{n,N+1} = \frac{\beta(\alpha-\lambda)^N}{(\nu-\lambda)\alpha^N}\pi_{0,0}.$$

6.1.3 Mean

Let $E[N_M]$ denotes the mean number of packets in the system and let $E[N_{A_M}]$ and $E[N_{S_M}]$ denotes the mean number of packets in an active mode and sleep mode,respectively. Then,

$$E[N_M] = E[N_{A_M}] + E[N_{S_M}],$$

where

$$E[N_{A_M}] = \sum_{n=0}^{\infty} n\pi_{n,0} = \left\{\frac{\mu}{(\lambda-\mu)}\left[\frac{\lambda}{(\mu-\lambda)}\right.\right.$$
$$\left.+\frac{\beta((\alpha-\lambda)^N - \alpha^N)}{\alpha^N(\mu-\lambda)}\right] + \frac{\beta(\alpha-\lambda)^N}{(\mu-\lambda)\alpha^N}\left[\frac{(\nu-\lambda)b^4}{(\nu b^2-\lambda)(1-b^2)^2}\right.$$
$$\left.\left.-\frac{\lambda^2}{(\nu-\lambda)(\nu b^2-\lambda)} - \frac{b^2}{(1-b^2)^2} + \frac{\lambda^2}{(\nu-\lambda)^2}\right]\right\}\pi_{0,0},$$

$$E[N_{S_M}] = \sum_{n=0}^{\infty} n\pi_{n,N+1} = \frac{\beta\lambda(\alpha-\lambda)^N}{(\nu-\lambda)^2\alpha^N}\pi_{0,0}.$$

Let $M(t)$ be the expected system size at any time $t, t > 0$. Then,

$$M(t) = \sum_{n=0}^{\infty} nP_{n,0}(t) + \sum_{n=0}^{\infty} nP_{n,N+1}(t).$$

6.1.4 Variance

Let $V(t)$ denotes the variance of system size at any time t, $t > 0$. Then

$$V(t) = M^2(t) - (M(t))^2,$$

where $M^2(t)$ is the second order moment for the number of packets in the system at any time t, $t > 0$.

$$M^2(t) = \sum_{n=0}^{\infty} n^2 P_{n,0}(t) + \sum_{n=0}^{\infty} n^2 P_{n,N+1}(t).$$

6.1.5 Energy saving

The energy saving factor is defined as the percentage of time the UE spends in sleep state i.e., the ratio of time UE spends in a sleep period to total time across all the states. Hence, using steady-state probabilities $\pi_{i,j}$ for $i = 0, 1, 2, ...;\ j = 1, 2, ..., N$ energy saving factor E_S is computed as [18,19]:

$$E_S = \sum_{j=1}^{N} \pi_{0,j} + \sum_{i=0}^{\infty} \pi_{i,N+1} = \left\{ \frac{\beta(\alpha^N - (\alpha-\lambda)^N)}{\lambda \alpha^N} + \frac{\beta(\alpha-\lambda)^N}{(\nu-\lambda)\alpha^N} \right\} \pi_{0,0}.$$

6.2 Semi-Markov model

In this Section, performance measures are computed for long run probabilities for the semi-Markov model.

6.2.1 Steady-state measures

Let A_{SM}, S_{SM} and L_{SM} be the steady-state probabilities for being in active state, short sleep and long sleep respectively for the semi-Markov model in a long run. Then

$$A_{SM} = \pi_{I_0} + \sum_{k=1}^{\infty} \pi_{W_k} = \frac{1}{M}\left[\frac{\mu}{\lambda(\mu-\lambda)} - \frac{e^{-\left(\frac{\lambda}{\beta}+\frac{\lambda N}{\alpha}+\frac{\lambda}{\nu}\right)}}{\mu-\lambda} - \frac{e^{-\frac{\lambda}{\beta}}}{\lambda} - \frac{\lambda}{\mu} \frac{e^{-\left(\frac{\lambda N}{\alpha}+\frac{\lambda}{\beta}\right)}\left(1-e^{-\frac{\lambda}{\nu}}\right)^2}{\left(\frac{\lambda}{\mu}-1+e^{-\frac{\lambda}{\nu}}\right)} \left[\frac{e^{\frac{\lambda}{\nu}}}{\mu} - \frac{1}{\mu-\lambda} \right] \right],$$

$$S_{SM} = \sum_{j=1}^{N} \pi_{S_j} = \frac{1}{M}\left[\frac{e^{-\frac{\lambda}{\beta}}\left(1-e^{-\frac{\lambda N}{\alpha}}\right)}{\lambda} \right],$$

$$L_{SM} = \sum_{i=0}^{\infty} \pi_{L_i} = \frac{1}{M}\left[\frac{e^{-\left(\frac{\lambda N}{\alpha}+\frac{\lambda}{\beta}\right)}\left(e^{\frac{\lambda}{\nu}}-1\right)}{\lambda} \right].$$

6.2.2 Mean

Let $E[N_{SM}]$ be the mean number of packets in the system and let $E[N_{A_{SM}}]$ and $E[N_{S_{SM}}]$ denotes the mean number of packets in an active mode and sleep mode, respectively. Then,

$$E[N_{SM}] = E[N_{A_{SM}}] + E[N_{S_{SM}}],$$

where

$$E[N_{A_{SM}}] = \sum_{n=1}^{\infty} n\pi_{W_n} = \frac{1}{M}\left[\frac{\mu\left(1-e^{-\left(\frac{\lambda}{\beta}+\frac{\lambda N}{\alpha}+\frac{\lambda}{\nu}\right)}\right)}{(\mu-\lambda)^2} + \frac{\lambda}{\mu^2} \frac{e^{-\left(\frac{\lambda N}{\alpha}+\frac{\lambda}{\beta}\right)}\left(1-e^{-\frac{\lambda}{\nu}}\right)}{\left(\frac{\lambda}{\mu}-1+e^{-\frac{\lambda}{\nu}}\right)} \left[1 - e^{\frac{2\lambda}{\nu}} - \frac{\left(1-e^{-\frac{\lambda}{\nu}}\right)\left(2-\frac{\lambda}{\mu}\right)}{\left(1-\frac{\lambda}{\mu}\right)^2} \right] \right],$$

$$E[N_{S_{SM}}] = \sum_{n=0}^{\infty} n\pi_{L_n} = \frac{1}{M}\left[\frac{e^{-\left(\frac{\lambda N}{\alpha}+\frac{\lambda}{\beta}\right)} e^{\frac{2\lambda}{\nu}} \left(1-e^{\frac{\lambda}{\nu}}\right)^2}{\lambda} \right].$$

6.2.3 Energy saving

The energy saving factor is defined as the percentage of time the UE spends in sleep state i.e., the ratio of time UE spends in a sleep period to total time across all the states. Hence, using steady-state probabilities π_m, $m \in \Omega$, energy saving factor E_S is computed as

$$E_S = \sum_{j=1}^{N} \pi_{S_j} + \sum_{i=0}^{\infty} \pi_{L_i} = \frac{1}{M}\left[\frac{e^{-\frac{\lambda}{\beta}}\left(1-e^{-\frac{\lambda N}{\alpha}}\right)}{\lambda} + \frac{e^{-\left(\frac{\lambda N}{\alpha}+\frac{\lambda}{\beta}\right)}\left(e^{\frac{\lambda}{\nu}}-1\right)}{\lambda} \right].$$

7 Discussion part

In this section, numerical results are discussed for both models. For this, we assume some reasonable values for the mean sojourn rates or times for numerical illustration. The graph for long-run probabilities for both the models are plotted by varying various parameters, and their behaviour is analyzed. To plot the graphs, the values of parameters are fixed based on the stability conditions of the proposed models $\frac{\lambda}{\mu} < 1$, $\frac{\lambda}{\nu} < 1$ and $\alpha \geq \lambda$, $\beta > 0$ and $\lambda > 0$. The values are as follows $\alpha = 1$, $\beta = 1.2$, $\lambda = 0.5$, $\mu = 1.5$ and $\nu = 1.8$.
To show the results of the Markov model, MM is written with the varying parameter values, and for the Semi-Markov model, SMM is written with the varying parameter values. Figure 5 exemplifies the behaviour of working state $\pi_{n,0}$ with respect to n, for varying values of number of short cycles N. For the Markov model, the graph is plotted for five different values of N, and the Semi-Markov model graph is plotted for $N = 5$. It is observed that the increase in the number of short cycles leads to an increase in the steady-state probability of working state $\pi_{n,0}$ and as the number of packets n increases, probability decreases and reaches the stable value.

