



Reliability Analysis and Effect of Propagation Failure on High Altitude Platforms in Wireless Communication

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Abstract

With the rapid advancements in Internet of Things, high-altitude platforms (HAPs) are revolutionizing various sectors, particularly in the realm of communication networks. As HAPs continue to exert their transformative influence across multiple domains, the aspect of reliability assumes a pivotal role in determining their triumph and widespread acceptance. The proposed study presents a comprehensive reliability analysis of HAPs using stochastic modeling. The investigation focuses on two distinct types of failures affecting a HAP: local failure (LF) and propagation failure (PF). To assess the reliability of the system employing a redundant HAP in the presence of these two failure types, a stochastic model has been developed. Considering the non-Markovian nature of the PF, the proposed model incorporates Markov and semi-Markov processes. Through extensive numerical examples, the presented study explores the influence of failure propagation probability on system reliability and throughput. This analysis provides insights into how the system's reliability and overall performance are affected by the likelihood of failure. Furthermore, the study offers a comparative analysis of the results obtained from the Markov and semi-Markov models which provides a deeper understanding of the strengths and limitations of each modeling approach, enabling them to make more informed decisions in practice.

Keywords Backup server · High altitude platforms · Local failure · Propagation failure · Reliability characteristics · Semi-Markov process · Transient analysis

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1 Introduction

The increasing need for fast, wide-bandwidth communication has captured the attention of researchers who are working on creating inventive solutions for the future of wireless communication. Optical wireless communication has arisen as a hopeful option that is free from protocol constraints, feasible, secure, and simple to implement, rendering it highly suitable for fulfilling the demands of upcoming communication technologies. Free-space optics (FSO) plays a pivotal role in optical wireless communication. Nonetheless, because it relies on direct line-of-sight connections, the dependability of FSO systems is markedly influenced by atmospheric conditions, limiting terrestrial link distances to only a few kilometers. To address this challenge, relaying techniques have been employed to mitigate the effects of atmospheric fading [1, 4]. FSO communication systems currently employ both serial (multi-hop) and parallel (cooperative) relaying methods to improve their performance.

However, an alternative approach involves leveraging high-altitude platforms (HAPs) positioned within the stratosphere, specifically between 17 and 32 kilometers [15, 29]. HAPs provide advantages such as increased distance capabilities, high capacity, low transmission delay, and acceptable power consumption. This makes HAPs a promising solution to overcome the limitations of terrestrial communication, enabling longer and more reliable communication links for various applications.

Reliability holds immense significance in the context of HAPs for a variety of reasons. To achieve the desired level of reliability, it is crucial to dedicate meticulous attention to various aspects of HAP design, manufacturing, and operation. This entails employing robust materials, incorporating redundant systems, implementing effective maintenance protocols, and conducting comprehensive testing and quality assurance procedures. Furthermore, continuous monitoring, data analysis, and predictive maintenance techniques play a vital role in maintaining reliability. These approaches aid in the early detection and resolution of potential issues before they escalate into major failures.

By proactively identifying and addressing these concerns, HAP operators can significantly improve the overall reliability of the system. In conclusion, as HAPs continue to revolutionize multiple domains, their reliability becomes a pivotal factor in ensuring their success and widespread adoption. By delivering consistent and dependable performance, HAPs can fulfill their intended missions while simultaneously enhancing operational efficiency and minimizing risks in challenging environments. The commitment to reliability not only inspires confidence in the users and stakeholders but also unlocks the full potential of HAPs in various applications. An extensive literature devoted to the study of the reliability of HAPs can be found in [35].

HAP is vulnerable to different types of failures, including local failures (LF) and propagation failures (PF). When an LF happens, it only disrupts the specific component that's affected. However, a PF has the potential to cause widespread damage across the entire system. In a communication system aided by a HAP where a PF is a factor, the impact of the PF on the source node depends on when it occurs. If the PF, like a virus, emerges before the HAP's LF, it triggers a cascading effect, impacting other parts of the communication network. In this scenario, the PF spreads and disrupts the operation of the entire system. Conversely, if the LF of the HAP occurs prior to the emergence of the source's PF, there is a probabilistic disconnection of communication from the source. As a result, the PF from the source node is prevented from affecting the rest of the system.

Therefore, the timing and sequence of LF and PF events in the HAP-aided communication system play a crucial role in determining the extent of failure propagation and the overall system performance. Understanding and managing these dynamics are important for ensuring the reliability and uninterrupted operation of the communication network [16]. FSO technology facilitates inter-HAP links between different HAPs within a network. Consequently, in the event of a failure of a HAP, a redundant HAP is deployed to seamlessly take over and continue providing the required services. The FSO technology enables reliable and high-speed communication between HAPs, allowing for efficient coordination and connectivity within the network.

When a HAP encounters a failure, such as a malfunction or outage, the redundant HAP is designed to swiftly and automatically step in and assume the responsibilities of the failed HAP. This redundancy ensures uninterrupted service provision and helps maintain the overall network performance and connectivity. By employing redundant HAPs and leveraging FSO technology, the network can effectively manage and mitigate the impact of failures on the system. The seamless transition from the failed HAP to the redundant HAP ensures continuous operation and minimizes disruptions, enhancing the reliability and availability of the network as a whole [30]. In the proposed modeling framework, the secondary HAP is assumed to operate in a cold standby configuration. This choice is motivated by practical considerations: keeping the redundant HAP in cold standby conserves energy, minimizes operational costs, and extends the service lifetime of the redundant platform. A hot standby configuration, in contrast, would require the redundant HAP to remain continuously active alongside the primary, which—although enabling faster failover—would significantly increase energy consumption, coordination complexity, and system cost. The cold standby assumption thus provides a more realistic balance between reliability enhancement and operational feasibility in large-scale HAP deployments.

The pictorial representation of the proposed model is given in Fig. 1.

There are a number of research studies conducted on the reliability analysis of HAPs. Although Markov models have been widely employed for reliability analysis of commu-

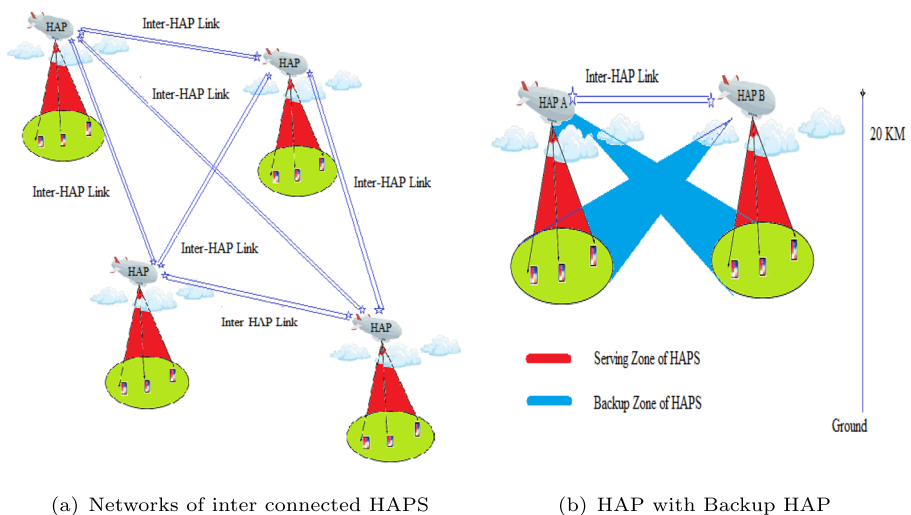


Fig. 1 Pictorial representation of the proposed model

nication and aerospace systems, they are inherently restricted by the assumption of exponentially distributed sojourn times. This assumption implies a constant failure rate, which often does not hold in practice. In the context of HAPs, certain failure mechanisms—particularly propagation failures—are progressive in nature and exhibit time-dependent hazard rates rather than memoryless behavior. Empirical studies in related aerospace domains (e.g., [16, 27]) have demonstrated that component lifetimes are more accurately represented by non-exponential distributions such as the Weibull or gamma distributions, reflecting wear-out and degradation effects. To capture these realistic dynamics, it becomes necessary to move beyond standard Markov modeling and employ semi-Markov processes, which allow for general lifetime distributions while retaining analytical tractability. By incorporating a gamma-distributed sojourn time for propagation failures, the proposed semi-Markov framework provides a closer approximation to real-world HAP operations and yields more reliable performance insights than a purely Markovian model.

1.1 Related Work

Researchers are more interested in analysing the reliability, performability, and energy analysis of HAPs [15, 23, 31]. The reliability of HAPs has been extensively analyzed through the utilization of k -out-of- n systems in numerous studies. One crucial application within this field involves examining the reliability of rotor-craft platforms [14, 20, 32]. These studies focus on evaluating the reliability of platforms with a multi-rotor architecture, enabling them to sustain operation even when $k - 1$ out of the n rotary-wing engines experience failures. This analysis provides valuable insights into the efficiency and dependability of rotor-craft platforms, contributing to advancements in HAP technology.

Dharmaraja et al. [5] proposed a Markov model design for determining the tether's reliability of a tethered HAP. They looked at the reliability of the Markov model by supposing that the tether consists of a number of wires, the failure rates of which depend on the number of failed wires. In a recent study by Wu et al. [34], a novel approach to reliability analysis was introduced. The proposed method involves consecutive- k -out-of- r -from- n subsystems, specifically focusing on a balanced system with load sharing. This innovative approach considers both linear and circular structures when evaluating the reliability of the system. Truong et al. [30] proposed an algorithm to identify a set of necessary HAPs and designed the HAP network topology to analyse the reliability of the system. In their study, Sharma et al. [27] put forward a network configuration for HAP that emphasizes both high capacity and reliability.

This configuration utilizes free-space optical links, enabling seamless communication between ground stations and consecutive HAPs positioned in the stratosphere. The researcher in [8] concentrated on theoretically deriving a channel model tailored for the communication link between HAP and terrestrial mobile users or stations. The channel model aimed to provide a mathematical framework for understanding and analyzing the characteristics of the communication channel in this context. Levitin et al. [17] examined heterogeneous fault-tolerant systems following a 1-out-of- n :G design. These systems incorporate a combination of hot and cold standby redundancies to strike a balance between the costs associated with restoration and operation of standby elements.

The research paper [14] presented a model that utilizes a multidimensional Markov process to analyze and evaluate the reliability characteristics of a tethered HAP. The platform is

specifically based on a hexacopter configuration, and the model aims to assess its reliability in various operational scenarios. The author of [32] investigated the reliability of the flight module in a tethered HAP using the k -out-of- n : F model. Various versions of this model were examined, including one that incorporates diverse failure logic, which accounts for the dependence of system failures on the configuration of failed components. The analysis aimed to assess the reliability and functioning of the flight module under different scenarios, considering the complex interplay of component failures and their impact on overall system performance.

The article [33] proposed a reliability analysis for a balanced system that encompasses multiple sequential stages and consists of several multi-state components. Li et al. [18] conducted an investigation on a k -out-of- n system comprising independent exponential components. The study assumed that when the system experienced a failure, some working components were immediately suspended, and repair operations commenced promptly upon component failure.

While several previous studies have advanced the understanding of HAP reliability, they generally exhibit significant limitations. In particular, most existing work does not explicitly distinguish between different types of failures that affect HAPs, nor does it adequately incorporate redundancy, which is a critical aspect of practical deployment. To the best of the authors' knowledge, no comprehensive study has been reported that simultaneously models LF and PF while accounting for the interdependent behavior of redundant HAPs. Furthermore, the majority of existing approaches rely on oversimplified assumptions, such as exponentially distributed failure times, which neglect the inherently non-Markovian characteristics of certain failure mechanisms—especially propagation failures. Such simplifications restrict the accuracy of reliability predictions and may lead to misleading assessments of system performance in real-world settings.

Therefore, there is a clear need for a more rigorous modeling framework that captures both Markovian and semi-Markovian dynamics, incorporates the probability of failure propagation, and evaluates the impact of redundancy on overall system reliability and throughput. Addressing this gap would provide a more realistic and holistic reliability analysis of HAPs, offering valuable insights for the design of resilient communication networks and other mission-critical applications enabled by these platforms.

The following Table 1 provides a comparison of the underlying model with the existing literature.

