

Partial Observed Fluid Queue Model for Rechargeable Batteries

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ABSTRACT

A stochastic fluid model is presented to study the charge level of a rechargeable battery of finite capacity. The net inflow rate is not always fully observed, since it depends on environmental factors and on the use, users have made of their devices. To describe this situation, the parameters determining the rate of the charging and discharging processes are assumed to be random variables with only partially observed laws. Thus, the transient analysis of the charge level of a battery is obtained and discussed using stochastic filtering techniques on a stochastic fluid model.

CCS CONCEPTS

• **Mathematics of computing** Markov processes; • **Mathematics of computing** Queueing theory;

KEYWORDS

Rechargeable Batteries, Fluid Queues, Partial Information, Filtering, Transient Distributions

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1 INTRODUCTION

The rechargeable batteries are used in mobiles, digital cameras, remote sensors and communication satellites and have emerged as the basic commodity in recent years. The major constraints for the users are the limited quantity of the stored energy of a battery and the short battery life due to use of various applications in the device. Refer [5–7] for the stochastic models concerning batteries of mobile communication devices transmitting the packets. To design an energy aware device, we have to learn how energy is being consumed by the hardware components. Improving battery life and battery capacity are features closely related to better understanding of user's interaction with his devices. Hardware resource monitoring have proved to offer valuable information in the field of performance analysis.

In the literature, fluid queue models have been applied to various real life situations and provide extensive information, like, for instance, in modeling the performance of a network switch, a router, the IEEE 802.11 protocol, [3], and

Asynchronous Transfer Model, [18]. Also, there are many applications that helps in understanding the bursty nature of traffic in high speed networks, [11], and in the area of telecommunications, [2]. The life of a battery using fluid queues was already modelled in [5]. Various methodologies have been studied in literature to obtain the transient buffer content distribution of fluid queues. Regarding the transient solution of fluid queues driven by a Markov process and using recurrence relations, we quote [15, 16]. Concerning fluid queues driven by $M/M/1$ queue, [17] represents the exact transient solution in terms of functions satisfying recurrence relations. Assuming that the input and output rates are modulated by a finite state irreducible Markov process, [12] analyzed transient and asymptotic behavior of general Markov fluid queue models. [10] analyzed a fluid queue driven by $M/M/1/N$ queue, obtaining a closed form transient solution, for specific values of net input rates. The transient buffer content distribution is derived in [13], for a fluid model with background birth-death process having linear state dependent rates.

The battery charge of a rechargeable device is a continuous quantity. The system is represented as a reservoir where battery charge get accumulated or is depleted gradually over time, subject to a random environment of background process. Thus, the battery charge available, at any time t , is considered as the fluid available in the reservoir.

In addition, we have to note that, in the real life situations, the intensities of the charge and discharge processes depend on several conditions, like weather, humidity and/or temperature, and habits of the different users. Not all of these factors can be fully observed. Therefore, different discharging and charging rates could be considered. To approach this task in this paper, instead of considering several different level of charge in a battery, the charge level is modelled as a fluid queue having intensities driven by some random variables having probability laws which are only partially observable. In order to take information on these laws, we introduce a stochastic filtering problem.

Inspired by the model discussed in [9], for a full information case, in this paper, we find a closed form solution for the amount of charge in a battery at any time. It is obtained for some fixed values of the random variables driving the intensities of the charge and discharge processes. Then, we deduce the conditional distribution of these random variables by using a classical stochastic filtering approach, [4, 19–21]. To the best of our knowledge, this is the first time in literature, that a procedure using fluid queues and stochastic filtering techniques is presented.

The paper is organized as follows. Sect. 2 gives a general description of the model and a transient analysis for it for fixed values of the random variables driving the intensities

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of the charge level. A closed form expression for the distribution law of the fluid queue is presented in Sect. 3. In Sect. 4, the continuous time filtering problem is set, the filtering equation is derived and a result of uniqueness is presented. Then, we give a recursive structure for the filter. Finally, Sect. 5 provides conclusions with future directions.