Figure 6 presents the graph of steady-state probability of long sleep $\pi_{n,N+1}$ with respect to n, with number of short

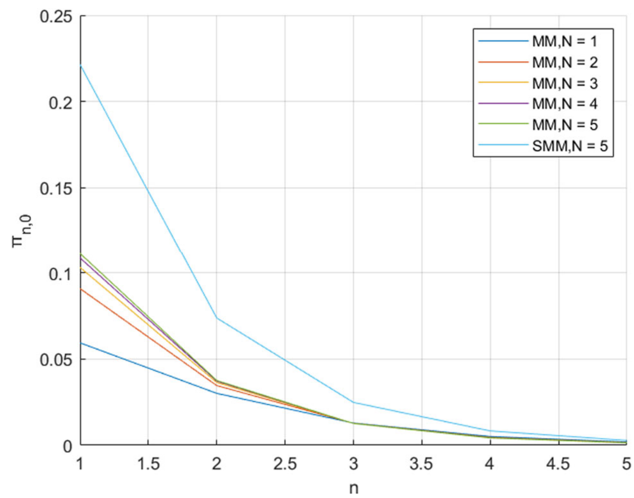

Fig. 5 Steady-state probability of working states varying number of packets and short cycles

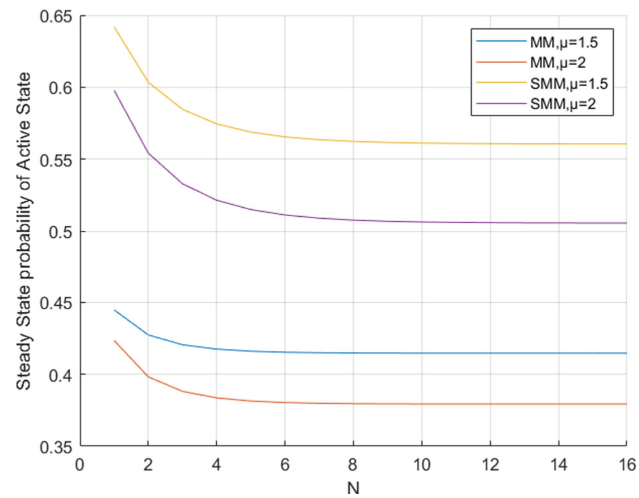

Fig. 7 Steady-state probability of being in active state varying number of short cycles and packet service rate

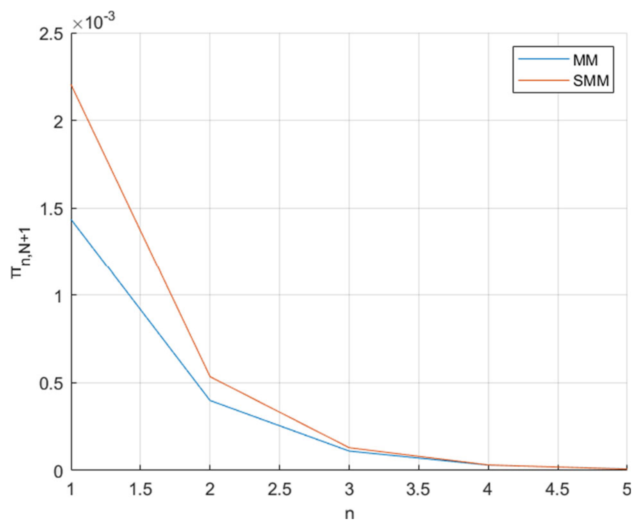

Fig. 6 Steady-state probability of Long sleep varying number of packets

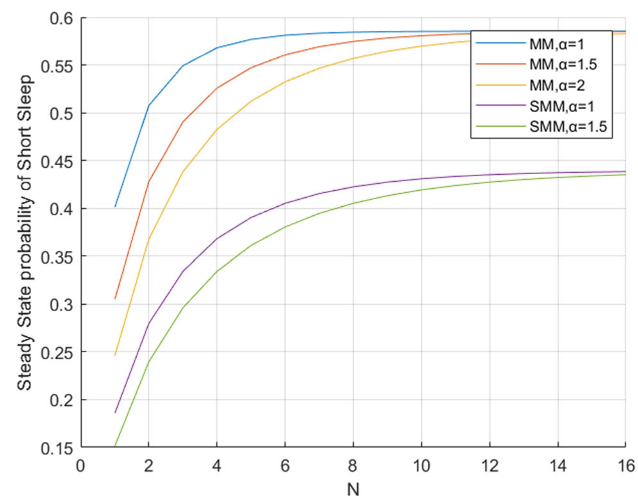

Fig. 8 Steady-state probability of being in Short Sleep varying the number of short cycles and the value of α

cycles $N = 5$ for both the models. It is evident from Fig. 6 that, rise in the number of packets leads to a decrease in the steady-state probability of long sleep and finally converges to zero for an increasing number of packets, i.e., the value of n.

The steady-state probability of being in a working state against the values of N for varying values of μ is plotted in Fig. 7 for both the models. This is plotted, keeping the other parameter values same as given except for the value of μ. It can be observed that the steady-state probability for being in a working state decreases with the increase in the number of short cycles. For the semi-Markov, the probability of the system remaining in a working state is more.

The steady-state probability of being in short sleep against the values of N for varying values of α is plotted in Fig. 8. For the Markov model, α is the expected sojourn rate, whereas for the semi-Markov model, $\frac{1}{\alpha}$ is the expected sojourn rate. This is also plotted, keeping the other parameter values same as given above except for a value of α. It can be observed that the steady-state probability for being in short sleep increases with the increase in the number of short cycles. It can be observed that less the value of α more is the probability.

The steady-state probability of being in long sleep against the values of N for varying values of ν is plotted in Fig. 9. This is also plotted, keeping the other parameter values same as given except for the value of ν. For the Markov model, ν is the expected sojourn rate, whereas for the semi-Markov model, $\frac{1}{\nu}$ is the expected sojourn rate. It can be observed that the steady-state probability for being in long sleep decreases with the increase in the number of short cycles. It can be observed that less the value of ν more is the probability.