1.2 Contribution

The proposed study investigates a reliability model for HAP. The existing literature proposes an architecture for HAPs in which two HAPs work together in tandem, serving as backups for each other [30]. This configuration ensures that if one HAP experiences a failure, the other HAP can seamlessly take over its functions and continue providing services. But, the reliability analysis of the proposed architecture has not been conducted. Moreover, the proposed stochastic approach is the kind of first to consider two types of failures: LF and PF of the HAP, while also incorporating redundancy. In the event of an LF, a few components of the HAP may malfunction, but they are promptly repaired, allowing the service to continue without interruption. However, in the case of a PF, the HAP experiences a gradual failure of its components, eventually leading to a complete breakdown of the system

Table 1 Comparison of existing literature with underlying study

Reference	Type of system	SMP analysis	Simulation results	Back-up HAP
D'Amico et al. [9]	k -out-of- n : F model for reliability analysis of HAP	Yes	No	No
Dong et al. [7]	Multiple HAP coverage is analyzed with link geometry	No	Yes	No
Jain et al. [12]	Reliability analysis of multiple HAPs system	Yes	No	No
Gautam and Jain [10]	Fluid queue model to illustrate HAPs Lithium Ion (Li-ion) battery reliability	No	No	No
Vishnevsky et al. [31]	k -out-of- n : F model for reliability analysis of HAP	No	No	No
Vishnevsky et al. [32]	Heterogeneous k -out-of- n system for reliability analysis of HAP	No	No	No
Choudhary et al. [3]	Dependability analysis of HAP VoIP network using semi-Markov process	No	Yes	No
Dharmaraja et al. [5]	Reliability analysis of tether in tethered HAP	No	No	No
Raj [21]	Performance analysis of HAP using Markov regenerative process	No	Yes	No
Raj and Dharmaraja [22]	Performance analysis of HAP using semi-Markov process	Yes	Yes	No
Raj et al. [24]	Dependability analysis of HAP using semi-Markov process	Yes	Yes	No
Dharmaraja et al. [6]	Reliability analysis of HAP using phased mission systems	No	No	No
Mittal et al. [19]	A k -out-of- n : system is proposed to assess the reliability of a HAPs	Yes	No	No
Proposed study	Reliability analysis of HAP considering different failure scenarios	Yes	Yes	Yes

and the cessation of service provision. Thus, the system will shut down and stop working. Therefore, the service will be handed over to a redundant HAP in the region. To address this scenario, a redundant HAP is employed, which takes over the service provision in the event of a PF. The redundant HAP continues to provide service to the packets, albeit at a degraded service rate. Once the repair process of the main HAP is completed, the redundant HAP ceases its operation, and the main HAP resumes its normal functioning. The deployment of redundant HAPs is technically feasible and has already been envisioned in real-world architectures, such as tandem HAP configurations and balloon-based platforms (e.g., Project Loon). Advances in solar-powered HAPs and FSO inter-HAP links make seamless service migration between primary and backup platforms increasingly practical.

It is important to consider that the transitions between different states within a system can be influenced by various network parameters and factors, resulting in random behaviour. To capture this randomness, stochastic models such as continuous time Markov chain (CTMC) and semi-Markov process (SMP) are assumed, as described in [2]. In one of the models discussed, all states in the system have lifetimes that follow exponential distributions. In the second model, however, the lifetimes of certain states are modeled using non-exponential distributions. To represent these characteristics, the first model is constructed using a CTMC, while the second model utilizes an SMP. It is assumed that propagation failures' lifetimes

will follow gamma-distribution to capture the non-Markovian nature of gradual degradation and wear-out processes. Unlike the exponential distribution, which assumes a constant failure rate, the gamma distribution allows for increasing hazard rates over time, thereby offering a more realistic representation of propagation failures in long-duration HAP missions. It can capture varying failure propagation speeds observed in real HAP systems (e.g., cascading software bugs or atmospheric-induced degradations). It is worth noting that each state in this model is considered to be regenerative, meaning that once an event occurs, it prevents any other event from happening in the same state. As a result, the Markov property is satisfied at every transition epoch within the proposed model. Consequently, the underlying stochastic process of the model can be effectively represented as an SMP. For the sake of mathematical tractability of the model and to obtain the solution, it has been assumed that the sojourn time of rest of the states follow an exponential distribution.

The steady-state probabilities of SMP are computed by using two-stage method. This method allows for a comprehensive examination of the system's behaviour. The Markov model has been analysed in the time dependent and time independent scenarios for transient and steady-state solutions, respectively. By using numerical techniques, the underlying study illustrates the impact of various factors on reliability indicators. Furthermore, a comparison is made between the stochastic models using MP and SMP. The comparative analysis provides insights into the performance and effectiveness of the two modeling approaches, shedding light on their respective strengths and limitations.

The rest of this article is divided into seven sections. In Sect. 2, we offer an overview of the potential queueing system's specifications. Section 3 presents an analysis of the system, covering both transient and steady-state aspects. In Sect. 5, we delve into key performance indicators related to the model we propose. Section 6 provides in-depth case studies, exploring specific scenarios. Lastly, Sect. 7 wraps up the article, summarizing the findings and providing recommendations for future research.

2 Model Description

The proposed model comprises four states: the working state of the main HAP (M), the local failure state of the main HAP (L), the propagation failure state of the main HAP (P), and the working state of the backup HAP (B).

2.1 Modeling Assumptions

To make the analysis tractable while still capturing the essential dynamics of HAP failures, we adopt the following modeling assumptions:

- Independence of LF and PF: LF and PF are treated as statistically independent events. This reflects the fact that LFs typically arise from component-level malfunctions, while PFs correspond to systemic degradation or cascading faults. Independence allows us to isolate their effects on reliability. We note that in some real systems, LF and PF may exhibit coupling; extending the model to capture such dependence is a possible extension.
- Sojourn-time distributions: Except for PF in the SMP, all sojourn times (arrivals, service, repair, and LF occurrence) are assumed to be exponentially distributed. This as-

sumption enables comparison with a CTMC baseline and provides analytical tractability. In contrast, PF sojourns are modeled with a gamma distribution, consistent with the progressive and non-memoryless nature of degradation observed in aerospace systems. This choice allows increasing hazard rates to be represented, unlike the exponential distribution.

- **Single redundant HAP:** The system consists of one primary and one redundant HAP operating in a cold-standby configuration. The redundant HAP is activated only in the event of a PF. This reflects a minimal yet practical redundancy strategy: minor local failures are repaired on board, whereas redundancy is reserved for severe propagation failures that necessitate a full switchover. Although we focus on a single redundant HAP for clarity and tractability, the framework can be extended to multiple or hierarchical backups.

A description of each state and the possible transitions is provided below.

- **Working state of main HAP (M):** At the initial stage, the main HAP is in the working state. In this condition, packets reach the system following a Poisson process, each with an exponentially distributed arrival rate of λ that is independent and identically distributed. The primary HAP handles these incoming packets, and the time it takes to serve them is exponentially distributed with a rate of μ_1 . The proposed model utilizes the process sharing discipline (PS) to manage the service of incoming packets. In a PS queue, the service rate is evenly distributed among all the packets present in the system. Therefore, when there are multiple packets in the system, each packet is served at a rate of μ_1 . In the busy state, the main HAP may experience two types of failures, namely local failure (LF) and propagation failure (PF). If the main HAP encounters a LF, it transitions to the LF state. Similarly, if the main HAP encounters a PF, it transitions to the PF state. These states are considered as the reliable states of the system as the system will be in working condition.
- **Local failure state of main HAP (L):** Upon encountering a LF event, the main HAP transitions to the L state with exponentially distributed rate α_L . In this state, the repair process for the failed components is initiated, and it is exponentially distributed with rate of β_L . In the meantime of repair of the main HAP, there may be the arrival of packets at the system, which will wait in a buffer of infinite size. Once the main HAP is repaired, the system state will move to the working state and the packets. Finally, these packets will be served.
- **Propagation failure state of main HAP (P):** Upon encountering a PF event, the main HAP transitions to the P state. In this model, it is considered that the lifetime of the P state can follow either an exponential distribution, resulting in an MP, or a non-exponential distribution, resulting in an SMP. In the MP case, the propagation failure rate follows an exponential distribution with a rate of α_P . Similar to the LF state, the repair process for the failed components in the P state follows an exponential distribution with a rate of β_P . However, according to [28], the lifetime of the HAP can be modeled using a gamma distribution. Therefore, in the SMP case, the propagation failure rate follows a gamma distribution with parameters k and α_P . In this state, it takes some time for the HAP to become fully functional again. Consequently, all arriving packets are transferred to the redundant HAP for service at a rate of γ , allowing uninterrupted service provision

during the repair process. When the main HAP experiences a propagation failure, arriving packets are queued in an infinite buffer and then transferred to the redundant HAP, which becomes active and serves them to ensure uninterrupted operation.

- Working state of redundant HAP (B): In this state, packets arrive at the system following a Poisson process with an identically independent exponentially distributed rate of λ . The redundant HAP serves the arriving packets, and the service time is exponentially distributed with parameter μ_2 ($\mu_1 > \mu_2$). These states are considered as the reliable states of the system, as the system will be in working condition.

In the proposed framework, the redundant HAP is activated only in the event of a PF, whereas during a LF the service is temporarily interrupted and resumes once the failed components are repaired. This modeling choice reflects the operational distinction between the two types of failures. A local failure typically corresponds to a minor malfunction or component-level fault in the primary HAP, which can often be repaired within a short duration without necessitating a full system switchover. Activating the backup HAP for every minor LF would introduce additional overhead in terms of coordination, energy consumption, and resource allocation, thereby reducing the efficiency of the redundant system. In contrast, a propagation failure represents a more severe and system-wide degradation, leading to an eventual shutdown of the primary HAP. In such cases, seamless migration of service to the redundant HAP becomes essential to ensure continuity. Thus, the backup HAP's limited role in our model is motivated by practical considerations: it is reserved for handling critical failure events (PFs), while less severe local failures are assumed to be addressed through on-board repair and maintenance protocols. Future work could extend the framework to scenarios where redundancy is triggered for both LF and PF, thereby providing a broader comparative analysis of redundancy activation policies and their impact on system reliability and throughput.

It is important to acknowledge the limitations of the present study. The model assumes fixed failure and repair rates, whereas in practice these may vary over time due to aging, environmental conditions, or operational stress. Furthermore, the independence assumed between local and propagation failures, as well as between failure and repair mechanisms, is an idealization that may not hold in real systems where coupling effects can arise. Finally, the analysis focuses on a single redundant HAP in cold standby; while this provides a useful baseline, it does not capture the dynamics of larger redundant constellations. Future work will address these aspects by incorporating non-stationary rates, statistical dependence, and more general redundancy strategies. Figure 2 shows the state transition diagram for the proposed model.

3 Markov Model

Let the underlying CTMC is denoted by $\{(N_1(t), N_2(t)); t \geq 0\}$ where $\{N_1(t); t \geq 0\}$ shows the number of packets in the system and $\{N_2(t); t \geq 0\}$ exhibits the state of the system at any given time t . The state space of the underlying stochastic process is provided as follows:

$$\Omega = \{(i, j); i = 0, 1, 2, \dots, j = M, B, P, L\}$$

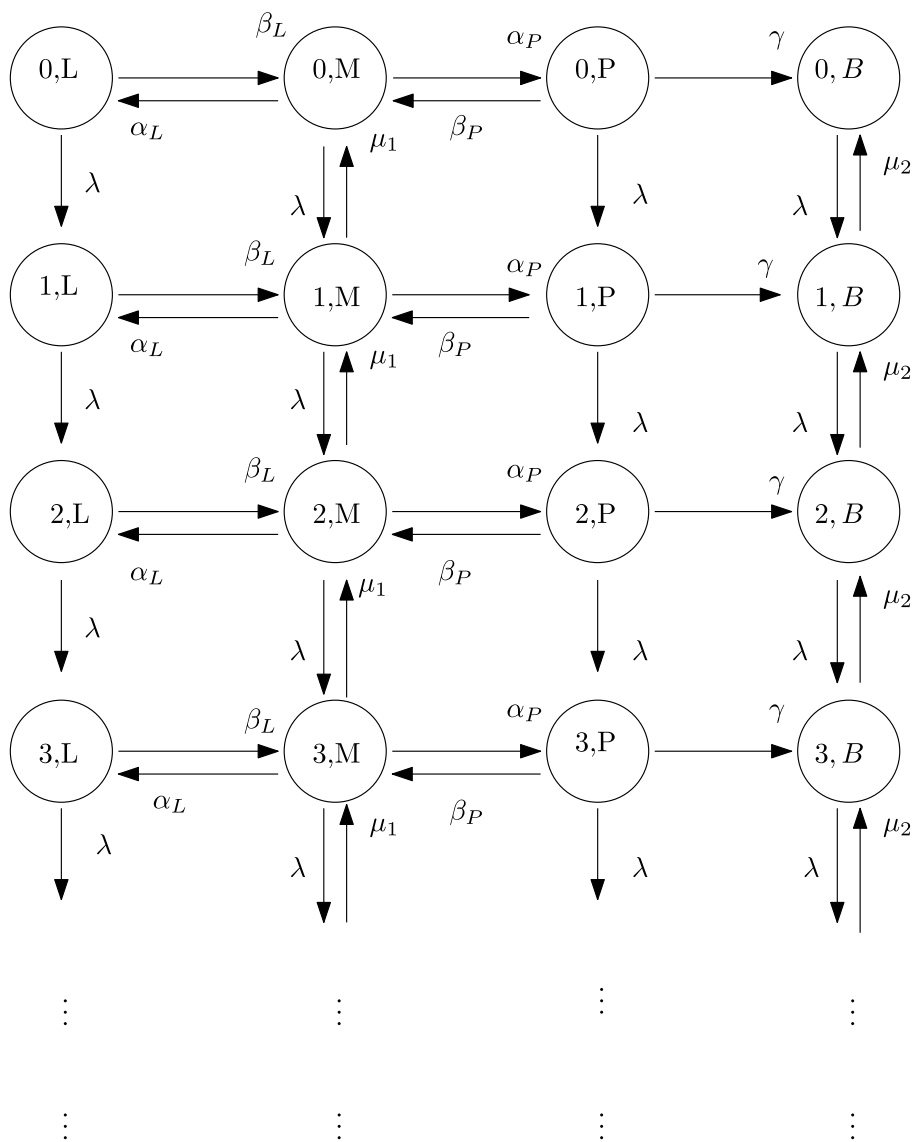


Fig. 2 State transition diagram for the proposed model

where

$$j = \begin{cases} M & \text{when the main HAP is in working state,} \\ B & \text{when the backup HAP is in working state,} \\ L & \text{when the local failures occurs in main HAP,} \\ P & \text{when the propagation failures occurs in main HAP.} \end{cases}$$