2 THE MODEL

At any time t , the level of charge in a rechargeable battery is a continuous quantity, say $C(t)$. Assume that the battery has a finite capacity B , $B \in \mathbb{R}^+$, $C(t) \in [0, B]$, bounded interval.

A stochastic fluid model is investigated. The background process is determined by a switch, which means that we have to deal with an on-off model. When the switch is on, the battery gets charged at a non negative rate α , the rate of charge of the battery. Otherwise, the battery gets discharged. The rate of discharge β is non negative and depends also on the level of charge in the battery.

In order to take into account the randomness of the scenario, instead of considering a fluid model with several states, we assume only two states with rates α and β as random variables. There are many papers in literature which concern with fluid queues but under complete information. The novelty of the present paper lies in the fact that we consider a partial observable scenario, in the sense that the law of α and β is assumed to be unknown. The observation is the level of $C(t)$ and to get information about the law of α and β , we have to solve a stochastic filtering problem.

To achieve this program, first step is studying the fluid queue model for fixed values of α and β , inspired by the procedure presented in [15, 16]. Then, the second step is in deriving the conditional law of α and β by setting a stochastic filtering problem, performed in Sect. 4. Thus, from now on, in this Section and in Sect. 3, α and β are assumed to be fixed positive constants.

The state of the background process is represented by $\{S(t), t \geq 0\}$, which take values in $\{0, 1\}$ and $Q = [q_{ij}]$, $q_{00} = -\lambda = -q_{01}$ and $q_{10} = \mu = -q_{11}$. Without loss of generality, λ , the birth rate, and μ , the death rate, are assumed such that $\lambda > \mu$. The fluid model is represented by the pair $\{(S(t), C(t)), t \geq 0\}$ and, for $t \geq 0$, $x \in [0, B]$ and $i \in \{0, 1\}$, the corresponding distribution function is

$$F_i(t, x) = P[S(t) = i, C(t) \leq x].$$

For $S(t) = i$, $i \in \{0, 1\}$, the net inflow rate r_i is defined as

$$r_i = (\alpha - \beta)i - \beta(1 - i).$$

The system of Kolmogorov forward differential equations governing the Markov process $\{S(t), C(t), t \geq 0\}$, [18], is given by

$$\frac{\partial F_i(t, x)}{\partial t} + r_i \frac{\partial F_i(t, x)}{\partial x} = \sum_{j \in \{0, 1\}} q_{ij} F_j(t, x),$$

and, as usual, $P[C(t) > x] = 1 - F(t, x) = 1 - \sum_{i \in \{0, 1\}} F_i(t, x)$.

In order to evaluate $P[C(t) > x]$, let $Z = \{Z_n : n = 0, 1, \dots\}$, the so-called process of uniformization, a discrete time Markov chain with finite state space $\{0, 1\}$, and one-step transition probability matrix is $P = I + \frac{Q}{\lambda + \mu}$. Let $\{N(t) : t \geq 0\}$ be a Poisson process with rate $\lambda + \mu$, and independent of Z , such that $S(t) = Z_{N(t)}$.

If the uniformized background process has n transitions during $[0, t]$, i.e., $N(t) = n$, at times $0 < \tau_1 < \tau_2 < \dots <$

$\tau_n < t$, the interval $[0, t]$ is split into $n + 1$ subintervals with lengths $L_1 = \tau_1, L_2 = \tau_2 - \tau_1, \dots, L_{n+1} = t - \tau_n$. For $i \in \{0, 1\}$, the number of intervals associated with the net input rate r_i is $V_n^i = \sum_{k=0, \dots, n} \mathbb{I}_{Z_k=i}$.