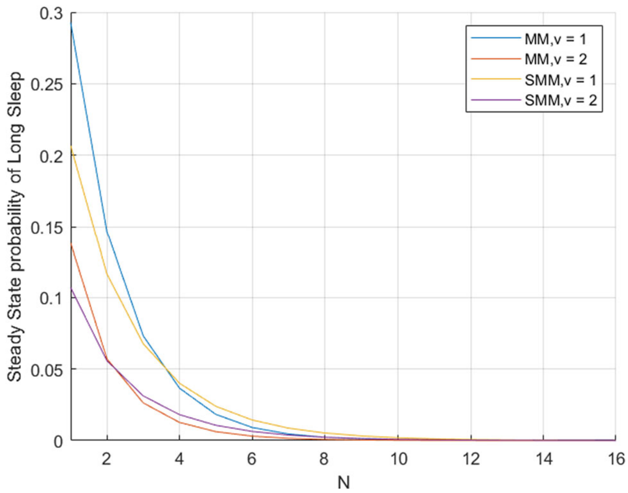

Fig. 9 Steady-state probability of being in long sleep varying the number of short cycles and the value of ν

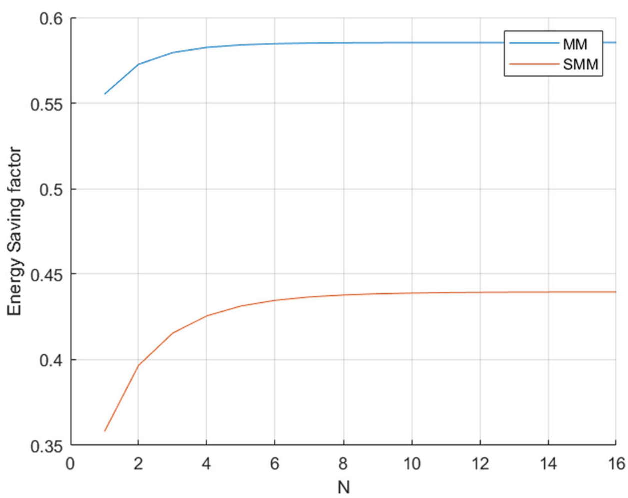

Fig. 10 Energy saving factor varying number of short cycles

The energy saving factor against the values of N is plotted in Fig. 10. All the other parameter values are kept same. For the semi-Markov model, as the steady-state probability of UE being in active state is more than the Markov model, hence the power saving factor is less for the semi-Markov model. The factor stabilizes as the value of N increases.

The percentage of difference in measures of both the models is computed as the measured value reaches the steady-state value. For comparison purpose, the values of parameters used are as follows $\alpha = 1$, $\beta = 1.2$, $\lambda = 0.5$, $\mu = 1.5$ and $\nu = 1.8$ for both the models. Comparison between the energy saving factor for both the models is given in Table 3. We have observed from the Table 3 that, the energy saving factor approaches the steady-state value for Markov model with $N \geq 12$ and for semi-Markov model with $N \geq 16$. Also, we have observed that the energy saving factor is atleast 33.19% more for the Markov model than the semi Markov model. Hence, energy saving can be maximized in the UE using the Markov modeling of the DRX mechanism. These percentage difference for other measures is computed from the observations presented in Figs. 7, 8, 9. From Fig. 7, we have observed that the active state steady-state probability is less for the Markov model by 36% than the semi-Markov model. From Fig. 8, we have observed that the short sleep steady-state probability is more for the Markov model by 37% than the semi-Markov model. Similarly, from Fig. 9, we have observed that the long sleep steady-state probability is less for the Markov model by 48% than the semi-Markov model.

Table 3 Values for energy saving factor by varying number of short cycles

N	Markov model(MM)	Semi-Markov model(SMM)	% Difference between MM and SMM
8	0.5852	0.4377	33.69
9	0.5853	0.4384	33.50
10	0.5853	0.4388	33.38
11	0.5853	0.4391	33.29
12	0.5854	0.4393	33.25
13	0.5854	0.4393	33.25
14	0.5854	0.4394	33.22
15	0.5854	0.4394	33.22
16	0.5854	0.4395	33.19
17	0.5854	0.4395	33.19
18	0.5854	0.4395	33.19

8 Conclusion and future work

In this paper, two stochastic models are investigated for the UE. The models are two dimensional infinite state continuous time models proposed with few assumption. The states of the models are based on different periods of the DRX mechanism. In models, energy saving in the UE is analyzed using the closed-form solutions of steady-state probability and the expected sojourn times for both the Markov model and semi-Markov model. The distributions for sojourn time in states used for the semi-Markov model are taken from literature. The paper has presented that the energy saving in the UE can be maximized in the UE using the Markov modeling of DRX mechanism rather than semi-Markov modelling. The energy saving factor stabilizes as the number of short sleep increases.

We propose that with some other distributions for sojourn time in states, there can be more improvement in the energy saving in the UE. One can also extend this stochastic modelling of energy saving in the UE to the base station for the

LTE-A networks. Stochastic modelling of the proposed work with 5G and NR can also be considered as the future work. Further, we plan to develop testbed for the DRX mechanism to validate and refine our proposed model.

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Declarations

Conflict of interest The authors have not disclosed any competing interests.

Appendix

Appendix 1

Let $f_{i,j}(s)$ represents the Laplace transform of $P_{i,j}(t)$. Taking Laplace transform of (4), we get

$$(s+\alpha) f_{0,1}(s) = \beta f_{0,0}(s). \tag{15}$$

On the inversion of the above equation, we have

$$P_{0,1}(t) = \beta e^{-\alpha t} * P_{0,0}(t),$$

where '$*$' represents the convolution of functions. Taking Laplace transform on Equation (5), we have

$$f_{0,j}(s) = \frac{(\alpha-\lambda)}{(s+\alpha)} f_{0,j-1}(s), \ \ j = 2, 3, \ldots, N$$

which recursively yields,

$$f_{0,j}(s) = \frac{(\alpha-\lambda)^{j-1}}{(s+\alpha)^{j-1}} f_{0,1}(s), \ \ j = 2, 3, \ldots, N.$$

Using equation (15) in above equation, we obtain

$$f_{0,j}(s) = \frac{\beta(\alpha-\lambda)^{j-1}}{(s+\alpha)^{j}} f_{0,0}(s), \ \ j = 1, 2, \ldots, N. \tag{16}$$

On inversion of equation (16), we obtain the required transient probabilities $P_{0,j}(t),\ j = 1, 2, \ldots, N$. Now, taking Laplace transform on (6), we get

$$f_{0,N+1}(s) = \frac{(\alpha-\lambda)}{(s+\nu)} f_{0,N}(s).$$

Using (16) for $j = N$ in above equation, we obtain

$$f_{0,N+1}(s) = \frac{\beta(\alpha-\lambda)^{N}}{(s+\nu)(s+\alpha)^{N}} f_{0,0}(s). \tag{17}$$

Inversion yields,

$$P_{0,N+1}(t) = \beta(\alpha-\lambda)^N e^{-\nu t} * e^{-\alpha t}\frac{t^{N-1}}{(N-1)!} * P_{0,0}(t).$$

Taking Laplace transform of (7), we get

$$f_{i,N+1}(s) = \frac{\lambda}{(s+\nu)} f_{i-1,N+1}(s), \ \ i = 1, 2, \ldots$$

which recursively yields,

$$f_{i,N+1}(s) = (\frac{\lambda}{s+\nu})^i f_{0,N+1}(s), \ \ i = 1, 2, \ldots.$$

Using (17) in above equation, we obtain

$$f_{i,N+1}(s) = \frac{\beta\lambda^i(\alpha-\lambda)^N}{(s+\nu)^{i+1}(s+\alpha)^N} f_{0,0}(s), \ \ i = 0, 1, \ldots. \tag{18}$$

Taking inverse Laplace transform of (18) yields the required transient probabilities $P_{i,N+1}(t), i = 0, 1, \ldots$.