Let $P_{i,j}(t) = P\{N_1(t) = i, N_2(t) = j\}$, $i = 0, 1, 2, \dots, j = M, L, P, B$ be the probability that the system is in state j with i number of packets in the system at any time t . Then, $P_{i,j}(t)$ satisfies the following forward Kolmogorov equations.

$$P'_{0,M}(t) = -(\alpha_P + \alpha_L + \lambda)P_{0,M}(t) + \beta_L P_{0,L}(t) + \beta_P P_{0,P}(t) + \mu_1 P_{1,M}(t), \quad (1)$$

$$P'_{n,M}(t) = -(\lambda + \mu_1 + \alpha_P + \alpha_L)P_{n,M}(t) + \beta_L P_{n,L}(t) + \beta_P P_{n,P}(t) + \lambda P_{n-1,M}(t) + \mu_1 P_{n+1,M}(t), n = 1, 2, \dots, \quad (2)$$

$$P'_{0,B}(t) = -\lambda P_{0,B}(t) + \gamma P_{0,P}(t) + \mu_2 P_{1,B}(t), \quad (3)$$

$$P'_{n,B}(t) = -(\lambda + \mu_2)P_{n,B}(t) + \lambda P_{n-1,B}(t) + \mu_2 P_{n+1,B}(t) + \gamma P_{n,P}(t), n = 1, 2, 3, \dots, \quad (4)$$

$$P'_{0,L}(t) = -(\lambda + \beta_L)P_{0,L}(t) + \alpha_L P_{0,M}(t), \quad (5)$$

$$P'_{n,L}(t) = -(\lambda + \beta_L)P_{n,L}(t) + \lambda P_{n-1,L}(t) + \alpha_L P_{n,M}(t), n = 1, 2, 3, \dots, \quad (6)$$

$$P'_{0,P}(t) = -(\lambda + \beta_P + \gamma)P_{0,P}(t) + \alpha_P P_{0,M}(t), \quad (7)$$

$$P'_{n,P}(t) = -(\lambda + \beta_P + \gamma)P_{n,P}(t) + \lambda P_{n-1,P}(t) + \alpha_P P_{n,M}(t), n = 1, 2, 3, \dots \quad (8)$$

Let us assume that initially the main HAP is working and there are no packets in the system. That is, $P_{0,M}(0) = 1$.

3.1 Transient Analysis

Theorem 1 *The time-dependent probabilities of the HAP encountering a local failure and a propagation failure are given by the following equations*

$$P_{n,L}(t) = \alpha_L e^{-(\lambda + \beta_L)t} * \sum_{r=0}^n \frac{\lambda^{n-r} t^{n-r-1}}{(n-r-1)!} e^{-(\lambda + \beta_L)t} * P_{r,M}(t), n = 1, 2, 3, \dots \quad (9)$$

$$P_{n,P}(t) = \alpha_P e^{-(\lambda + \gamma + \beta_P)t} * \sum_{r=0}^n \frac{\lambda^{n-r} t^{n-r-1}}{(n-r-1)!} e^{-(\lambda + \gamma + \beta_P)t} * P_{r,M}(t), n = 1, 2, 3, \dots \quad (10)$$

Proof Please see Appendix 7 for the proof.

Theorem 2 The time-dependent probabilities of the main HAP being in working state are given by for $n = 1, 2, 3, \dots$,

$$\begin{aligned}
 P_{n,M}(t) = & \mu_1 \beta^{n+1} \int_0^t P_{0,M}(y) [I_{n-1}(\cdot) - I_{n+1}(\cdot)] e^{-(\alpha_P + \alpha_L + \lambda + \mu_1)(t-y)} dy \\
 & + \beta_L \int_0^t \left[\sum_{r=1}^n \beta^{n-r} [I_{n-r}(\cdot) - I_{n+r}(\cdot)] P_{r,L}(y) \right. \\
 & + \left. \sum_{r=1}^{\infty} \beta^{-r} [I_r(\cdot) - I_{2n+r}(\cdot)] P_{n+r,L}(y) \right] e^{-(\alpha_P + \alpha_L + \lambda + \mu_1)(t-y)} dy \\
 & + \beta_P \int_0^t \left[\sum_{r=1}^n \beta^{n-r} [I_{n-r}(\cdot) - I_{n+r}(\cdot)] P_{r,P}(y) \right. \\
 & + \left. \sum_{r=1}^{\infty} \beta^{-r} [I_r(\cdot) - I_{2n+r}(\cdot)] P_{r,P}(y) \right] e^{-(\alpha_P + \alpha_L + \lambda + \mu_1)(t-y)} dy
 \end{aligned} \quad (11)$$

where $I_n(\cdot) = I_n(\alpha(t-y))$, $\alpha = 2\sqrt{\lambda\mu_1}$ and $\beta = \sqrt{\frac{\lambda}{\mu_1}}$.

Proof Please see Appendix 7 for the proof.

Theorem 3 The time-dependent probabilities of the backup HAP being in working state are given by for $n = 1, 2, 3, \dots$,

$$\begin{aligned}
 P_{n,B}(t) = & \mu_2 \beta_1^{n+1} \int_0^t P_{0,B}(y) [I_{n-1}(\cdot) - I_{n+1}(\cdot)] e^{-(\lambda + \mu_2)(t-y)} dy \\
 & + \gamma \int_0^t \left[\sum_{r=1}^n \beta^{n-r} [I_{n-r}(\cdot) - I_{n+r}(\cdot)] P_{r,P}(y) \right.
 \end{aligned} \quad (12)$$

$$\left. + \sum_{r=1}^{\infty} \beta^{-r} [I_r(\cdot) - I_{2n+r}(\cdot)] P_{n+r,P}(y) \right] e^{-(\lambda + \mu_2)(t-y)} dy. \quad (13)$$

where $I_n(\cdot) = I_n(\alpha_1(t-y))$, $\alpha_1 = 2\sqrt{\lambda\mu_2}$, and $\beta_1 = \sqrt{\frac{\lambda}{\mu_2}}$.

Proof Let us define the generating function as follows:

$$G_B(z, t) = \sum_{n=0}^{\infty} z^n P_{n,B}(t). \quad (14)$$

Since $P_{0,B}(0) = 0$, which implies $G_B(z, 0) = 0$.

From equations (3) and (4), to prove the result given in equation (13), we establish an analogy between the previous theorems and we can apply similar techniques, one can obtain the probabilities $P_{n,B}(t)$ by solving the equation

$$\frac{\partial G_B}{\partial t} = -(\lambda + \mu_2)G_B + \left(\lambda z + \frac{\mu_2}{z}\right)G_B + \mu_2 \left(1 - \frac{1}{z}\right)P_{0,B}(t) + \gamma \sum_{n=0}^{\infty} z^n P_{n,P}(t)$$

and which gives a solution

$$G_B = \mu_2 \left(1 - \frac{1}{z}\right) \int_0^t P_{0,B}(y) e^{-(\lambda + \mu_2)(t-y)} e^{(\lambda z + \frac{\mu_2}{z})(t-y)} dy \\ + \gamma \sum_{n=0}^{\infty} z^n \int_0^t P_{n,P}(y) e^{-(\lambda + \mu_2)(t-y)} e^{(\lambda z + \frac{\mu_2}{z})(t-y)} dy$$

From the above equation, we obtain the time-dependent system size probabilities on backup HAP $P_{n,B}(t)$ is given in equation (13) where $\alpha_1 = 2\sqrt{\lambda\mu_2}$, $\beta_1 = \sqrt{\frac{\lambda}{\mu_2}}$.

3.2 Steady-State Analysis

Using the time-dependent probabilities of the system given in Section 3.1, the steady-state probabilities can be computed. Since, $\lim_{t \rightarrow \infty} P_{i,j}(t) = \pi_{i,j}$ and $\lim_{t \rightarrow \infty} P'_{i,j}(t) = 0$, equations (1)-(8) can be written as

$$0 = -(\alpha_P + \alpha_L + \lambda)\pi_{0,M} + \beta_L\pi_{0,L} + \beta_P\pi_{0,P} + \mu_1\pi_{1,M} \quad (15)$$

$$0 = -(\lambda + \mu_1 + \alpha_P + \alpha_L)\pi_{n,M} + \beta_L\pi_{n,L} + \beta_P\pi_{n,P} + \lambda\pi_{n-1,M} + \mu_1\pi_{n+1,M}, \quad (16) \\ n = 1, 2, \dots$$

$$0 = -\lambda\pi_{0,B} + \gamma\pi_{0,P} + \mu_2\pi_{1,B} \quad (17)$$

$$0 = -(\lambda + \mu_2)\pi_{n,B} + \lambda\pi_{n-1,B} + \mu_2\pi_{n+1,B} + \gamma\pi_{n,P}, \quad n = 1, 2, 3, \dots \quad (18)$$

$$0 = -(\lambda + \beta_L)\pi_{0,L} + \alpha_L\pi_{0,M} \quad (19)$$

$$0 = -(\lambda + \beta_L)\pi_{n,L} + \lambda\pi_{n-1,L} + \alpha_L\pi_{n,M}, \quad n = 1, 2, 3, \dots \quad (20)$$

$$0 = -(\lambda + \beta_P + \gamma)\pi_{0,P} + \alpha_P\pi_{0,M} \quad (21)$$

$$0 = -(\lambda + \beta_P + \gamma)\pi_{n,P} + \lambda\pi_{n-1,P} + \alpha_P\pi_{n,M}, \quad n = 1, 2, 3, \dots \quad (22)$$

We will make use of Tauberian theorem on the Laplace transform, from equations (1)-(8), we obtain the following results of steady-state probabilities.

It is known that $\lim_{s \rightarrow 0} s f_{i,j}(s) = \lim_{t \rightarrow \infty} P_{i,j}(t) = \pi_{ij}$. Then from the Laplace transform of certain equation yield

$$\begin{aligned}\sum_{n=1}^{\infty} \pi_{n,L} &= \frac{\lambda}{\beta_L} \pi_{0,L} + \frac{\alpha_L}{\beta_L} \sum_{n=1}^{\infty} \pi_{n,M} \\ \sum_{n=1}^{\infty} \pi_{n,P} &= \frac{\lambda}{\beta_P + \gamma} \pi_{0,P} + \frac{\alpha_P}{\gamma + \beta_P} \sum_{n=1}^{\infty} \pi_{n,M} \\ \pi_{0,L} &= \frac{\alpha_L}{\lambda + \beta_L} \pi_{0,M} \\ \pi_{0,P} &= \frac{\alpha_P}{\lambda + \beta_P + \gamma} \pi_{0,M} \\ \pi_{0,M} &= \frac{\beta_L}{\lambda + \alpha_P + \alpha_L} \pi_{0,L} + \frac{\beta_P}{\lambda + \alpha_P + \alpha_L} \pi_{0,P} + \frac{\mu_1}{\lambda + \alpha_P + \alpha_L} \pi_{1,M} \\ \pi_{0,M} &= \frac{\alpha_L \beta_L}{(\lambda + \alpha_P + \alpha_L)(\lambda + \beta_L)} \pi_{0,M} + \frac{\alpha_P \beta_P}{(\lambda + \alpha_P + \alpha_L)(\lambda + \beta_P + \gamma)} \pi_{0,M} \\ &\quad + \frac{\mu_1}{\lambda + \alpha_P + \alpha_L} \pi_{1,M}.\end{aligned}$$

Therefore,

$$\pi_{0,M} = \left[1 - \frac{\alpha_L \beta_L}{(\lambda + \alpha_P + \alpha_L)(\lambda + \beta_L)} - \frac{\alpha_P \beta_P}{(\lambda + \alpha_P + \alpha_L)(\lambda + \beta_P + \gamma)} \right]^{-1} \frac{\mu_1}{\lambda + \alpha_P + \alpha_L} \pi_{1,M}. \quad (23)$$

Also,

$$\begin{aligned}
\sum_{n=1}^{\infty} \pi_{n,M} &= \frac{\lambda}{\alpha_P + \alpha_L} \pi_{0,M} + \frac{\beta_P}{\alpha_L + \alpha_P} \sum_{n=1}^{\infty} \pi_{n,P} + \frac{\beta_L}{\alpha_L + \alpha_P} \sum_{n=1}^{\infty} \pi_{n,L} - \frac{\mu_1}{\alpha_P + \alpha_L} \pi_{1,M} \\
\sum_{n=1}^{\infty} \pi_{n,M} &= \frac{\beta_L}{\alpha_L + \alpha_P} \left(\frac{\lambda}{\beta_L} \pi_{0,L} + \frac{\alpha_L}{\beta_L} \sum_{n=1}^{\infty} \pi_{n,M} \right) + \frac{\beta_P}{\alpha_L + \alpha_P} \\
&\quad \left(\frac{\lambda}{\gamma + \beta_P} \pi_{0,P} + \frac{\alpha_P}{\gamma + \beta_P} \sum_{n=1}^{\infty} \pi_{n,M} \right) + \frac{\lambda}{\alpha_P + \alpha_L} \pi_{0,M} - \frac{\mu_1}{\alpha_P + \alpha_L} \pi_{1,M} \\
\sum_{n=1}^{\infty} \pi_{n,M} &\left(1 - \frac{\alpha_L}{\alpha_L + \alpha_P} - \frac{\alpha_P \beta_P}{(\alpha_L + \alpha_P)(\gamma + \beta_P)} \right) = \left(\frac{\lambda \alpha_L}{(\alpha_L + \alpha_P)(\lambda + \beta_L)} \right. \\
&\quad \left. + \frac{\lambda \alpha_P \beta_P}{(\alpha_L + \alpha_P)(\gamma + \beta_P)(\lambda + \beta_P + \gamma)} + \frac{\lambda}{\alpha_P + \alpha_L} \right) \pi_{0,M} - \frac{\mu_1}{\alpha_P + \alpha_L} \pi_{1,M} \\
\sum_{n=1}^{\infty} \pi_{n,M} &= \left[1 - \frac{\alpha_L}{\alpha_L + \alpha_P} - \frac{\alpha_P \beta_P}{(\alpha_L + \alpha_P)(\gamma + \beta_P)} \right]^{-1} \times \left[\left(\frac{\lambda \alpha_L}{(\alpha_L + \alpha_P)(\lambda + \beta_L)} \right. \right. \\
&\quad \left. \left. + \frac{\lambda \alpha_P \beta_P}{(\alpha_L + \alpha_P)(\gamma + \beta_P)(\lambda + \beta_P + \gamma)} + \frac{\lambda}{(\alpha_L + \alpha_P)} \right) \pi_{0,M} - \frac{\mu_1}{\alpha_P + \alpha_L} \pi_{1,M} \right]
\end{aligned}$$