Let k_i be the number of intervals associated with r_i . Then $\mathbf{k} = (k_0, k_1)$ is a partition of $n + 1$, and $\|\mathbf{k}\| := k_0 + k_1 = n + 1$. Conditioning on the number of transitions n and on the partition $V_n = \mathbf{k}$ for $V_n = (V_n^0, V_n^1)$, and $\mathbf{k} = (k_0, k_1)$,

$$\begin{aligned} P[C(t) > x] &= \sum_{n=0}^{\infty} \sum_{\|\mathbf{k}\|=n+1} P[N(t) = n, V_n = \mathbf{k}] \\ &\quad P[C(t) > x | N(t) = n, V_n = \mathbf{k}] \\ &= \sum_{n=0}^{\infty} e^{-(\lambda+\mu)t} \frac{[(\lambda+\mu)t]^n}{n!} \sum_{k=1, \dots, n+1} G[n, k] M(t, x, n, k). \end{aligned}$$

In particular,

$$M(t, x, n, k) = P[C(t) > x | N(t) = n, V_n = (k, n - k + 1)],$$

$$\begin{aligned} G[n, k] &= P[V_n = (k, n - k + 1) | N(t) = n] \\ &= G_0[n, k] + G_1[n, k], \end{aligned} \quad (1)$$

where $G_i[n, k]$ is the probability of the partition to be $(k, n - k + 1)$ if the number of transitions are n and the state visited after the last transition is i .

Thus, given $G_0[0, 1] = 1, G[0, 0] = 0$, we have

$$\begin{aligned} G_0[n, k] &= \{G_0[n - 1, k - 1] + G_1[n - 1, k - 1]\} \frac{\mu}{\lambda + \mu}, \\ G_1[n, k] &= \{G_0[n - 1, k] + G_1[n - 1, k]\} \frac{\lambda}{\lambda + \mu}. \end{aligned}$$

3 CLOSED FORM EXPRESSIONS OF G , M AND F FOR FIXED α AND β

For $n \in \mathbb{N}$, $n \geq 1$, a closed form for G_i , $i \in \{0, 1\}$, is derived recursively, since $G_0[n, 1] = G_1[n, n + 1] = 0$, $G_1[n, 1] = \left(\frac{\lambda}{\lambda + \mu}\right)^n$, $G_0[n, n + 1] = \left(\frac{\mu}{\lambda + \mu}\right)^n$, and, for $k = 2, \dots, n$,

$$\begin{aligned} G_0[n, k] &= \binom{n-1}{k-2} \frac{\lambda^{n-k+1} \mu^{k-1}}{(\lambda + \mu)^n}, \\ G_1[n, k] &= \binom{n-1}{n-k} \frac{\lambda^{n-k+1} \mu^{k-1}}{(\lambda + \mu)^n}. \end{aligned}$$

Consequently, for $k = 1, 2, \dots, n, n + 1$,

$$G[n, k] = \binom{n}{k-1} \frac{\lambda^{n-k+1} \mu^{k-1}}{(\lambda + \mu)^n}. \quad (2)$$

To compute $M(t, x, n, k)$, let $U_1, U_2, \dots, U_n$ be independent random variables, uniform distributed on $(0, 1)$. Let their order statistics be defined by setting $U_{(0)} = 0, U_{(n+1)} = 1$, and $U_{(1)} \leq U_{(2)} \leq \dots \leq U_{(n)}$. Then, τ_i , the time of the i th transition has the distribution as $tU_{(i)}$, see [14]. Now, $\{L_i\}_{i=1, \dots, n+1}$ defined by setting

$$L_1 \equiv tU_{(1)}, L_2 \equiv t(U_{(2)} - U_{(1)}), \dots, L_{n+1} \equiv t(1 - U_{(n)}),$$

is a sequence of exchangeable random variables, and the background process is $M/M/1/1$ queue. The time spent in any state follows exponential property. On rearranging the $n + 1$ intervals such that the first k are associated with the rate r_0 , and the next $n - k + 1$ with r_1 ,

$$\begin{aligned} M(t, x, n, k) &= P[C(t) > x | N(t) = n, V_n^0 = k, V_n^1 = n - k + 1], \\ &= P[(r_0 - r_1)tU_{(k)} + r_1 t > x]. \end{aligned}$$