Appendix 2

Define a generating function with coefficients as the transient probabilities of working states,

$$G(z,t) = P_{0,0}(t) + \sum_{i=1}^{\infty} P_{i,0}(t) z^i.$$

Then,

$$\frac{\partial G(z,t)}{\partial z} = P^{'}_{0,0}(t) + \sum_{i=1}^{\infty} P^{'}_{i,0}(t) z^i.$$

Substituting (1), (2) and (3) in the above equation, we get

$$\begin{aligned}\frac{\partial G(z,t)}{\partial z} & -\left(\lambda z + \frac{\mu}{z} - (\mu+\lambda)\right) G(z,t) \\ &= -\left(\beta + \frac{\mu}{z} - \mu\right) P_{0,0}(t) \\ &+(\nu-\lambda)\sum_{i=0}^{\infty} P_{i,N+1}(t) z^i + \lambda z \sum_{i=1}^{N} P_{0,i}(t).\end{aligned}$$

Solving the above partial differential equation, we obtain

$$\begin{aligned}G(z,t) &= -\left(\beta + \frac{\mu}{z} - \mu\right)\int_0^t P_{0,0}(y) e^{\left(\lambda z + \frac{\mu}{z} - (\mu+\lambda)\right)(t-y)} dy \\ &+(\nu-\lambda)\int_0^t \sum_{i=0}^{\infty} P_{i,N+1}(y) z^i e^{\left(\lambda z + \frac{\mu}{z} - (\mu+\lambda)\right)(t-y)} dy\end{aligned}$$

$$+\lambda z \int_0^t \sum_{i=1}^{N} P_{0,i}(y) e^{\left(\lambda z+\frac{\mu}{z}-(\mu+\lambda)\right)(t-y)} dy + e^{-\left(\mu+\lambda-(\lambda z+\frac{\mu}{z})\right)t}.$$

If $a = 2\sqrt{\lambda\mu}$ and $b = \sqrt{\frac{\lambda}{\mu}}$ then

$$e^{\left(\lambda z+\frac{\mu}{z}\right)(t-y)} = \sum_{n=-\infty}^{\infty} (bz)^n I_n(a(t-y)),$$

where $I_n(.)$ is the modified bessel function of first kind. Then,

$$G(z,t) = -\left(\beta + \frac{\mu}{z} - \mu\right) \int_0^t P_{0,0}(y) e^{-(\mu+\lambda)(t-y)} \left[\sum_{n=-\infty}^{\infty} (bz)^n I_n(a(t-y))\right] dy + (\nu-\lambda) \int_0^t \sum_{i=0}^{\infty} P_{i,N+1}(y) z^i \left[e^{-(\mu+\lambda)(t-y)} \sum_{n=-\infty}^{\infty} (bz)^n I_n(a(t-y))\right] dy$$
$$+\lambda z \int_0^t \sum_{i=1}^{N} P_{0,i}(y) e^{-(\mu+\lambda)(t-y)} \sum_{n=-\infty}^{\infty} (bz)^n I_n(a(t-y)) dy + e^{-(\mu+\lambda)t} \sum_{n=-\infty}^{\infty} (bz)^n I_n(a(t)).$$

Further comparing the coefficient of z^n for $n = 1, 2, \ldots$ from both sides in above equation, we find

$$P_{n,0}(t) = -\int_0^t P_{0,0}(y) e^{-(\mu+\lambda)(t-y)} b^n [(\beta-\mu) I_n(.) + \mu b I_{n+1}(.)] dy + (\nu-\lambda) \int_0^t \sum_{i=0}^{n} P_{i,N+1}(y) e^{-(\mu+\lambda)(t-y)} b^{n-i} I_{n-i}(.) dy$$
$$+(\nu-\lambda) \int_0^t \sum_{i=1}^{\infty} P_{n+i,N+1}(y) e^{-(\mu+\lambda)(t-y)} b^{-i} I_{-i}(.) dy + \lambda \int_0^t \sum_{i=1}^{N} P_{0,i}(y) e^{-(\mu+\lambda)(t-y)} b^{n-1} I_{n-1}(.) dy + e^{-(\mu+\lambda)t} b^n I_n(at), \tag{19}$$

where $I_n(.) = I_n(a(t-y))$.
And comparing the coefficient of z^{-n} for n = 1,2,3,.. on both sides, we have,

$$0 = -\int_0^t P_{0,0}(y) e^{-(\mu+\lambda)(t-y)} b^{-n} [(\beta-\mu) I_{-n}(.) + \mu b I_{-n+1}(.)] dy + (\nu-\lambda) \int_0^t \sum_{i=0}^{\infty} P_{i,N+1}(y) e^{-(\mu+\lambda)(t-y)} \times b^{-n-i} I_{-n-i}(.) dy + \lambda \int_0^t \sum_{i=1}^{N} P_{0,i}(y) e^{-(\mu+\lambda)(t-y)} \times b^{-n-1} I_{-n-1}(.) dy + e^{-(\mu+\lambda)t} b^{-n} I_{-n}(at). \tag{20}$$

Using the property of modified Bessel function that $I_n(.) = I_{-n}(.)$ and Eqs. (19) and (20) [(19) - $b^{2n} \times$ (20)], we have for $n = 1, 2, \ldots$

$$P_{n,0}(t) = -\int_0^t P_{0,0}(y) e^{-(\mu+\lambda)(t-y)} \mu b^{n+1} \times [I_{n+1}(.) - I_{n-1}(.)] dy + (\nu-\lambda) \int_0^t \sum_{i=0}^{n} P_{i,N+1}(y) e^{-(\mu+\lambda)(t-y)} \times b^{n-i} [I_{n-i}(.) - I_{n+i}(.)] dy + (\nu-\lambda) \int_0^t \sum_{i=1}^{\infty} P_{n+i,N+1}(y) \times e^{-(\mu+\lambda)(t-y)} b^{-i} [I_i(.) - I_{2n+i}(.)] dy + \lambda \int_0^t \sum_{i=1}^{N} P_{0,i}(y) \times e^{-(\mu+\lambda)(t-y)} b^{n-1} [I_{n-1}(.) - I_{n+1}(.)] dy. \tag{21}$$

After doing some simple manipulation on the above equation, we get

$$P_{n,0}(t) = -\mu b^{n+1} P_{0,0}(t) \times [I_{n+1}(at) - I_{n-1}(at)] e^{-(\mu+\lambda)(t)} + \sum_{i=0}^{n} (\nu-\lambda) b^{n-i} P_{i,N+1}(t) \times [I_{n-i}(at) - I_{n+i}(at)] e^{-(\mu+\lambda)(t)} + \sum_{i=1}^{\infty} (\nu-\lambda) b^{-i} P_{n+i,N+1}(t) \times [I_i(at) - I_{2n+i}(at)] e^{-(\mu+\lambda)(t)} + \sum_{i=1}^{N} \lambda b^{n-1} P_{0,i}(t) \times [I_{n-1}(at) - I_{n+1}(at)] e^{-(\mu+\lambda)(t)}.$$

Using $I_{n-1}(\alpha x) - I_{n+1}(\alpha x) = \frac{2n}{\alpha x} I_n(\alpha x)$ in (21) yields the required transient probabilities $P_{n,0}(t), n \geq 1$.