From the equations (1) - (4), one can derive the following recurrence relations

$$\pi_{n,M} = \frac{\lambda}{\mu_1} \pi_{n-1,M} - \frac{\alpha_P}{\mu_1} \pi_{0,M} - \frac{\beta_L}{\mu_1} \sum_{r=0}^{n-1} \pi_{r,L} - \frac{\beta_P}{\mu_1} \sum_{r=0}^{n-1} \pi_{r,P}, n = 1, 2, \dots \quad (24)$$

$$\pi_{n,B} = \frac{\lambda}{\mu_2} \pi_{n-1,B} - \frac{\gamma}{\mu_2} \sum_{r=0}^{n-1} \pi_{r,P}, n = 1, 2, \dots \quad (25)$$

From the above recurrence relations, one can easily find the probabilities $\pi_{n,M}$ and $\pi_{n,B}$ in terms of $\pi_{0,M}$ and $\pi_{0,B}$ respectively. Also, one can reach the stability condition for the existence of steady-state probabilities, which are $\frac{\lambda}{\mu_1} < 1$ and $\frac{\lambda}{\mu_2} < 1$. Now, we express all the steady-state probabilities $\pi_{n,M}$ and $\pi_{n,B}$ in terms of $\pi_{0,M}$ or $\pi_{1,M}$. Then, using the normalization principle, i.e., $\sum_{i,j \in \Omega} \pi_{i,j} = 1$, we will obtain all the steady-state probabilities in explicit form.

4 Semi-Markov Model

Let $\{X(t), t \geq 0\}$ be the stochastic process for the underlying system. In this case, it has been considered that the number of packets waiting in the queue has a finite capacity, say N . Here, N is a large finite number. Because of the gamma distribution governing the mean sojourn time of certain states and the time epochs' Markov properties, we can appropriately

represent the underlying stochastic process as an SMP, as described in [2]. The state space for the proposed SMP model is defined as

$$\Omega = \{M_i, P_i, B_i, L_i; i = 0, 1, 2, \dots, N\}.$$

To analyze the underlying SMP model, we utilize a two-stage method. In this approach, a kernel matrix $K(t)$ is computed, with its elements denoted as $K_{m,n}(t) = P\{Y_1 = n, T_1 \leq t \mid Y_0 = m\}$, $m, n \in \Omega$, where $\{(Y_n, T_n), n \geq 0\}$ represents the underlying Markov renewal sequence of random variables. Table 2 illustrates the mean sojourn time distributions of the various transitions of the proposed model.

Let $F_{m,n}$ denotes the cumulative distribution function (CDF) for transitioning from state m to state n . The non-zero values of this function are determined based on the CDF definitions provided in Table 2. The entries of $K(t)$ can be further determined using the CDFs provided in Table 2 as follows:

$$\begin{aligned} K_{M_0, M_1}(t) &= \int_0^t \bar{F}_{M_0, P_0}(x) \bar{F}_{M_0, L_0}(x) dF_{M_0, M_1}(x) \\ &= \frac{\lambda}{(k-1)!} \frac{1 - e^{-(\lambda + \alpha_L)t}}{\lambda + \alpha_P + \alpha_L} - \left[\frac{1 - e^{-(\lambda + \alpha_L)t}}{\lambda + \alpha_L} - \frac{1 - e^{-(\lambda + \alpha_L + \alpha_P)t}}{\lambda + \alpha_P + \alpha_L} \right] \\ &\quad [\lambda t^{k-1} - \alpha_P \lambda (t^{k-1})], \end{aligned} \quad (26)$$

$$\begin{aligned} K_{M_i, M_{i+1}}(t) &= \int_0^t \bar{F}_{M_i, P_i}(x) \bar{F}_{M_i, L_i}(x) \bar{F}_{M_i, M_{i-1}}(x) dF_{M_i, M_{i+1}}(x) \\ &= \frac{\lambda}{(k-1)!} \frac{1 - e^{-(\lambda + \alpha_L + \mu_1)t}}{\lambda + \alpha_P + \alpha_L + \mu_1} - \left[\frac{1 - e^{-(\lambda + \alpha_L + \mu_1)t}}{\lambda + \alpha_L + \mu_1} \right. \\ &\quad \left. - \frac{1 - e^{-(\lambda + \alpha_L + \alpha_P + \mu_1)t}}{\lambda + \alpha_P + \alpha_L + \mu_1} \right] [\lambda t^{k-1} - \alpha_P \lambda (t^{k-1})], i = 1, 2, \dots, N \end{aligned} \quad (27)$$

$$\begin{aligned} K_{M_0, P_0}(t) &= \int_0^t \bar{F}_{M_0, M_1}(x) \bar{F}_{M_0, L_0}(x) dF_{M_0, P_0}(x) \\ &= \frac{(\alpha_P)^k (1 - e^{-(\lambda + \alpha_L + \alpha_P)t})}{(k-1)! (\lambda + \alpha_P + \alpha_L)} (t^{k-1}), \end{aligned} \quad (28)$$

Table 2 Parameters with their interpretation

Notations	Description	Distribution
λ	Arrival of packets	Exponential
α_L	Local failure rate of main HAP	Exponential
β_L	Repair rate of main HAP	Exponential
μ_1	Service of packets by main HAP	Exponential
k, α_P	Propagation failure rate	Gamma
β_P	Repair rate of main HAP for propagation failure	Exponential
μ_2	Service of packets by backup HAP	Exponential
γ	Replacement rate of failed HAP to backup HAP	Exponential

$$\begin{aligned}
 K_{M_i, P_i}(t) &= \int_0^t \bar{F}_{M_{i-1}, M_i}(x) \bar{F}_{M_i, M_{i+1}}(x) \bar{F}_{M_i, L_i}(x) dF_{M_i, P_i}(x) \\
 &= \frac{(\alpha_P)^k (1 - e^{-(\lambda + \alpha_L + \alpha_P + \mu_1)t})}{(k-1)! (\lambda + \alpha_P + \alpha_L + \mu_1)} (t^{k-1}), i = 1, 2, \dots, N
 \end{aligned} \quad (29)$$

$$\begin{aligned}
 K_{M_0, L_0}(t) &= \int_0^t \bar{F}_{M_0, M_1}(x) \bar{F}_{M_0, P_0}(x) dF_{M_0, L_0}(x) \\
 &= \frac{\alpha_L}{(k-1)!} \frac{1 - e^{-(\lambda + \alpha_L + \alpha_P)t}}{\lambda + \alpha_P + \alpha_L} \\
 &\quad - \left[\frac{1 - e^{-(\lambda + \alpha_L)t}}{\lambda + \alpha_L} - \frac{1 - e^{-(\lambda + \alpha_L + \alpha_P)t}}{\lambda + \alpha_P + \alpha_L} \right] [\lambda t^{k-1} - \alpha_P \alpha_L (t^{k-1})],
 \end{aligned} \quad (30)$$

$$\begin{aligned}
 K_{M_i, L_i}(t) &= \int_0^t \bar{F}_{M_i, M_{i+1}}(x) \bar{F}_{M_i, M_{i-1}}(x) \bar{F}_{M_i, P_i}(x) dF_{M_i, L_i}(x) \\
 &= \frac{\alpha_L}{(k-1)!} \frac{1 - e^{-(\lambda + \alpha_L + \alpha_P + \mu_1)t}}{\lambda + \alpha_P + \alpha_L + \mu_1} - \left[\frac{1 - e^{-(\lambda + \alpha_L + \mu_1)t}}{\lambda + \alpha_L + \mu_1} \right. \\
 &\quad \left. - \frac{1 - e^{-(\lambda + \alpha_L + \alpha_P + \mu_1)t}}{\lambda + \alpha_P + \alpha_L + \mu_1} \right] [\lambda t^{k-1} - \alpha_P \alpha_L (t^{k-1})], i = 1, 2, \dots, N
 \end{aligned} \quad (31)$$

$$\begin{aligned}
 K_{M_i, M_{i-1}}(t) &= \int_0^t \bar{F}_{M_i, P_i}(x) \bar{F}_{M_i, L_i}(x) \bar{F}_{M_i, M_{i+1}}(x) dF_{M_i, M_{i-1}}(x) \\
 &= \frac{\mu_1}{(k-1)!} \frac{1 - e^{-(\lambda + \alpha_L + \alpha_P + \mu_1)t}}{\lambda + \alpha_P + \alpha_L + \mu_1} - \left[\frac{1 - e^{-(\lambda + \alpha_L + \mu_1)t}}{\lambda + \alpha_L + \mu_1} \right. \\
 &\quad \left. - \frac{1 - e^{-(\lambda + \alpha_L + \alpha_P + \mu_1)t}}{\lambda + \alpha_P + \alpha_L + \mu_1} \right] [\lambda t^{k-1} - \mu_1 \alpha_P (t^{k-1})], i = 1, 2, \dots, N
 \end{aligned} \quad (32)$$

$$\begin{aligned}
 K_{P_i, P_{i+1}}(t) &= \int_0^t \bar{F}_{P_i, B_i}(x) \bar{F}_{P_i, M_i}(x) dF_{P_i, P_{i+1}}(x) \\
 &= \frac{\lambda}{\lambda + \beta_P + \gamma} (1 - e^{-(\lambda + \gamma + \beta_P)t}),
 \end{aligned} \quad (33)$$

$$\begin{aligned}
 K_{P_i, B_i}(t) &= \int_0^t \bar{F}_{P_i, P_{i+1}}(x) \bar{F}_{P_i, M_i}(x) dF_{P_i, B_i}(x) \\
 &= \frac{\gamma}{\lambda + \beta_P + \gamma} (1 - e^{-(\lambda + \gamma + \beta_P)t}),
 \end{aligned} \quad (34)$$

$$\begin{aligned}
 K_{P_i, M_i}(t) &= \int_0^t \bar{F}_{P_i, P_{i+1}}(x) \bar{F}_{P_i, B_i}(x) dF_{P_i, M_i}(x) \\
 &= \frac{\beta_P}{\lambda + \beta_P + \gamma} (1 - e^{-(\lambda + \gamma + \beta_P)t}),
 \end{aligned} \quad (35)$$

$$K_{B_0, B_1}(t) = \int_0^t dF_{B_0, B_1}(x) = 1, \quad (36)$$

$$\begin{aligned} K_{B_i, B_{i+1}}(t) &= \int_0^t \bar{F}_{B_i, B_{i+1}}(x) dF_{B_i, B_{i+1}}(x) \\ &= \frac{\lambda}{\lambda + \mu_2} (1 - e^{-(\lambda + \mu_2)t}), i = 1, 2, \dots, N \end{aligned} \quad (37)$$

$$\begin{aligned} K_{B_i, B_{i-1}}(t) &= \int_0^t \bar{F}_{B_i, B_{i+1}}(x) dF_{B_i, B_{i-1}}(x) \\ &= \frac{\mu_2}{\lambda + \mu_2} (1 - e^{-(\lambda + \mu_2)t}), i = 1, 2, \dots, N \end{aligned} \quad (38)$$

$$\begin{aligned} K_{L_i, L_{i+1}}(t) &= \int_0^t \bar{F}_{L_i, M_i}(x) dF_{L_i, L_{i+1}}(x) \\ &= \frac{\lambda}{\lambda + \beta_L} (1 - e^{-(\lambda + \beta_L)t}), i = 0, 1, 2, \dots, N \end{aligned} \quad (39)$$

$$\begin{aligned} K_{L_i, M_i}(t) &= \int_0^t \bar{F}_{L_i, L_{i+1}}(x) dF_{L_i, M_i}(x) \\ &= \frac{\beta_L}{\lambda + \beta_L} (1 - e^{-(\lambda + \beta_L)t}), i = 0, 1, 2, \dots, N. \end{aligned} \quad (40)$$

Define $K_{m,n}(\infty) = \lim_{t \rightarrow \infty} K_{m,n}(t)$, where $m, n \in \Omega$. The matrix $K(\infty)$ represents the one-step transition probabilities for the underlying model whose row sum is 1. The $K(\infty)$ matrix is provided as follows:

$$K(\infty) = \begin{pmatrix} M_M & M_P & M_L & O \\ P_M & P_P & O & P_B \\ O & O & O & B_B \\ L_M & O & O & L_L \end{pmatrix}$$

where O is the zero matrix of appropriate order and rest of the entries are defined as follows.