Since there are $n+1$ intervals, $k \in \{1, 2, \dots, n, n+1\}$. Recalling that α and $\beta \in \mathbb{R}^+$, $r_1 = \alpha - \beta$ and $r_0 = -\beta$, then $r_1 \geq r_0$ and

$$M(t, x, n, n+1) = P[r_0 t > x] = 0.$$

If $U_1, U_2, \dots, U_n$ are independent random variables uniform on $(0, 1)$, for $k = 1, \dots, n$, the k^{th} order statistics $U_{(k)}$ is a Beta random variable of parameters k and $n - k + 1$, $\beta(k, n - k + 1)$. Therefore, for $t \geq 0$ and $-\beta t \leq x < (\alpha - \beta)t$

$$M(t, x, n, k) = F_{Beta(k, n-k+1)}\left(1 - \frac{x + \beta t}{\alpha t}\right).$$

Note that, recursively,

$$\int_0^y v^{k-1} (1-v)^{n-k} dv = \frac{1}{\binom{n}{k} k} \sum_{i=k}^n \binom{n}{i} y^i (1-y)^{n-i}.$$

Setting, as usual, for $u \in \mathbb{R}$, $\Gamma(u) = \int_0^{+\infty} v^{u-1} e^{-v} dv$, for $x \in (-\beta t, (\alpha - \beta)t)$ and $k = 1, 2, \dots, n$,

$$M(t, x, n, k) = \sum_{i=k}^n \binom{n}{i} \left(1 - \frac{x + \beta t}{\alpha t}\right)^i \left(\frac{x + \beta t}{\alpha t}\right)^{n-i}.$$

Summing up,

$$F(t, x) = \sum_{n=1}^{\infty} e^{-(\lambda+\mu)t} \frac{[(\lambda+\mu)t]^n}{n!} \sum_{k=1}^n G[n, k] M(t, x, n, k).$$

4 A FILTERING PROBLEM

In the framework of the study of a rechargeable batteries of finite capacity, the charge or discharge processes depend on several external conditions, like humidity, temperature, or on how the user uses his device. On the other hand, the distribution of the fluid queue model, $F(t, x)$ depends on the variables t and x , on the parameters λ and μ , and on the random variables α and β . This is a system with behaviour that can only be partially observed by the users, in the sense that the probability law of α and β is unknown. The problem lies in extracting informations related to the system by just observing the level of charge in the battery of the device. Usually, this level is displayed in percentage terms, hence, we have to consider a discretized fluid process or a, so called, quasi birth death process. Several papers concern with this task, see between others [1], and references therein.

Thence, we consider $\{Y(t), t \geq 0\}$, a stochastic jump process, taking values in $\Theta = \{1, \dots, 99, 100\}$, nonexplosive, with initial condition $Y(0) = y_0 \in \Theta$. This is an observable process, and, for each time t , and $y \in \Theta$,

$$Y(t) = y \iff \frac{(y-1)B}{100} < C(t) \leq \frac{yB}{100}.$$

To build a formal problem, let $(\Omega, \{F_t\}, P)$ be a probability space equipped with an increasing family of σ -algebras satisfying the standard properties. On this space, let α and β be random variables, such that, $0 \leq \alpha \leq A$ and $0 \leq \beta \leq B$, for some real positive constants A and B . The σ -algebra F_t contains all the informations related to the random variables α and β , but it is not directly observable. All the available information is contained in the σ -algebra generated by the history of the process Y until the time t , defined as $F_t^Y = \sigma\{Y(s), s \leq t\}$ and $F_t^Y \subseteq F_t$.

The structure of the F_t -intensity function of the process Y is given by the Proposition 4.1 below.