Appendix 3

In Eq. (19), substituting n = 0, we have

$$P_{0,0}(t) = -\int_0^t P_{0,0}(y) e^{-(\mu+\lambda)(t-y)} [(\beta-\mu) I_0(.) + \mu b I_1(.)] dy + (\nu-\lambda) \int_0^t \sum_{i=0}^{\infty} P_{i,N+1}(y) e^{-(\mu+\lambda)(t-y)} b^{-i} I_i(.) dy + \frac{\lambda}{b} \times \int_0^t \sum_{i=1}^{N} P_{0,i}(y) e^{-(\mu+\lambda)(t-y)} I_1(.) dy + e^{-(\mu+\lambda)t} I_0(at). \tag{22}$$

As

$$\sum_{n=0}^{\infty} P_{n,0}(t) + \sum_{j=1}^{N} P_{0,j}(t) + \sum_{i=0}^{\infty} P_{i,N+1}(t) = 1.$$

Taking Laplace transform of above equation, we get

$$\sum_{n=0}^{\infty} f_{n,0}(s) + \sum_{j=1}^{N} f_{0,j}(s) + \sum_{i=0}^{\infty} f_{i,N+1}(t) = \frac{1}{s}. \tag{23}$$

Taking Laplace transform of transient probabilities obtained in Theorem 2 and using $L[I_n(\alpha t)] = \frac{1}{\sqrt{s^2-\alpha^2}}\left(\frac{s-\sqrt{s^2-\alpha^2}}{\alpha}\right)^n$ and $L\left[\frac{2n}{\alpha t} I_n(\alpha t)\right] = \frac{2\alpha^{n-1}}{(s+\sqrt{s^2-\alpha^2})^n}$, we get

$$\begin{aligned}
f_{n,0}(s) &= \mu b^{n+1} f_{0,0}(s) \frac{2\alpha^{n-1}}{(d+\sqrt{d^2-\alpha^2})^n} \\
&+(\nu-\lambda)\sum_{i=0}^{n} b^{n-i} f_{i,N+1}(s) \frac{1}{\sqrt{d^2-\alpha^2}} \\
&\times\left[\left(\frac{d-\sqrt{d^2-\alpha^2}}{\alpha}\right)^{n-i} - \left(\frac{d-\sqrt{d^2-\alpha^2}}{\alpha}\right)^{n+i}\right] \\
&+(\nu-\lambda)\sum_{i=1}^{\infty} b^{-i} f_{n+i,N+1}(s) \frac{1}{\sqrt{d^2-\alpha^2}} \\
&\times\left[\left(\frac{d-\sqrt{d^2-\alpha^2}}{\alpha}\right)^{i} - \left(\frac{d-\sqrt{d^2-\alpha^2}}{\alpha}\right)^{2n+i}\right] \\
&+\lambda b^{n-1}\sum_{i=1}^{N} f_{0,i}(s) \frac{2\alpha^{n-1}}{(d+\sqrt{d^2-\alpha^2})^n}
\end{aligned} \tag{24}$$

where $d = s + \lambda + \mu$.
Using (16), (18) and (24) in equation (23), we obtain the required Laplace transform of transient probability of idle state (0,0). Inverse Laplace transform of the above equation yields the idle state probability $P_{0,0}(t)$.

Appendix 4

From Eq. (11), we get

$$\pi_{0,1} = \frac{\beta}{\alpha}\pi_{0,0}. \tag{25}$$

From Eq. (12), we get

$$\alpha\pi_{0,j} = (\alpha-\lambda)\pi_{0,j-1}.,\ j = 2, 3, \ldots, N$$

which recursively yields,

$$\pi_{0,j} = \frac{(\alpha-\lambda)^{j-1}}{\alpha^{j-1}}\pi_{0,1},\ j = 2, 3, \ldots, N.$$

Using the value of $\pi_{0,1}$ obtained in equation (25), we get

$$\pi_{0,j} = \frac{\beta(\alpha-\lambda)^{j-1}}{\alpha^{j}}\pi_{0,0},\ j = 1, 2, \ldots, N. \tag{26}$$

From Eq. (13), we have

$$\pi_{0,N+1} = \frac{(\alpha-\lambda)}{\nu}\pi_{0,N}$$

and using Eq. (26) we obtain, for $j = N$

$$\pi_{0,N+1} = \frac{\beta(\alpha-\lambda)^{N}}{\nu\alpha^{N}}\pi_{0,0}. \tag{27}$$

From Eq. (14), we get

$$\pi_{i,N+1} = \frac{\lambda}{\nu}\pi_{i-1,N+1},\ i = 1, 2, \ldots$$

which recursively yields

$$\pi_{i,N+1} = \frac{\lambda^{i}}{\nu^{i}}\pi_{0,N+1},\ i = 1, 2, \ldots$$

and using value of $\pi_{0,N+1}$ obtained in equation (27), we have

$$\pi_{i,N+1} = \frac{\beta\lambda^{i}(\alpha-\lambda)^{N}}{\nu^{i+1}\alpha^{N}}\pi_{0,0},\ i = 0, 1, \ldots. \tag{28}$$

Taking Laplace transform of (21) for $n = 1, 2, \ldots$ we have

$$\begin{aligned}
f_{n,0}(s) &= f_{0,0}(s)\mu b^{n+1}\left[\frac{1}{\sqrt{d^2-a^2}}\left(\frac{d-\sqrt{d^2-a^2}}{a}\right)^{n+1}\right. \\
&\left.-\frac{1}{\sqrt{d^2-a^2}}\left(\frac{d-\sqrt{d^2-a^2}}{a}\right)^{n-1}\right] \\
&+(\nu-\lambda)\sum_{i=0}^{n} f_{i,N+1}(s) b^{n-i}\left[\frac{1}{\sqrt{d^2-a^2}}\left(\frac{d-\sqrt{d^2-a^2}}{a}\right)^{n-i}\right. \\
&\left.-\frac{1}{\sqrt{d^2-a^2}}\left(\frac{d-\sqrt{d^2-a^2}}{a}\right)^{n+i}\right] \\
&+(\nu-\lambda)\sum_{i=1}^{\infty} f_{n+i,N+1}(s) b^{-i}\left[\frac{1}{\sqrt{d^2-a^2}}\left(\frac{d-\sqrt{d^2-a^2}}{a}\right)^{i}\right. \\
&\left.-\frac{1}{\sqrt{d^2-a^2}}\left(\frac{d-\sqrt{d^2-a^2}}{a}\right)^{2n+i}\right] \\
&+\lambda\sum_{i=1}^{N} f_{0,i}(s) b^{n-1}\left[\frac{1}{\sqrt{d^2-a^2}}\left(\frac{d-\sqrt{d^2-a^2}}{a}\right)^{n-1}\right.
\end{aligned}$$

$$-\frac{1}{\sqrt{d^2-a^2}}\left(\frac{d-\sqrt{d^2-a^2}}{a}\right)^{n+1}\Bigg]$$

where $d = s + \lambda + \mu$, $a = 2\sqrt{\lambda\mu}$ and $b = \sqrt{\frac{\lambda}{\mu}}$. Using Eqs. (16) and (18) in above equation, we get

$$\begin{aligned}
f_{n,0}(s) = {} & \frac{\mu b^{n+1}}{\sqrt{d^2-a^2}}\Bigg[\left(\frac{d-\sqrt{d^2-a^2}}{a}\right)^{n+1} \\
& -\left(\frac{d-\sqrt{d^2-a^2}}{a}\right)^{n-1}\Bigg] f_{0,0}(s) \\
& +(\nu-\lambda)\sum_{i=0}^{n}\frac{b^{n-i}}{\sqrt{d^2-a^2}}\frac{\beta\lambda^i(\alpha-\lambda)^N}{(s+\nu)^{i+1}(s+\alpha)^N} \\
& \times\left[\left(\frac{d-\sqrt{d^2-a^2}}{a}\right)^{n-i}-\left(\frac{d-\sqrt{d^2-a^2}}{a}\right)^{n+i}\right] f_{0,0}(s) \\
& +(\nu-\lambda)\sum_{i=1}^{\infty}\frac{b^{-i}}{\sqrt{d^2-a^2}}\frac{\beta\lambda^{n+i}(\alpha-\lambda)^N}{(s+\nu)^{n+i+1}(s+\alpha)^N} \\
& \times\left[\left(\frac{d-\sqrt{d^2-a^2}}{a}\right)^{i}-\left(\frac{d-\sqrt{d^2-a^2}}{a}\right)^{2n+i}\right] f_{0,0}(s) \\
& +\lambda\sum_{i=1}^{N}\frac{b^{n-1}}{\sqrt{d^2-a^2}}\frac{\beta(\alpha-\lambda)^{i-1}}{(s+\alpha)^i}\Bigg[\left(\frac{d-\sqrt{d^2-a^2}}{a}\right)^{n-1} \\
& -\left(\frac{d-\sqrt{d^2-a^2}}{a}\right)^{n+1}\Bigg] f_{0,0}(s).
\end{aligned}$$