$$\begin{aligned}
M_M &= \begin{matrix} & M_0 & M_1 & M_2 & M_3 & \cdots & M_N \\ \begin{matrix} M_0 \\ M_1 \\ M_2 \\ M_3 \\ \vdots \\ M_N \end{matrix} & \begin{pmatrix} 0 & K_{M_0, M_1} & 0 & \cdots & \cdots & 0 \\ K_{M_1, M_0} & 0 & K_{M_1, M_2} & \cdots & \cdots & 0 \\ 0 & K_{M_2, M_1} & 0 & \cdots & \cdots & 0 \\ 0 & 0 & K_{M_3, M_2} & \ddots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & K_{M_N, M_{N-1}} & 0 \end{pmatrix} \end{matrix}, \\
M_P &= \begin{matrix} & P_0 & P_1 & P_2 & P_3 & \cdots & P_N \\ \begin{matrix} M_0 \\ M_1 \\ M_2 \\ M_3 \\ \vdots \\ M_N \end{matrix} & \begin{pmatrix} K_{M_0, P_0} & 0 & 0 & \cdots & \cdots & 0 \\ 0 & K_{M_1, P_1} & 0 & \cdots & \cdots & 0 \\ 0 & 0 & K_{M_2, P_2} & \cdots & \cdots & 0 \\ 0 & 0 & 0 & \ddots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & K_{M_N, P_N} \end{pmatrix} \end{matrix}, \\
M_L &= \begin{matrix} & L_0 & L_1 & L_2 & L_3 & \cdots & L_N \\ \begin{matrix} M_0 \\ M_1 \\ M_2 \\ M_3 \\ \vdots \\ M_N \end{matrix} & \begin{pmatrix} K_{M_0, L_0} & 0 & 0 & \cdots & \cdots & 0 \\ 0 & K_{M_1, L_1} & 0 & \cdots & \cdots & 0 \\ 0 & 0 & K_{M_2, L_2} & \cdots & \cdots & 0 \\ 0 & 0 & 0 & \ddots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & K_{M_N, L_N} \end{pmatrix} \end{matrix}, \\
P_M &= \begin{matrix} & M_0 & M_1 & M_2 & M_3 & \cdots & M_N \\ \begin{matrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ \vdots \\ P_N \end{matrix} & \begin{pmatrix} K_{P_0, M_0} & 0 & 0 & \cdots & \cdots & 0 \\ 0 & K_{P_1, M_1} & 0 & \cdots & \cdots & 0 \\ 0 & 0 & K_{P_2, M_2} & \cdots & \cdots & 0 \\ 0 & 0 & 0 & \ddots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & K_{P_N, M_N} \end{pmatrix} \end{matrix}
\end{aligned}$$

$$\begin{aligned}
P_P &= \begin{matrix} & P_0 & P_1 & P_2 & P_3 & \cdots & P_N \\ \begin{matrix} P_0 \\ P_1 \\ P_2 \\ \vdots \\ P_{N-1} \\ P_N \end{matrix} & \begin{pmatrix} 0 & K_{P_0, P_1} & 0 & \cdots & \cdots & 0 \\ 0 & 0 & K_{P_1, P_2} & \cdots & \cdots & 0 \\ 0 & 0 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & K_{P_{N-1}, P_N} \\ 0 & 0 & 0 & \cdots & \cdots & 0 \end{pmatrix} \end{matrix}, \\
P_B &= \begin{matrix} & B_0 & B_1 & B_2 & B_3 & \cdots & B_N \\ \begin{matrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ \vdots \\ P_N \end{matrix} & \begin{pmatrix} K_{P_0, B_0} & 0 & 0 & \cdots & \cdots & 0 \\ 0 & K_{P_1, B_1} & 0 & \cdots & \cdots & 0 \\ 0 & 0 & K_{P_2, B_2} & \cdots & \cdots & 0 \\ 0 & 0 & 0 & \ddots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & K_{P_N, B_N} \end{pmatrix} \end{matrix}, \\
B_B &= \begin{matrix} & B_0 & B_1 & B_2 & B_3 & \cdots & B_{N-1} & B_N \\ \begin{matrix} B_0 \\ B_1 \\ B_2 \\ B_3 \\ \vdots \\ B_{N-1} \\ B_N \end{matrix} & \begin{pmatrix} 0 & K_{B_0, B_1} & 0 & \cdots & \cdots & 0 & 0 \\ K_{B_1, B_0} & 0 & K_{B_1, B_2} & \cdots & \cdots & 0 & 0 \\ 0 & K_{B_2, B_1} & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & 0 & \ddots & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & 0 & K_{B_{N-1}, B_N} \\ 0 & 0 & 0 & \cdots & \cdots & K_{B_N, B_{N+1}} & 0 \end{pmatrix} \end{matrix}, \\
L_M &= \begin{matrix} & M_0 & M_1 & M_2 & M_3 & \cdots & M_N \\ \begin{matrix} L_0 \\ L_1 \\ L_2 \\ L_3 \\ \vdots \\ L_N \end{matrix} & \begin{pmatrix} K_{L_0, M_0} & 0 & 0 & \cdots & \cdots & 0 \\ 0 & K_{L_1, M_1} & 0 & \cdots & \cdots & 0 \\ 0 & 0 & K_{L_2, M_2} & \cdots & \cdots & 0 \\ 0 & 0 & 0 & \ddots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & K_{L_N, M_N} \end{pmatrix} \end{matrix}, \\
L_L &= \begin{matrix} & L_0 & L_1 & L_2 & L_3 & \cdots & L_N \\ \begin{matrix} B_0 \\ B_1 \\ B_2 \\ B_3 \\ \vdots \\ B_{N-1} \\ B_N \end{matrix} & \begin{pmatrix} 0 & K_{B_0, L_1} & 0 & \cdots & \cdots & 0 \\ 0 & 0 & K_{B_1, L_2} & \cdots & \cdots & 0 \\ 0 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 0 & \ddots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & K_{B_{N-1}, L_N} \\ 0 & 0 & 0 & \cdots & \cdots & 0 \end{pmatrix} \end{matrix},
\end{aligned}$$

The above defined block matrices are square matrices of order $N + 1$. For the embedded Markov chain (EMC) of the SMP, the steady-state probabilities (v_i) are computed as follows

$$v = vK(\infty), \sum_{i \in \Omega} v_i = 1. \quad (41)$$

v'_i s can be computed by solving equations (41). Now, the steady-state probabilities of SMP are calculated using the formula

$$\pi_i = \frac{v_i h_i}{\sum_{i \in \Omega} v_j h_j}, i \in \Omega \quad (42)$$

where h_i denotes the mean sojourn time, spent by the process at each state i and can be computed using the formulas

$$h_{M_0} = \frac{1}{(k-1)!} \left[\frac{(k-1)!}{\lambda + \alpha_L} - \left(\frac{1}{\lambda + \alpha_L} - \frac{1}{\lambda + \alpha_L + \alpha_P} \right) \right]. \quad (43)$$

$$h_{M_i} = \frac{1}{(k-1)!} \left[\frac{(k-1)!}{\lambda + \alpha_L + \mu_1} - \left(\frac{1}{\lambda + \alpha_L + \mu_1} - \frac{1}{\lambda + \alpha_L + \alpha_P + \mu_1} \right) \right], \quad (44)$$

$$i = 1, 2, 3, \dots, N.$$

$$h_{P_i} = \frac{1}{\lambda + \gamma + \beta_P}, i = 0, 1, 2, \dots, N. \quad (45)$$

$$h_{B_0} = \frac{1}{\lambda}. \quad (46)$$

$$h_{B_i} = \frac{1}{\lambda + \mu_2}, i = 1, 2, \dots, N. \quad (47)$$

$$h_{L_i} = \frac{1}{\lambda + \beta_L}, i = 0, 1, 2, \dots, N. \quad (48)$$

5 Performance Metrics

This section outlines the measures related to the extended-term probabilities for the underlying model.

1. The reliability of the system at time t is given by from equations (1) and (2),

$$\sum_{n=0}^{\infty} P'_{n,M}(t) = -(\alpha_P + \alpha_L) \sum_{n=0}^{\infty} P_{n,M}(t) + \beta_L \sum_{n=0}^{\infty} P_{n,L}(t) + \beta_P \sum_{n=0}^{\infty} P_{n,P}(t).$$

The Laplace transform of the above equation yields

$$(s + \alpha_P + \alpha_L) \sum_{n=0}^{\infty} f_{n,M}(s) = 1 + \beta_L \sum_{n=0}^{\infty} f_{n,L}(s) + \beta_P \sum_{n=0}^{\infty} f_{n,P}(s).$$

From equations (3) and (8), and after taking the Laplace transform, one can reach the following expressions:

$$\begin{aligned} \sum_{n=0}^{\infty} s f_{n,B}(s) &= \gamma \sum_{n=0}^{\infty} f_{n,P}(s) \\ \sum_{n=0}^{\infty} (s + \beta_L) f_{n,L}(s) &= \alpha_L \sum_{n=0}^{\infty} f_{n,M}(s) \\ \sum_{n=0}^{\infty} (s + \beta_P + \gamma) f_{n,P}(s) &= \alpha_P \sum_{n=0}^{\infty} f_{n,M}(s). \end{aligned}$$

Therefore,

$$\begin{aligned} \sum_{n=0}^{\infty} (s + \alpha_P + \alpha_L) f_{n,M}(s) &= 1 + \frac{\beta_L \alpha_L}{s + \beta_L} \sum_{n=0}^{\infty} f_{n,M}(s) + \frac{\beta_P \alpha_P}{s + \beta_P + \gamma} \sum_{n=0}^{\infty} f_{n,M}(s) \\ \sum_{n=0}^{\infty} f_{n,M}(s) &= \frac{1}{s + \alpha_P + \alpha_L} + \left[\frac{\beta_L \alpha_L}{(s + \beta_L)(s + \alpha_P + \alpha_L)} \right. \\ &\quad \left. + \frac{\beta_P \alpha_P}{(s + \beta_P + \gamma)(s + \alpha_P + \alpha_L)} \sum_{n=0}^{\infty} f_{n,M}(s) \right] \\ \sum_{n=0}^{\infty} f_{n,M}(s) &= \frac{1}{s + \alpha_P + \alpha_L} \sum_{n=0}^{\infty} [F(s)]^n. \end{aligned}$$

$$\begin{aligned} R(s) &= \sum_{n=0}^{\infty} f_{n,M}(s) + \sum_{n=0}^{\infty} f_{n,B}(s) \\ &= \left[1 + \frac{\gamma \alpha_P}{s(s + \gamma + \beta_P)} \right] \sum_{n=0}^{\infty} f_{n,M}(s) \\ &= \left[1 + \frac{\gamma \alpha_P}{s(s + \gamma + \beta_P)} \right] \left[\frac{1}{s + \alpha_P + \alpha_L} \sum_{n=0}^{\infty} [F(s)]^n \right], \end{aligned}$$

where

$$F(s) = \left[\frac{\beta_L \alpha_L}{(s + \beta_L)(s + \alpha_P + \alpha_L)} + \frac{\beta_P \alpha_P}{(s + \beta_P + \gamma)(s + \alpha_P + \alpha_L)} \right].$$

By taking the inverse Laplace transform, we can obtain $F(t)$ and $R(t)$ as follows.

$$\begin{aligned}
 F(t) &= \frac{e^{-(\alpha_P + \alpha_L)t} (\beta_L \alpha_L^2 + \beta_P \alpha_P^2 - \beta_L \beta_P \alpha_L - \beta_L \gamma \alpha_L + \beta_L \alpha_L \alpha_P + \beta_P \alpha_L \alpha_P)}{(\alpha_L - \beta_L + \alpha_P)(\beta_P + \gamma - \alpha_L - \alpha_P)} \\
 &\quad + \frac{\beta_L \alpha_L e^{-(\beta_L t)}}{(\alpha_L - \beta_L + \alpha_P)} - \frac{\beta_P \alpha_P e^{-(\beta_P + \gamma)t}}{(\beta_P + \gamma - \alpha_L - \alpha_P)} \\
 R(t) &= \left[\frac{\gamma \alpha_P}{(\beta_P + \gamma)(\alpha_L + \alpha_P)} + \frac{\gamma \alpha_P e^{-(\beta_P + \gamma)t}}{(\beta_P + \gamma)(\beta_P + \gamma - \alpha_L - \alpha_P)} \right. \\
 &\quad \left. - \frac{e^{-(\alpha_L + \alpha_P)t} (\alpha_L^2 + \alpha_P^2 - \beta_P \alpha_L - \beta_P \alpha_P - \gamma \alpha_L + 2\alpha_L \alpha_P)}{(\alpha_L + \alpha_P)(\beta_P + \gamma - \alpha_L - \alpha_P)} \right] * \sum_{n=0}^{\infty} [F(t)]^{*n}.
 \end{aligned} \tag{49}$$

2. The mean time between failures is computed by using equation (49) as follows:

$$MTBF = \int_0^{\infty} R(t) dt.$$

3. The probability that the main HAP encounters the local failure at time t :

$$P_L = \sum_{i=0}^{\infty} P_{i,L}(t) = \frac{\alpha_L}{\alpha_P + \alpha_L - \beta_L} \left[e^{-\beta_L t} - e^{-(\alpha_P + \alpha_L)t} \right] * \sum_{n=0}^{\infty} [F(t)]^{*n}.$$