PROPOSITION 4.1. *Given the parameters λ and μ , the intensity Λ_t of the process $\{Y(t), t \geq 0\}$, is a function of the time t , of the random variables α and β , $\Lambda_t = \Lambda_t(\alpha, \beta)$, where*

$\Lambda_t(\cdot)$ is a bounded deterministic function which is explicitly computable.

PROOF. Since, for $y \in \Theta$,

$$\begin{aligned} P[Y(t) = y] &= P\left[y - 1 < \frac{C(t)}{B} 100 \leq y\right] \\ &= F\left(t, y \frac{B}{100}\right) - F\left(t, (y-1) \frac{B}{100}\right), \end{aligned}$$

by the expression obtained for $F(t, x)$ and the continuity of F in t ,

$$\begin{aligned} \Lambda_t &= \lim_{\delta \rightarrow 0} \frac{1}{\delta} E[Y(t+\delta) - Y(t)], \\ &= \sum_{y \in \Theta} y \frac{\partial}{\partial t} \left[F\left(t, y \frac{B}{100}\right) - F\left(t, (y-1) \frac{B}{100}\right) \right]. \end{aligned}$$

Taking into account that, for $n = 1, 2, \dots$ and $k = 1, 2, \dots, n$

$$\begin{aligned} &\frac{\partial}{\partial t} M(t, x, n, k) \\ &= \frac{nx}{\alpha^n t^{n+1}} \binom{n-1}{k-1} [(\alpha - \beta)t - x]^{k-1} (x + \beta t)^{n-k}, \end{aligned}$$

the claim follows just evaluating

$$\frac{\partial}{\partial t} \left[F\left(t, y \frac{B}{100}\right) - F\left(t, (y-1) \frac{B}{100}\right) \right].$$

□

Since, the conditional law of $C(t)$, given α and β , $L(C(t)|\alpha, \beta)$, was obtained previously. Now, we have to set up a stochastic filtering problem in which we derive the conditional distribution of the random variables α and β , given the observation $\{Y(t) : t \geq 0\}$.

Hence, our knowledge reduces to the filter $\pi_t(g)$, which is the cadlag version of the conditional distribution of any measurable bounded function g of α and β on $[0, A] \times [0, B]$, given F_t^Y , i.e., $\pi_t(g) = E[g|F_t^Y]$. Let π_0 denotes their joint initial probability measure and let Π be the set of the probability measures on $[0, A] \times [0, B]$, endowed with the weak topology. Consequently,

$$P\left[C(t) \leq x | F_t^Y\right] = \int_{[0, A] \times [0, B]} P[C(t) \leq x | \alpha, \beta] \pi_t(d\alpha, d\beta),$$

and a stochastic problem with partial observation arises.

4.1 Filtering Equation

By the minimal representation for the intensity Λ of Y , we get

$$\pi_t(\Lambda) = E[\Lambda_t(\alpha, \beta) | F_t^Y] = \int_{[0, A] \times [0, B]} \Lambda_t(a, b) \pi_t(da, db).$$

To evaluate the quantities related to the present model, instead of using the knowledge of the parameters α and β , we have to consider their conditional distribution π , which coincides with the required filter, one of the solutions of the Kushner-Stratonovich equation, stated in Equation (3).

THEOREM 4.2. *For any g , measurable bounded function on $[0, A] \times [0, B]$, setting*

$$\Phi_s(g\Lambda) = \pi_s(g\Lambda) - \pi_s(g)\pi_s(\Lambda),$$

the Kushner-Stratonovich equation is

$$\pi_t(g) = \pi_0(g) + \int_0^t [\pi_s(\Lambda)]^+ \Phi_s(g\Lambda) [dY(s) - \pi_s(\Lambda)ds] \quad (3)$$

denoting by $[z]^+ = 1/z$ if $z \neq 0$, and $[z]^+ = 0$ otherwise.

The proof can be obtained by a standard procedure applying the classical innovation method, refer to [4, 19, 21]. The uniqueness for the solution to (3) holds by Theorem 4.3 below.