The steady-state probability $\pi_{n,0}$ for $n = 1, 2, \ldots$ can be found using the well known properties of the Laplace transform. It is well known that for steady-state

$$\frac{\lambda}{\mu} < 1$$

and

$$\lim_{t\to\infty} P_{n,0}(t) = \pi_{n,0} = \lim_{s\to 0} s f_{n,0}(s),$$

where $n = 1, 2, \ldots$. Hence,

$$\begin{aligned}
\pi_{n,0} = {} & \Bigg\{\frac{\mu b^{n+1}}{(\mu-\lambda)}\left[b^{n+1}-b^{n-1}\right]+\sum_{i=0}^{n}\frac{(\nu-\lambda)b^{n-i}}{(\mu-\lambda)} \\
& \times\frac{\beta\lambda^i(\alpha-\lambda)^N}{\nu^{i+1}\alpha^N}\left[b^{n-i}-b^{n+i}\right]+(\nu-\lambda)\sum_{i=1}^{\infty}\frac{b^{-i}}{(\mu-\lambda)} \\
& \times\frac{\beta\lambda^{n+i}(\alpha-\lambda)^N}{\nu^{n+i+1}\alpha^N}\left[b^{i}-b^{2n+i}\right]+\lambda\sum_{i=1}^{N}\frac{b^{n-1}}{(\mu-\lambda)} \\
& \times\frac{\beta(\alpha-\lambda)^{i-1}}{\alpha^i}\left[b^{n-1}-b^{n+1}\right]\Bigg\}\pi_{0,0}.
\end{aligned}$$

$$\begin{aligned}
\pi_{n,0} = {} & \Bigg\{\left[\frac{\mu b^{n+1}}{(\mu-\lambda)}+\frac{\beta b^{n-1}((\alpha-\lambda)^N-\alpha^N)}{\alpha^N(\mu-\lambda)}\right] \\
& \times\left[b^{n+1}-b^{n-1}\right]+\frac{\beta b^n(\alpha-\lambda)^N}{(\mu-\lambda)\alpha^N} \\
& \times\Bigg[\frac{(\nu-\lambda)b^{n+2}(1-(\frac{\lambda}{\nu b^2})^{n+1})}{(\nu b^2-\lambda)}-b^n\left(1-\left(\frac{\lambda}{\nu}\right)^{n+1}\right) \\
& +\left(\frac{\lambda}{\nu}\right)^{n+1}(b^{-n}-b^n)\Bigg]\Bigg\}\pi_{0,0}. \\
\pi_{n,0} = {} & A_n\pi_{0,0},
\end{aligned} \tag{29}$$

where

$$\begin{aligned}
A_n = {} & \left[\frac{\mu b^{n+1}}{(\mu-\lambda)}+\frac{\beta b^{n-1}((\alpha-\lambda)^N-\alpha^N)}{\alpha^N(\mu-\lambda)}\right]\left[b^{n+1}-b^{n-1}\right] \\
& +\frac{\beta b^n(\alpha-\lambda)^N}{(\mu-\lambda)\alpha^N}\Bigg[\frac{(\nu-\lambda)b^{n+2}(1-(\frac{\lambda}{\nu b^2})^{n+1})}{(\nu b^2-\lambda)} \\
& -b^n\left(1-\left(\frac{\lambda}{\nu}\right)^{n+1}\right)+\left(\frac{\lambda}{\nu}\right)^{n+1}(b^{-n}-b^n)\Bigg].
\end{aligned}$$

And as

$$\sum_{i=0}^{\infty}\pi_{i,0}+\sum_{j=1}^{N}\pi_{0,j}+\sum_{i=0}^{\infty}\pi_{i,N+1}=1.$$

Substituting (26), (28) and (29) in above equation, we get

$$\begin{aligned}
& \pi_{0,0}+\sum_{i=1}^{\infty}A_i\pi_{0,0}+\sum_{j=1}^{N}\frac{\beta(\alpha-\lambda)^{j-1}}{\alpha^j}\pi_{0,0} \\
& +\sum_{i=0}^{\infty}\frac{\beta\lambda^i(\alpha-\lambda)^N}{\nu^{i+1}\alpha^N}\pi_{0,0}=1.
\end{aligned}$$

Hence,

$$\pi_{0,0}=\frac{1}{1+\sum_{i=1}^{\infty}A_i+\sum_{j=1}^{N}\frac{\beta(\alpha-\lambda)^{j-1}}{\alpha^j}+\sum_{i=0}^{\infty}\frac{\beta\lambda^i(\alpha-\lambda)^N}{\nu^{i+1}\alpha^N}}.$$

Here, $\pi_{0,0}$ exist as $\lambda \le \alpha$, $\lambda < \mu$ and $\lambda < \nu$.

Appendix 5

The two stage method is used to solve the SMP model which is described by its kernel matrix [5]. Let $K_{m,n}(t)$ denotes the elements of kernel matrix $K(t)$, where $K_{m,n}(t)$ is the probability that the system has just entered the state m and in the next transition is going the enter the state n within time t. In this Section, we present the steady-state analysis for the semi-Markov model. The non-zero elements for kernel matrix are given as:

$$K_{I_0,S_1}(t) = \int_0^t (1 - F_{I_0,W_1}(x))dF_{I_0,S_1}(x)$$

$$K_{I_0,S_1}(t) = \begin{cases} 0 & \text{if } t < \frac{1}{\beta} \\ e^{-\frac{\lambda}{\beta}} & \text{if } t \geq \frac{1}{\beta} \end{cases}$$

$$K_{I_0,W_1}(t) = \int_0^t (1 - F_{I_0,S_1}(x))dF_{I_0,W_1}(x)$$

$$K_{I_0,W_1}(t) = \begin{cases} 1 - e^{-\lambda t} & \text{if } t < \frac{1}{\beta} \\ 1 - e^{-\frac{\lambda}{\beta}} & \text{if } t \geq \frac{1}{\beta} \end{cases}$$

$$K_{S_j,S_{j+1}}(t) = \int_0^t (1 - F_{S_j,W_1}(x))dF_{S_j,S_{j+1}}(x)$$

$$K_{S_j,S_{j+1}}(t) = \begin{cases} 0 & \text{if } t < \frac{1}{\alpha} \\ e^{-\frac{\lambda}{\alpha}} & \text{if } t \geq \frac{1}{\alpha} \end{cases}, j = 1, 2, \ldots, N-1$$