4. The probability that the main HAP encounters the propagation failure at time t : Since,

$$\begin{aligned}
 \sum_{n=0}^{\infty} f_{n,P}(s) &= \frac{\alpha_P}{s + \gamma + \beta_P} \sum_{n=0}^{\infty} f_{n,M}(s) \\
 \sum_{n=0}^{\infty} f_{n,P}(s) &= \frac{\alpha_P}{(s + \gamma + \beta_P)(s + \alpha_P + \alpha_L)} \sum_{n=0}^{\infty} [F(s)]^n \\
 \sum_{n=0}^{\infty} P_{n,P}(t) &= \alpha_P e^{-(\gamma + \beta_P)t} * e^{-(\alpha_P + \alpha_L)t} * \sum_{n=0}^{\infty} [F(t)]^{*n} \\
 P_P &= \sum_{n=0}^{\infty} P_{n,P}(t) \\
 &= \alpha_P e^{-(\gamma + \beta_P)t} * e^{-(\alpha_P + \alpha_L)t} * \sum_{n=0}^{\infty} [F(t)]^{*n}.
 \end{aligned}$$

5. The throughput of the system at time t is given by:
From equation (49),

$$\begin{aligned}
\text{Throughput} &= \left(\sum_{n=0}^{\infty} P_{n,M}(t) + \sum_{n=0}^{\infty} P_{n,B}(t) \right) * (\mu_1 + \mu_2) \\
&= \left[\frac{\gamma \alpha_P}{(\beta_P + \gamma)(\alpha_L + \alpha_P)} \right. \\
&\quad - \frac{e^{-(\alpha_L + \alpha_P)t} (\alpha_L^2 + \alpha_P^2 - \beta_P \alpha_L - \beta_P \alpha_P - \gamma \alpha_L + 2\alpha_L \alpha_P)}{(\alpha_L + \alpha_P)(\beta_P + \gamma - \alpha_L - \alpha_P)} \\
&\quad \left. + \frac{\gamma \alpha_P e^{-(\beta_P + \gamma)t}}{(\beta_P + \gamma)(\beta_P + \gamma - \alpha_L - \alpha_P)} \right] * \sum_{n=0}^{\infty} [F(t)]^{*n} * (\mu_1 + \mu_2).
\end{aligned}$$

6 Numerical Illustration

In this section, we conduct numerical analyses to assess the practicality of the suggested model for achieving stable results. Table 3 contains the necessary parameter values for generating the results. The numerical values used in this section are chosen for sensitivity analysis and to illustrate the qualitative impact of model parameters on system reliability and throughput. They are not intended to represent any single deployed HAP product or operator. The chosen parameter sets span a broad range of failure/repair and arrival/service rates so as to demonstrate model behaviour under low, medium, and high stress regimes (e.g., slow vs. aggressive failure environments and slow vs. rapid repairs). Specifically, Reference [3] provides a realistic and structured methodology for evaluating the reliability of HAPs, including representative failure and repair parameters derived from aerospace reliability practices. Our study aligns with these ranges to ensure that the selected values are representative of modern HAP systems. At the same time, we emphasize that the parameters used in our numerical examples are partly hypothetical and intended primarily to illustrate the qualitative impact of local and propagation failures on system performance.

6.1 Sensitivity Analysis in Transient State for CTMC

Figures 3-4 exhibit the behaviour of reliability of the system ($R(t)$) with respect to time for varying values of failure and repair rates. Figure 3(a) and (b) illustrate the decreasing trend of the reliability $R(t)$ over time. When the system is in the initial state at $t = 0$, the reliability is 100%, denoted by $R(t) = 1$. However, as the system continues to provide services, the reliability gradually diminishes. Notably, after a certain time period, there is a minimal change in the values of $R(t)$, indicating that the system is approaching a steady-state. Figure 3(a) highlights that as the local failure rate (α_L) increases, the reliability of the system

Table 3 Default values of parameters taken from Reference [3]

Parameters	λ	μ_1	μ_2	α_P
Possible values	0.5 – 2 per unit time	0.5 – 5 per unit time	0.5 – 5 per unit time	0.01 – 0.2 per unit time
Parameters	β_P	α_L	β_L	γ
Possible values	0.01 – 0.2 per unit time	0.01 – 0.2 per unit time	0.01 – 0.2 per unit time	0.01 – 0.2 per unit time

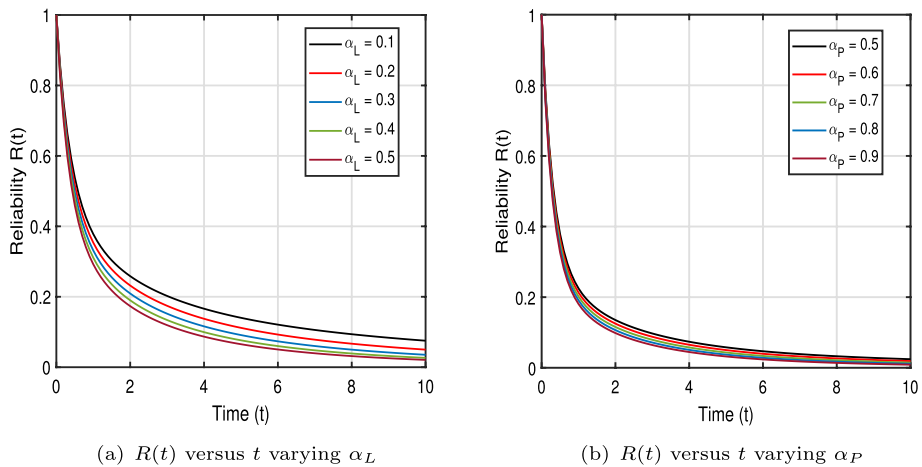


Fig. 3 Behaviour of $R(t)$ with respect to time for varying values of α_L and α_P

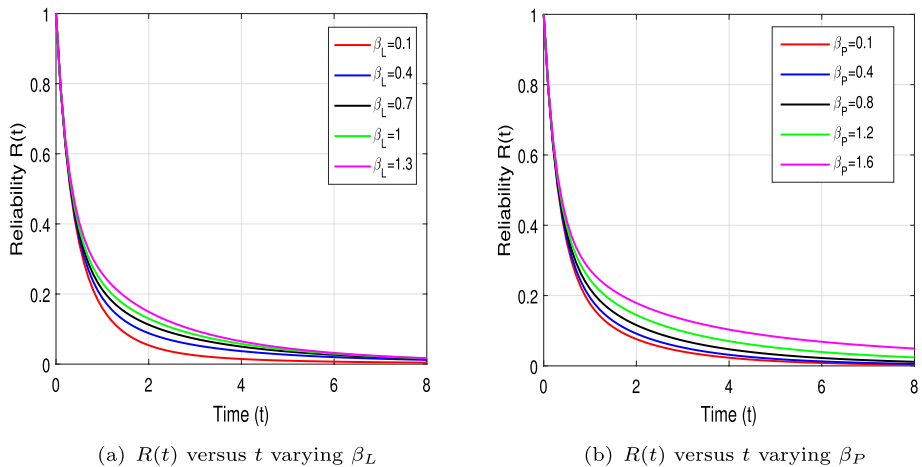


Fig. 4 Behaviour of $R(t)$ with respect to time for varying values of β_L and β_P

decreases. This outcome can be attributed to the higher likelihood of system failures with a higher local failure rate. Similarly, Fig. 3(b) demonstrates that $R(t)$ decreases as the propagation failure rate α_P increases, following the same reasoning. These observations emphasize the critical role of failure rates in determining the reliability of the system and provide valuable insights into the system's behaviour over time.

In contrast, Fig. 4(a) and (b) demonstrate that $R(t)$ is an increasing function with respect to the repair rates β_L and β_P , respectively. As the local failure rate β_L increases in the system, the values of $R(t)$ also increases. When the repair rate increases while keeping the failure rate fixed, the system becomes more reliable. Consequently, $R(t)$ increases as the values of β_L increase. A similar explanation can be provided for Fig. 4(b), with respect to the repair rate β_P . As β_P increases, the reliability of the system improves, leading to an increase in

$R(t)$. These findings highlight the positive impact of higher repair rates on the reliability of the system, offering insights into strategies for enhancing system performance and stability.

Figure 5(a) and (b) depict the behaviour of the system throughput over time, denoted as t . Both graphs illustrate that the throughput decreases as time progresses, primarily due to the decreasing reliability of the system. As the reliability declines, there is an increased likelihood of the system failing to provide services at any given time, leading to a decrease in throughput. From Fig. 5(a), it can be observed that with a fixed service rate, an increase in the arrival rate results in a decrease in throughput. This can be explained by the resource sharing policy, where all packets are served but with a reduced service rate. Consequently, as the arrival rate increases, more packets are present in the system, and each packet receives a lower service rate, resulting in a decrease in throughput. Conversely, when the service rate is increased in Fig. 5(b), the throughput also increases, which is an expected scenario. With a higher service rate, more packets can be served within a given time, leading to an improved throughput. These observations emphasize the interplay between arrival rate, service rate, and system reliability in determining the system throughput, providing valuable insights into the dynamics of the system's performance over time.

Figures 6 and 7 depict the behaviour of failure probabilities, namely the local failure probability (P_L) and the propagation failure probability (P_P), with respect to time t and varying values of failure and repair rates. In Fig. 6(a) and (b), the behaviour of P_L with respect to α_L and β_L is demonstrated. It is observed that P_L increases with time t and α_L . This can be attributed to the decreasing reliability of the system over time, leading to an increased probability of failure. Moreover, as the local failure rate α_L increases, there is a noticeable increment in the value of P_L , indicating a higher probability of local failure. Conversely, with increasing values of β_L , the failure probability P_L decreases. This can be attributed to the higher repair rate, which reduces the time required to repair failed components and thus decreases the probability of failure. Additionally, after a certain time period, the probabilities tend to approach a steady-state, indicating a more stable system behaviour. Similar observations can be made for Fig. 7(a) and (b), which illustrate the behaviour of P_P with respect to α_P and β_P . The same explanations can be applied to the behaviour of P_P as for P_L . These findings provide insights into the relationship between failure probabilities,

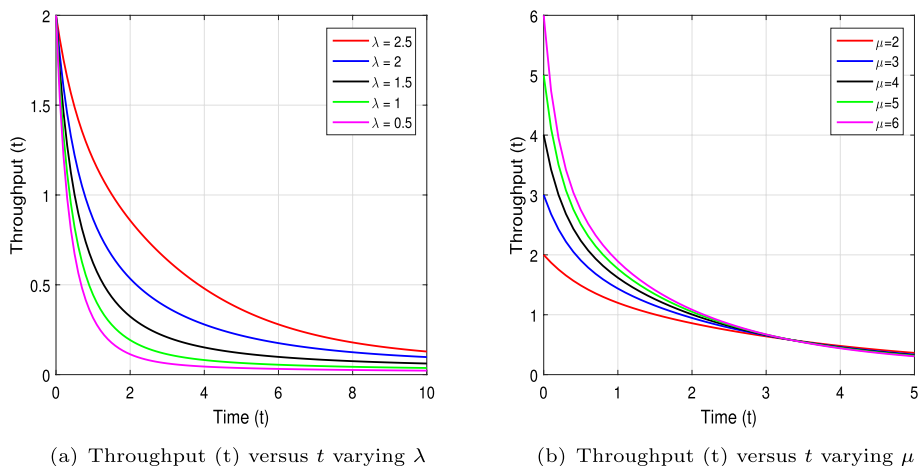


Fig. 5 Behaviour of throughput (t) with respect to time for varying values of λ and μ

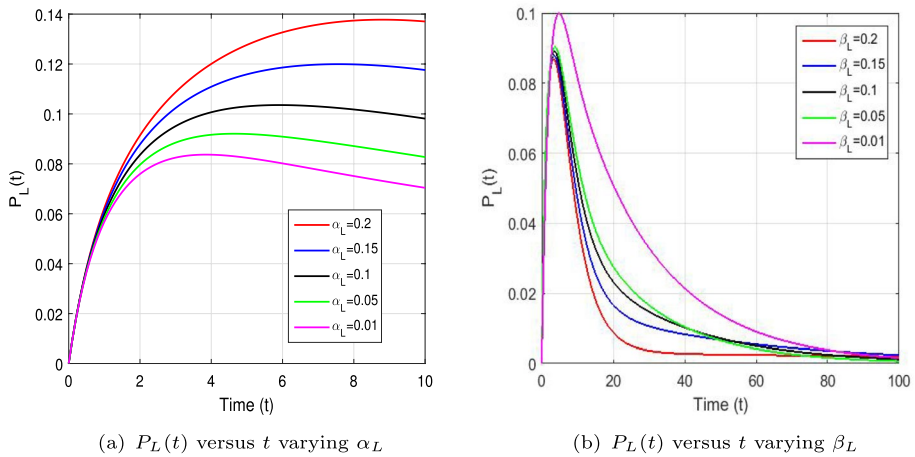


Fig. 6 Behaviour of $P_L(t)$ with respect to time for varying values of α_L and β_L

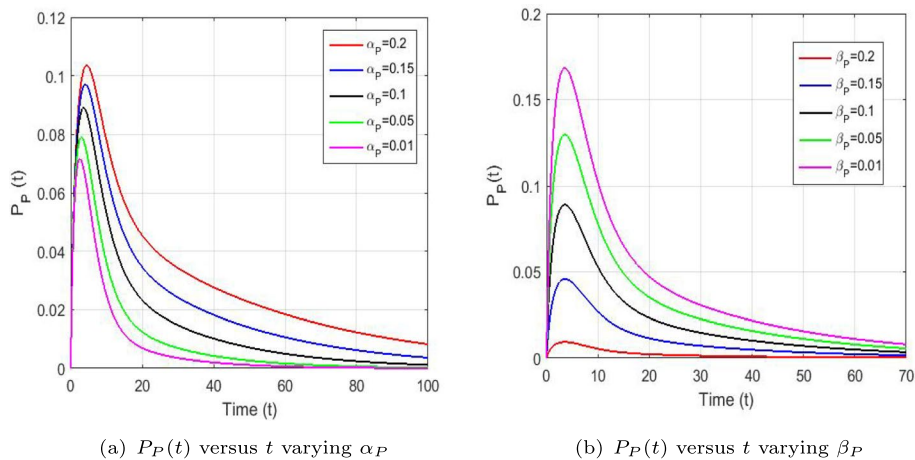


Fig. 7 Behaviour of $P_P(t)$ with respect to time for varying values of α_P and β_P

time, and the rates of failure and repair. They emphasize the impact of these factors on the likelihood of failure, highlighting the importance of effective repair strategies to mitigate failures and improve the reliability of the system.