THEOREM 4.3. *The Kushner-Stratonovich equation (3) admits a unique pathwise solution, necessarily F_t^Y -adapted with paths in the space of the right continuous with left limits functions $D_{\Pi}[0, T]$.*

PROOF. Inspired by the results obtained in [19, 21], let $\{\tau_n\}_{n \geq 0}$ be the sequence of the jump times of the process Y . At any τ_n , the filter is uniquely determined by the knowledge of π_{τ_n-} , i.e.

$$\pi_{\tau_n}(g) = [\pi_{\tau_n-}(\Lambda)]^+ \pi_{\tau_n-}(g\Lambda). \quad (4)$$

For $t \in [\tau_n, \tau_{n+1})$,

$$\pi_t(g) = \pi_{\tau_n}(g) + \int_{\tau_n}^t \{\pi_s(g)\pi_s(\Lambda) - \pi_s(g\Lambda)\} ds. \quad (5)$$

Given π and π' , any two solutions of (5) such that $\pi_{\tau_n}(g) = \pi'_{\tau_n}(g)$ and denoting by $\|\cdot\|$, the bounded variation norm,

$$\|\pi_t(g) - \pi'_t(g)\| \leq 3\|g\|\|\Lambda\| \int_{\tau_n}^t \|\pi_s - \pi'_s\| ds.$$

Hence, (5) is Lipschitz with respect to the bounded variation norm. Consequently, the solution at the jump time is uniquely determined by (4), while, in between the jump times, the solution is unique and necessarily F_t^Y -adapted using the Lipschitz property. $\square$

4.2 The Recursive Structure

The Kushner-Stratonovich equation has a natural recursive structure, following the sequence of the jump times $\{\tau_n\}_{n \geq 0}$. The structure of the equation shows that $\pi_t(g)$ has a deterministic behaviour between two consecutive jump times of Y , while at any jump time of Y , say τ_n , $\pi_t(g)$ jumps and its jump size is given by (4).

Therefore, if, at time $\tau_0 = 0$, we set π_0 as the joint initial probability measure of the random variables α and β , then, for $t \in [\tau_n, \tau_{n+1})$ and $n \geq 1$,

$$\begin{aligned} \pi_t(da, db) &= \pi_{\tau_n}(da, db) \\ &+ \int_{\tau_n}^t \{\pi_s(\Lambda) - \Lambda_s(a, b)\} \pi_s(da, db) ds, \end{aligned} \quad (6)$$

and at the jump time τ_n ,

$$\begin{aligned} \pi_{\tau_n}(da, db) &= (\pi_{\tau_n-}(\Lambda))^+ \Lambda_{\tau_n-}(a, b) \pi_{\tau_n-}(da, db) \\ &= \frac{\Lambda_{\tau_n-}(a, b) \pi_{\tau_n-}(da, db)}{\int_{[0, A] \times [0, B]} \Lambda_{\tau_n-}(a, b) \pi_{\tau_n-}(da, db)}. \end{aligned}$$

5 CONCLUSIONS

In this paper, we have analyzed the application of a partial observed fluid queue model for rechargeable batteries. There are several papers dealing with the area of rechargeable batteries, but, to the best of our knowledge, this is the first time that a partially observed scenario has been investigated.

Inspired by the model discussed in [9], for a full information case, starting from that procedure, we assume that the parameters driving the intensities of the charge and discharge processes are random variables. Then, we deduce a probabilistic prediction of their law by the observations.

To this end, first step is to assume some fixed values of the random variables driving the intensities of the charge and discharge processes. Then, we find a closed form solution for the amount of charge in a battery at any time. As the second step, we deduce the conditional law of these random variables by using a classical stochastic filtering approach, [4]. The representation and the approximation of the filter, which is the request conditional law, can be obtained, for instance, see [19–21] and references therein. In addition, asymptotic results about the fluid queue model can be obtained, for instance, see results obtained in [8].

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