$$K_{S_N,L_0}(t) = \int_0^t (1 - F_{S_N,W_1}(x))dF_{S_N,L_0}(x)$$

$$K_{S_N,L_0}(t) = \begin{cases} 0 & \text{if } t < \frac{1}{\nu} \\ e^{-\frac{\lambda}{\alpha}} & \text{if } t \geq \frac{1}{\alpha} \end{cases}$$

$$K_{W_k,W_{k+1}}(t) = \int_0^t (1 - F_{W_k,W_{k-1}}(x))dF_{W_k,W_{k+1}}(x)$$

$$K_{W_k,W_{k+1}}(t) = \frac{\lambda}{\lambda+\mu}(1 - e^{-(\lambda+\mu)t}), k = 1, 2, \ldots$$

$$K_{W_1,I_0}(t) = \int_0^t (1 - F_{W_1,W_2}(x))dF_{W_1,I_0}(x)$$

$$K_{W_1,I_0}(t) = \frac{\mu}{\lambda+\mu}(1 - e^{-(\lambda+\mu)t})$$

$$K_{W_k,W_{k-1}}(t) = \int_0^t (1 - F_{W_k,W_{k+1}}(x))dF_{W_k,W_{k-1}}(x)$$

$$K_{W_k,W_{k-1}}(t) = \frac{\mu}{\lambda+\mu}(1 - e^{-(\lambda+\mu)t}), k = 2, 3, \ldots$$

$$K_{S_j,W_1}(t) = \int_0^t (1 - F_{S_j,S_{j+1}}(x))dF_{S_j,W_1}(x)$$

$$K_{S_j,W_1}(t) = \begin{cases} 1 - e^{-\lambda t} & \text{if } t < \frac{1}{\alpha} \\ 1 - e^{-\frac{\lambda}{\alpha}} & \text{if } t \geq \frac{1}{\alpha} \end{cases}, j = 1, 2, \ldots, N$$

$$K_{L_i,L_{i+1}}(t) = \int_0^t (1 - F_{L_i,W_k}(x))dF_{L_i,L_{i+1}}(x)$$

$$K_{L_i,L_{i+1}}(t) = \begin{cases} 1 - e^{-\lambda t} & \text{if } t < \frac{1}{\nu} \\ 1 - e^{-\frac{\lambda}{\nu}} & \text{if } t \geq \frac{1}{\nu} \end{cases}, i = 0, 1, \ldots$$

$$K_{L_0,I_0}(t) = \int_0^t (1 - F_{L_0,L_1}(x))dF_{L_0,I_0}(x)$$

$$K_{L_0,I_0}(t) = \begin{cases} 0 & \text{if } t < \frac{1}{\nu} \\ e^{-\frac{\lambda}{\nu}} & \text{if } t \geq \frac{1}{\nu} \end{cases}$$

$$K_{L_i,W_i}(t) = \int_0^t (1 - F_{L_i,L_{i+1}}(x))dF_{L_i,W_i}(x)$$

$$K_{L_i,W_i}(t) = \begin{cases} 0 & \text{if } t < \frac{1}{\nu} \\ e^{-\frac{\lambda}{\nu}} & \text{if } t \geq \frac{1}{\nu} \end{cases}, i = 1, 2, \ldots$$

By the two stage analysis of semi-Markov model, $K_{m,n}(\infty) = \lim_{t\to\infty} K_{m,n}(t)$, where $m, n \in \Omega$, where matrix $K(\infty)$ with entries $K_{m,n}(\infty)$ gives the one step transition probabilities for the EMC(Embedded Markov Chain) of the SMP model and row sum of matrix $K(\infty)$ is 1. Hence, the one step transition probabilities for the model are given as:

$K_{I_0,S_1}(\infty) = e^{-\frac{\lambda}{\beta}}$

$K_{I_0,W_1}(\infty) = 1 - e^{-\frac{\lambda}{\beta}}$

$K_{S_j,S_{j+1}}(\infty) = e^{-\frac{\lambda}{\alpha}}, j = 1, 2, \ldots, N-1$

$K_{S_N,L_0}(\infty) = e^{-\frac{\lambda}{\alpha}}$

$K_{W_k,W_{k+1}}(\infty) = \frac{\lambda}{\lambda+\mu}, k = 1, 2, \ldots$

$K_{W_1,I_0}(\infty) = \frac{\mu}{\lambda+\mu}$

$K_{W_k,W_{k-1}}(\infty) = \frac{\mu}{\lambda+\mu}, k = 2, 3, \ldots$

$K_{S_j,W_1}(\infty) = 1 - e^{-\frac{\lambda}{\alpha}}, j = 1, 2, \ldots, N$

$K_{L_i,L_{i+1}}(\infty) = 1 - e^{-\frac{\lambda}{\nu}}, i = 0, 1, \ldots$

$K_{L_0,I_0}(t) = e^{-\frac{\lambda}{\nu}}$

$K_{L_i,W_i}(\infty) = e^{-\frac{\lambda}{\nu}}, i = 1, 2, \ldots$

The steady-state probabilities for states of EMC, given by vector **H** and is defined as $(H_I, H_{W_1}, \ldots, H_{S_1}, \ldots, H_{S_N}, H_{L_0}, H_{L_1}, \ldots)$ can be obtained by solving [35]:

$$H = HK(\infty), \sum_{m\in\Omega} H_m = 1.$$

Hence, using $H = HK(\infty)$ and the entries of matrix $K(\infty)$, we get the system of linear equations, i.e.,

$H_{I_0} = \frac{\mu}{\lambda+\mu} H_{W_1} + e^{-\frac{\lambda}{\nu}} H_{L_0}$

$H_{W_1} = \left(1 - e^{-\frac{\lambda}{\beta}}\right)H_{I_0} + \frac{\mu}{\lambda+\mu} H_{W_2} + e^{-\frac{\lambda}{\nu}} H_{L_1} + \left(1 - e^{-\frac{\lambda}{\alpha}}\right)(H_{S_1} + H_{S_2} + \ldots + H_{S_N})$

$H_{W_k} = \frac{\lambda}{\lambda+\mu} H_{W_{k-1}} + \frac{\mu}{\lambda+\mu} H_{W_{k+1}} + e^{-\frac{\lambda}{\nu}} H_{L_i}, k = 2, \ldots$

$H_{S_1} = e^{-\frac{\lambda}{\beta}} H_{I_0}$

$H_{S_{j+1}} = e^{-\frac{\lambda}{\alpha}} H_{S_j}, j = 1, 2, \ldots, N-1$

$H_{L_0} = e^{-\frac{\lambda}{\alpha}} H_{S_N}$

$H_{L_{i+1}} = \left(1 - e^{-\frac{\lambda}{\nu}}\right) H_{L_i}, i = 0, 1, \ldots.$

Solving this system of equations, we get the steady-state probabilities of each state, $H_m, m \in \Omega$ in terms of H_{I_0}, i.e.,

$H_{S_j} = e^{-\frac{\lambda}{\beta}} \left(e^{-\frac{\lambda}{\alpha}}\right)^{j-1} H_{I_0}, j = 1, 2, \ldots, N$

$H_{L_i} = e^{-\left(\frac{\lambda}{\beta} + \frac{\lambda N}{\alpha}\right)} \left(1 - e^{-\frac{\lambda}{\nu}}\right)^i H_{I_0}, i = 0, 1, \ldots$

$H_{W_1} = \left(\frac{\lambda+\mu}{\mu}\right)\left(1 - e^{-\left(\frac{\lambda}{\beta} + \frac{\lambda N}{\alpha} + \frac{\lambda}{\nu}\right)}\right) H_{I_0}$