Figure 8(a) and (b) illustrate the behaviour of the mean and variance of the system with varying values of α_P , representing the failure rate. Both graphs demonstrate that as the failure rate increases, the mean system size decreases. This indicates that more packets are lost during the HAP's failure, resulting in a smaller average number of packets in the system. Additionally, the values of both the mean and variance increase as the failure rate increases and reach a steady-state after a certain time period. This observation is consistent with the notion that higher failure rates lead to more unpredictable fluctuations in the system's behaviour, reflected in the increased variability captured by the variance. These findings highlight the impact of the failure rate on the system's mean size and variability.

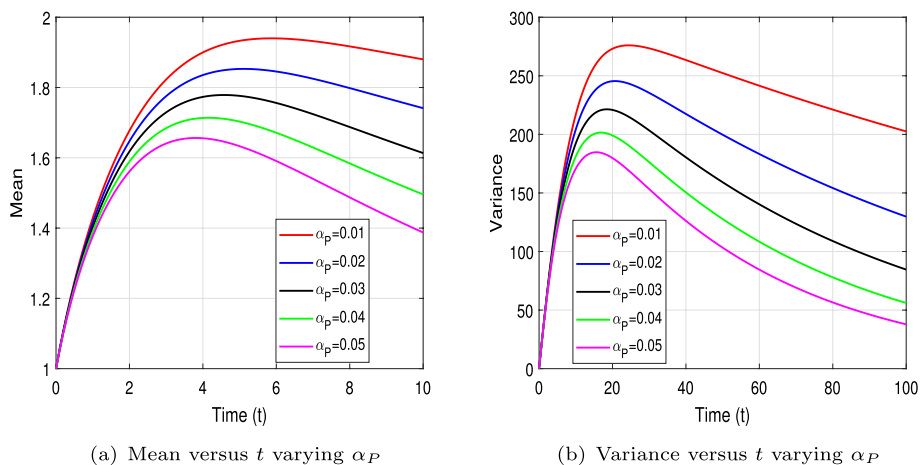
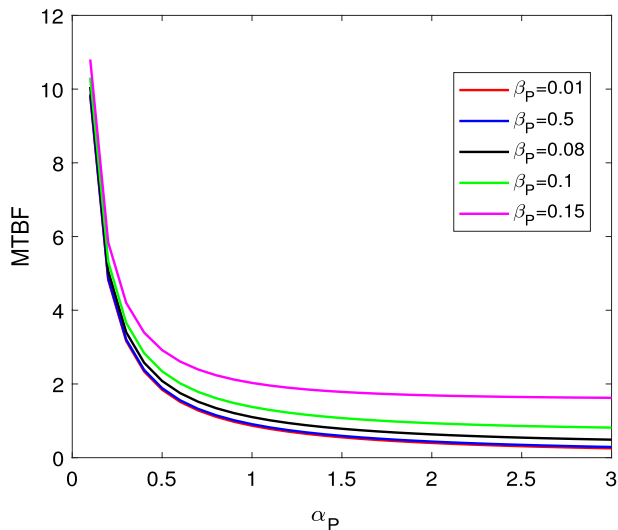


Fig. 8 Behaviour of mean and variance with respect to time for varying values of α_P

Fig. 9 Behaviour of MTBF with respect to time for varying values of α_P



Understanding these dynamics can assist in optimizing system performance and resource allocation to minimize packet loss and improve overall system efficiency.

Figure 9 illustrates the behaviour of MTBF with respect to β_P and α_P . The MTBF represents the average time elapsed between consecutive failures of the system. If the reliability of the system decreases over time, it indicates that failures occur more frequently, leading to a decrease in the MTBF. This suggests a deteriorating system performance. Conversely, if the repair rate is increasing, the MTBF curve tends to increase over time. This implies that the system is becoming more reliable, with longer average time intervals between failures.

6.2 Sensitivity Analysis in Steady-State for SMP

Figure 10(a) and 0(b) present the behaviour of the probability of the system being in the busy state, denoted as P_B , and the throughput with respect to the arrival rate (λ) and service rate (μ). Figure 10(a) illustrates that P_B increases with both λ and μ . This can be attributed to the fact that as the arrival rate or service rate increases, there are more packets in the system, leading to a higher likelihood of the system being in the busy state. In other words, a higher arrival rate means more packets are entering the system, while a higher service rate implies that packets are being served at a faster rate, which in turn increases the chances of the system being occupied. Similarly, Fig. 10(b) demonstrates that the throughput increases with both λ and μ . As the arrival rate or service rate increases, more packets are being processed and delivered by the system within a given time period. Therefore, the throughput, which represents the rate at which packets are successfully transmitted, also increases. This behaviour is intuitive, as a higher arrival rate or service rate allows for more efficient packet transmission and a greater overall throughput. Understanding the relationship between arrival rate, service rate, system occupancy, and throughput is essential for optimizing system performance and ensuring efficient utilization of resources. These findings provide insights into how changes in arrival and service rates impact the system's behaviour and its ability to handle incoming packets effectively.

Figure 11(a) and (b) illustrate the behaviour of the propagation failure probability, denoted as P_P , and system throughput with respect to α_P and β_P . Figure 11(a) shows that P_P exhibits a decreasing trend as β_P increases. This can be attributed to the fact that a higher repair rate leads to a faster recovery of failed components in the system. Consequently, the probability of the system experiencing propagation failures decreases. On the other hand, an increasing trend can be observed for P_P with respect to α_P , indicating that a higher propagation failure rate results in a greater likelihood of encountering propagation failures. In contrast, Fig. 11(b) demonstrates the opposite behaviour in system throughput. As β_P increases, the system's ability to recover from failures improves, resulting in a higher throughput. Conversely, an increasing α_P leads to more propagation failures, which negatively impact the system's performance and decrease the throughput. Similar patterns can be

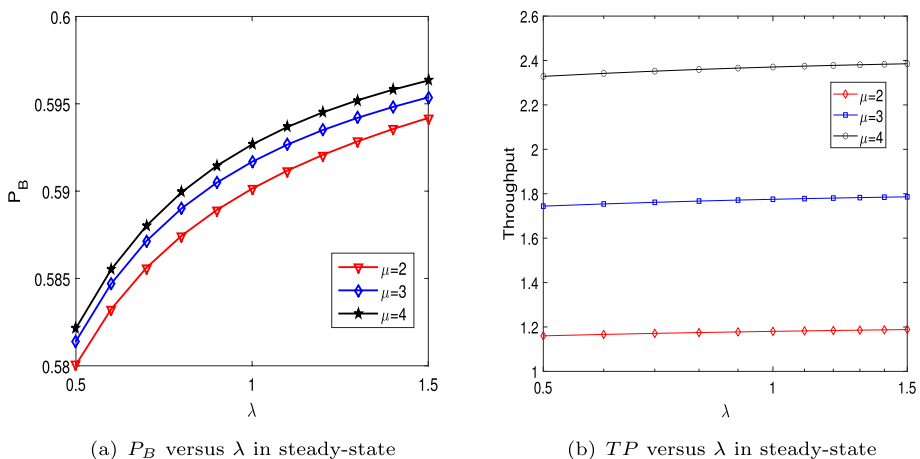


Fig. 10 Behaviour of P_B and TP with respect λ in steady-state

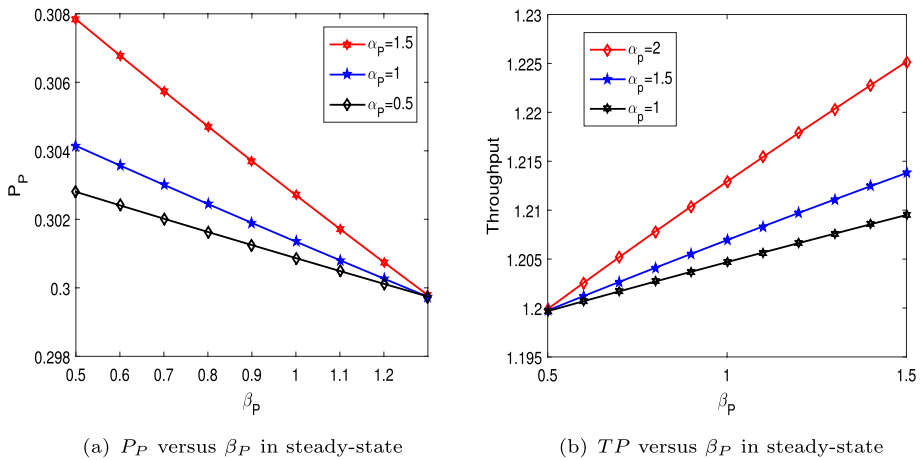


Fig. 11 Behaviour of P_P and TP with respect β_P in steady-state

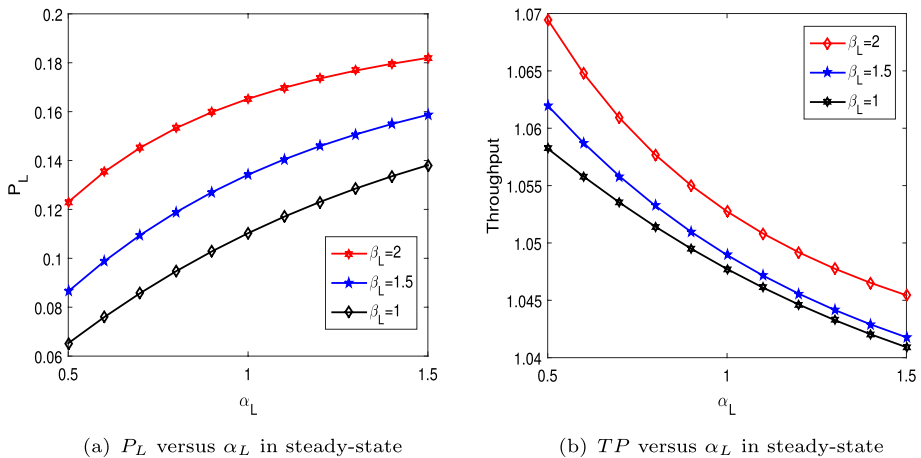
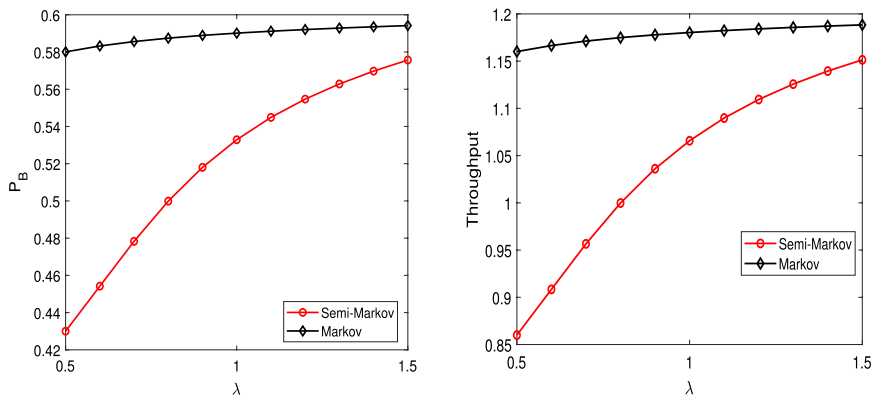


Fig. 12 Behaviour of P_L and TP with respect α_L in steady-state

observed in Fig. 12(a) and (b), where the behaviour of the local failure probability P_L and the system throughput with respect to α_L and β_L is depicted. Understanding the relationship between failure probabilities, repair rates, and system throughput is crucial for optimizing system performance and reliability. The results presented in these figures provide valuable insights into the impact of different factors on the system's behaviour, enabling better decision-making and resource allocation to improve system performance and reliability.