$H_{W_k} = \left(\frac{\lambda+\mu}{\mu}\right)\left[\left(\frac{\lambda}{\mu}\right)^{k-1}\left(1 - e^{-\left(\frac{\lambda}{\beta} + \frac{\lambda N}{\alpha} + \frac{\lambda}{\nu}\right)}\right) + \sum_{j=2}^{k} e^{-\left(\frac{\lambda}{\beta} + \frac{\lambda N}{\alpha}\right)}\left[\left(\frac{\lambda}{\mu}\right)^{k-j}(1 - e^{-\frac{\lambda}{\nu}})^j\right]\right] H_{I_0}, k = 2, \ldots$

Using equation $\sum_{m\in\Omega} H_m = 1$ and the probabilities, $H_m, m \in \Omega$ in terms of H_{I_0}, we obtain

$$H_{I_0} = \frac{1}{D},$$

where,

$$\begin{aligned} D = {} & \frac{2\mu}{\mu-\lambda} - \frac{\lambda+\mu}{\mu-\lambda} e^{-\left(\frac{\lambda}{\beta}+\frac{\lambda N}{\alpha}+\frac{\lambda}{\nu}\right)} \\ & + \frac{e^{-\frac{\lambda}{\beta}} - e^{-\left(\frac{\lambda N}{\alpha}+\frac{\lambda}{\beta}\right)}}{1-e^{-\frac{\lambda}{\alpha}}} + \frac{e^{-\left(\frac{\lambda N}{\alpha}+\frac{\lambda}{\beta}\right)}}{e^{-\frac{\lambda}{\nu}}} \\ & - \frac{\lambda(\lambda+\mu)}{\mu} \frac{e^{-\left(\frac{\lambda N}{\alpha}+\frac{\lambda}{\beta}\right)}(1-e^{-\frac{\lambda}{\nu}})^2}{\left(\frac{\lambda}{\mu}-1+e^{-\frac{\lambda}{\nu}}\right)} \left[\frac{e^{\frac{\lambda}{\nu}}}{\mu} - \frac{1}{\mu-\lambda}\right]. \end{aligned}$$

Let T_m denotes the sojourn time in state $m \in \Omega$. Since, the EMC obtained for this model is irreducible, aperiodic and positive recurrent, i.e, ergodic, hence the steady-state probabilities of each state $m \in \Omega$ of semi-Markov model [35] can be obtained as:

$$\pi_m = \frac{H_m E[T_m]}{\sum_{n\in\Omega} H_n E[T_n]}, m \in \Omega,$$

where $E[T_m]$ denotes the expected sojourn time in state $m \in \Omega$. Then the expected sojourn time spent in each state can be obtained as:

$$\begin{aligned} E[T_{I_0}] &= \int_0^\infty (1-F_{I_0,S_1}(t))(1-F_{I_0,W_1}(t))dt \\ E[T_{I_0}] &= \frac{1-e^{\frac{-\lambda}{\beta}}}{\lambda} \\ E[T_{W_k}] &= \int_0^\infty (1-F_{W_k,W_{k-1}}(t))(1-F_{W_k,W_{k+1}}(t))dt \\ E[T_{W_k}] &= \frac{1}{\lambda+\mu}, k = 1, 2, \ldots \\ E[T_{S_j}] &= \int_0^\infty (1-F_{S_j,W_1}(t))(1-F_{S_j,S_{j+1}}(t))dt \\ E[T_{S_j}] &= \frac{1-e^{\frac{-\lambda}{\alpha}}}{\lambda}, j = 1, 2, \ldots, N-1 \\ E[T_{S_N}] &= \int_0^\infty (1-F_{S_N,W_1}(t))(1-F_{S_N,L_0}(t))dt \\ E[T_{S_N}] &= \frac{1-e^{\frac{-\lambda}{\alpha}}}{\lambda} \\ E[T_{L_i}] &= \int_0^\infty (1-F_{L_i,L_{i+1}}(t))(1-F_{L_i,W_k}(t))dt \\ E[T_{L_i}] &= \frac{1-e^{\frac{-\lambda}{\nu}}}{\lambda}, i = 0, 1, \ldots. \end{aligned}$$

The stability conditions for the existence of steady-state probabilities of the semi-Markov model are given as $\lambda < \mu, \lambda \le \alpha$ and $\lambda < \nu$. Hence, under stability conditions using expected sojourn times and the steady-state probabilities of EMC, we get

$$\sum_{n\in\Omega} H_n E[T_n] = M H_{I_0},$$

where

$$\begin{aligned} M = {} & \frac{\mu}{\lambda(\mu-\lambda)} - \frac{e^{-\left(\frac{\lambda}{\beta}+\frac{\lambda N}{\alpha}+\frac{\lambda}{\nu}\right)}}{\mu-\lambda} - \frac{e^{-\left(\frac{\lambda N}{\alpha}+\frac{\lambda}{\beta}\right)}}{\lambda}\left[2-e^{\frac{\lambda}{\nu}}\right] \\ & - \frac{\lambda}{\mu} \frac{e^{-\left(\frac{\lambda N}{\alpha}+\frac{\lambda}{\beta}\right)}(1-e^{-\frac{\lambda}{\nu}})^2}{\left(\frac{\lambda}{\mu}-1+e^{-\frac{\lambda}{\nu}}\right)} \left[\frac{e^{\frac{\lambda}{\nu}}}{\mu} - \frac{1}{\mu-\lambda}\right]. \end{aligned}$$

Hence, the steady-state probabilities for the semi-Markov model are given as:

$$\begin{aligned} \pi_{I_0} &= \frac{1-e^{-\frac{\lambda}{\beta}}}{\lambda M}, \\ \pi_{W_1} &= \frac{1-e^{-\left(\frac{\lambda}{\beta}+\frac{\lambda N}{\alpha}+\frac{\lambda}{\nu}\right)}}{\mu M}, \\ \pi_{W_k} &= \frac{1}{\mu M}\Bigg[\left(\frac{\lambda}{\mu}\right)^{k-1}\left(1-e^{-\left(\frac{\lambda}{\beta}+\frac{\lambda N}{\alpha}+\frac{\lambda}{\nu}\right)}\right) \\ &\quad + \frac{\left(\frac{\lambda}{\mu}\right)\left(1-e^{-\frac{\lambda}{\nu}}\right)^2}{\left(\frac{\lambda}{\mu}-1+e^{-\frac{\lambda}{\nu}}\right)} e^{-\left(\frac{\lambda}{\beta}+\frac{\lambda N}{\alpha}\right)} \\ &\quad \left[\left(\frac{\lambda}{\mu}\right)^{k-2} - \left(1-e^{-\frac{\lambda}{\nu}}\right)^{k-2}\right]\Bigg], \; k = 2, 3, \ldots \\ \pi_{S_j} &= \frac{e^{-\frac{\lambda}{\beta}}\left(e^{-\frac{\lambda}{\alpha}}\right)^{j-1}\left(1-e^{-\frac{\lambda}{\alpha}}\right)}{\lambda M}, \; j = 1, 2, \ldots, N \\ \pi_{L_i} &= \frac{e^{-\left(\frac{\lambda}{\beta}+\frac{\lambda N}{\alpha}\right)}\left(1-e^{-\frac{\lambda}{\nu}}\right)^{i+1}}{\lambda M}, \; i = 0, 1, \ldots \end{aligned}$$

The steady-state probabilities $\pi_m \forall\, m \in S$ exist as $\lambda < \mu$, $\lambda \le \alpha$ and $\lambda < \nu$.

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