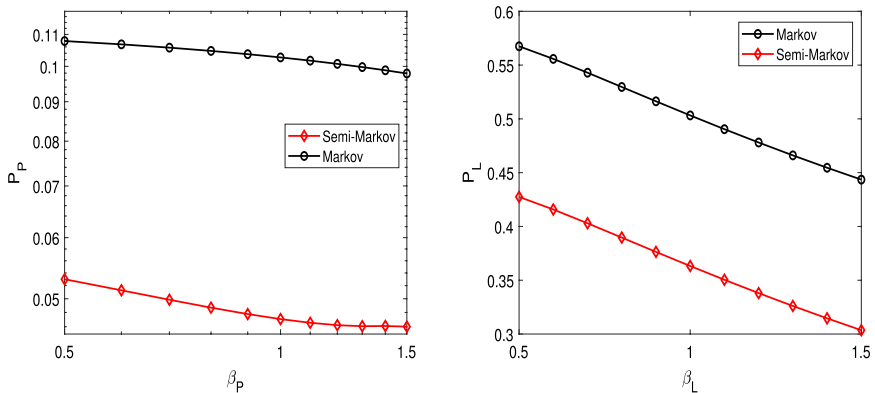
6.3 Comparison of Results for MP and SMP

Figures 13 and 14 present a comparison of performance measures between the CTMC and SMP cases. The behaviour of the performance indicators is consistent with the explanations provided earlier. In all four graphs, there is a notable disparity in the values of P_B ,



(a) P_B versus λ for semi-Markov and Markov models (b) TP versus λ for semi-Markov and Markov models

Fig. 13 Behaviour of P_B and TP versus λ for semi-Markov and Markov models



(a) P_P versus β_L for semi-Markov and Markov models (b) P_L versus β_L for semi-Markov and Markov models

Fig. 14 Behaviour of P_P and P_L versus β_P and β_L for semi-Markov and Markov models

throughput, P_P , and P_L between the CTMC and SMP models. The inclusion of different failure distributions in the model aims to capture the realistic scenario through appropriate modeling. The significant differences observed in the values of performance measures can be attributed to the variations in the rates of these distributions. Hence, it can be inferred that each metric under consideration is greatly influenced by the choice of failure distribution employed. This comparison highlights the importance of selecting an appropriate stochastic model to accurately represent the system's behaviour. The choice between CTMC and SMP models should be made based on the specific characteristics and requirements of the system under study. By using the most suitable model, researchers and practitioners can obtain more precise and reliable insights into the system's performance, enabling informed decision-making and optimization efforts.

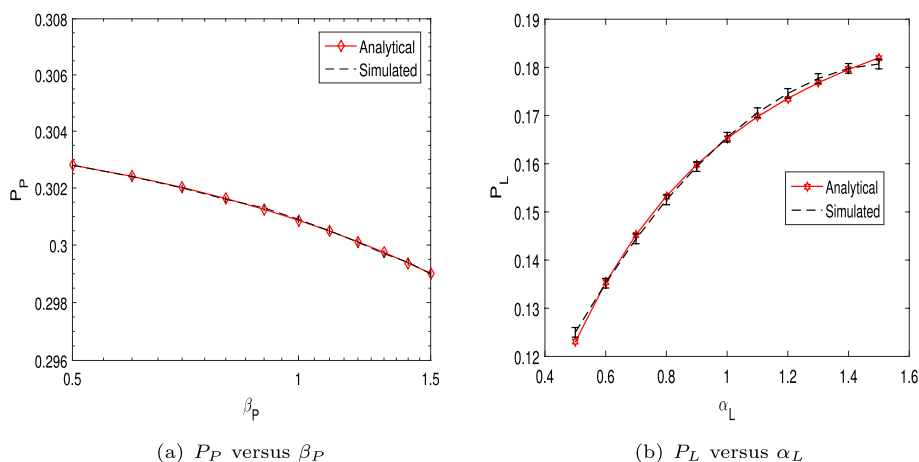


Fig. 15 Comparison of analytical and simulated results

6.4 Validation of Analytical and Simulated Results for SMP

We use a simulation approach to confirm the reliability of the proposed model in terms of its accuracy. The parameters used for analytical calculations, as detailed in Table 3, are input into a MATLAB program. We run the simulation 1000 times with these parameters to evaluate the precision of the simulation results, employing a t-distribution test for this purpose. “The t distribution is used for smaller sample sizes to account for the uncertainty in the standard deviation of the sample”. Subsequently, the 95% confidence interval is computed using the formula:

$$C.I. = \bar{X} \pm \frac{s * t_{0.05, k-1}}{\sqrt{k}}$$

where s is the standard deviation, $\bar{X}$ is sample mean, $t_{0.05, k-1}$ is tabulated value for 95% C.I., and k is sample size.

We present a comparison between the theoretical and simulated results for assessing the reliability measures of SMP models, as illustrated in Fig. 15. Figure 15(a) exhibits the comparison of analytical and simulated results of P_P with respect to β_P . Figure 15(b) demonstrates the behaviour of P_L with respect to α_L . The graph shows the comparison of obtained analytical and simulated results. The graphs display the results of the analysis using solid lines and the simulated results using dotted lines. The vertical lines represent the confidence interval. The graphs illustrate a close correspondence between the results obtained through both methods, confirming the validity of the hypothesized model.

The analytical and simulation results show close agreement overall, with minor deviations arising from the simplifying assumptions in the analytical model (e.g., exponential sojourns for LF and repair) and from sampling variability in simulations. The SMP model improves accuracy for non-exponential PF lifetimes, while simulations remain more reliable in extreme parameter regimes where analytical assumptions may not fully capture system dynamics.

7 Conclusions and Future Direction

The objective of this study is to analyze the reliability of HAPs in the event of failures. To capture the realistic functioning of HAPs, the study considers two types of failures that have different impacts on the system. LF affects only the affected component, while PF can result in widespread damage and a complete system shutdown. In the case of LF, a repair process is initiated to quickly restore the affected component. On the other hand, PF is addressed by utilizing a redundant HAP to continue packet services. The life time of PF is modeled using both exponential and gamma distributions, resulting in two different stochastic models: MP and SMP. The study provides time-dependent solutions for performance indicators in the MP model, along with the derivation of steady-state solutions. Moreover, the study conducts a comparative analysis of the results obtained from the MP and SMP models. This comparison enables a deeper understanding of the strengths and limitations of each modeling approach, assisting practitioners in making more informed decisions based on their specific requirements and scenarios. In a conscientious effort to validate the precision and effectiveness of the proposed model, a simulation approach is employed for SMP model. The main motivation of the proposed study is to evaluate the reliability of HAP in the presence of redundant HAP.

Finally, although the present work focuses on a system with a single primary and one redundant HAP, scalability to larger constellations remains an open challenge. Extending the model to multiple redundant platforms or full-scale HAP networks would significantly increase state-space complexity, potentially making exact analytical solutions intractable. Approximate methods, such as fluid approximations, mean-field techniques, or simulation-based approaches, may provide feasible alternatives for analyzing large-scale systems. Investigating such extensions forms an important avenue for future research.

The obtained results are benchmark and can be used by the communication network designer to implement. The study can be further extended to include relay HAP networks, where the occurrence of PF before the relay HAP's LF can lead to failure propagation, affecting other components of the communication network. In contrast, when the relay HAP's LF happens before the source's PF, it probabilistically severs the communication from the source, ensuring that its PF does not impact the rest of the network. To address the sequence-dependence and dynamic competing processes in HAP-aided systems and networks, new stochastic models and analysis methods will need to be developed. These models should consider the unique characteristics and interactions between different components, such as the relay HAP, source, and other network elements. By incorporating these factors, a more comprehensive understanding of the reliability dynamics in HAP-aided systems can be obtained, leading to improved design and optimization strategies.

The proposed model offers practical guidance for future HAP system design and planning. By quantifying the impact of local and propagation failures, it helps to determine when redundancy provides meaningful reliability gains and how standby strategies should be selected. The SMP model highlights the importance of modeling non-exponential failure processes, enabling more realistic fault-tolerance policies and predictive maintenance planning. Furthermore, the integration of throughput metrics allows network designers to balance reliability with service quality, making the model a useful tool for redundancy sizing, maintenance scheduling, and resilient HAP network planning.

8 Appendix

8.1 Proof of Theorem 1

Proof Let us define $f_{n,j}(s) = \int_0^\infty e^{-st} P_{n,j}(t) dt$, $j = M, B, L, P$. Here, $f_{n,j}(s)$ is the Laplace transform of $P_{n,j}(t)$. The Laplace transform of equations (5) and (6) yield

$$(s + \lambda + \beta_L) f_{0,L}(s) = \alpha_L f_{0,M}(s)$$

and

$$(s + \lambda + \beta_L) f_{n,L}(s) = \lambda f_{n-1,L}(s) + \alpha_L f_{n,M}(s).$$

After rewriting the above equations

$$f_{0,L}(s) = \frac{\alpha_L}{s + \lambda + \beta_L} f_{0,M}(s), \quad (50)$$

$$f_{n,L}(s) = \frac{\lambda}{s + \lambda + \beta_L} f_{n-1,L}(s) + \frac{\alpha_L}{s + \lambda + \beta_L} f_{n,M}(s). \quad (51)$$

After some simple algebraic calculation, we obtain

$$f_{n,L}(s) = \frac{\alpha_L}{s + \lambda + \beta_L} \sum_{r=0}^n \left(\frac{\lambda}{s + \lambda + \beta_L} \right)^{n-r} f_{r,M}(s), \quad n = 1, 2, 3, \dots \quad (52)$$

Applying the Laplace transform on the equations (7) and (8),

$$(s + \lambda + \beta_P + \gamma) f_{0,P}(s) = \alpha_P f_{0,M}(s)$$

and which yields

$$f_{0,P}(s) = \frac{\alpha_P}{s + \lambda + \beta_P + \gamma} f_{0,M}(s).$$

Also,

$$(s + \lambda + \beta_P + \gamma) f_{n,P}(s) = \alpha_P f_{n,M}(s) + \lambda f_{n-1,P}(s)$$

and which gives

$$f_{n,P}(s) = \frac{\lambda}{s + \lambda + \beta_P + \gamma} f_{n-1,P}(s) + \frac{\alpha_P}{s + \lambda + \beta_P + \gamma} f_{n,M}(s).$$

Iterating the above equation, one can obtain

$$f_{n,P}(s) = \frac{\alpha_P}{s + \lambda + \gamma + \beta_P} \sum_{r=0}^n \left(\frac{\lambda}{s + \lambda + \gamma + \beta_P} \right)^{n-r} f_{r,M}, \quad n = 1, 2, 3, \dots \quad (53)$$

On inversion of the equations (52) and (53) yield the required time-dependent system size probabilities $P_{n,L}(t)$ and $P_{n,P}(t)$, which are given in equation (9) and equation (10).

8.2 Proof of Theorem 2

Proof Let us define the generating function as follows:

$$G_M(z, t) = \sum_{n=0}^{\infty} z^n P_{n,M}(t). \quad (54)$$

Since $P_{0,M}(0) = 1$, which implies $G_M(z, 0) = 1$.

From equations (1) and (2), we get

$$\begin{aligned} \frac{\partial G_M(z, t)}{\partial t} &= \left[-(\lambda + \mu_1 + \alpha_P + \alpha_L) + \lambda z + \frac{\mu_1}{z} \right] G_M(z, t) \\ &\quad + \sum_{n=0}^{\infty} z^n [\beta_L P_{n,L}(t) + \beta_P P_{n,P}(t)] + \mu_1 \left(1 - \frac{1}{z} \right) P_{0,M}(t). \end{aligned}$$

Solving the above partial differential equation, we obtain

$$\begin{aligned} G_M(z, t) &= \mu_1 \left(1 - \frac{1}{z} \right) \int_0^t P_{0,M}(y) e^{[-(\lambda + \mu_1 + \alpha_P + \alpha_L) + (\lambda z + \frac{\mu_1}{z})](t-y)} dy \\ &\quad + \sum_{n=0}^{\infty} z^n \int_0^t [\beta_L P_{n,L}(y) + \beta_P P_{n,P}(y)] e^{[-(\lambda + \mu_1 + \alpha_P + \alpha_L) + (\lambda z + \frac{\mu_1}{z})](t-y)} dy \\ &\quad + e^{-(\lambda + \mu_1 + \alpha_P + \alpha_L)t} e^{(\lambda z + \frac{\mu_1}{z})t}. \end{aligned}$$

We know that

$$e^{(\lambda z + \frac{\mu_1}{z})t} = \sum_{n=-\infty}^{\infty} (\beta z)^n I_n(\alpha t), \quad \text{where } \alpha = 2\sqrt{\lambda\mu_1}, \quad \beta = \sqrt{\frac{\lambda}{\mu_1}}$$

and $I_n(\alpha t)$ is modified Bessel function of first kind.

Comparing the coefficient of z^n on both sides,

$$\begin{aligned}
P_{n,M}(t) = & \mu_1 \int_0^t P_{0,M}(y) [\beta I_n(\cdot) - \beta^{n-1} I_{n-1}(\cdot)] e^{-(\lambda+\mu_1+\alpha_P+\alpha_L)(t-y)} dy \\
& + \beta_L \int_0^t \left[\beta^n I_n(\cdot) P_{0,L}(y) + \beta^{n-1} I_{n-1}(\cdot) P_{1,L}(y) + \dots + \beta^0 I_0(\cdot) P_{n,L}(y) \right. \\
& \left. + \beta^{-1} I_{-1}(\cdot) P_{n+1,L}(y) + \beta^{-2} I_{-2}(\cdot) P_{n+2,L}(y) + \dots \right] e^{-(\lambda+\mu_1+\alpha_P+\alpha_L)(t-y)} dy \\
& + \beta_P \int_0^t \left[\beta^n I_n(\cdot) P_{0,P}(y) + \beta^{n-1} I_{n-1}(\cdot) P_{1,P}(y) + \dots + \beta^0 I_0(\cdot) P_{n,P}(y) \right. \\
& \left. + \beta^{-1} I_{-1}(\cdot) P_{n+1,P}(y) + \beta^{-2} I_{-2}(\cdot) P_{n+2,P}(y) + \dots \right] e^{-(\lambda+\mu_1+\alpha_P+\alpha_L)(t-y)} dy \\
& + e^{-(\lambda+\mu_1+\alpha_P+\alpha_L)t} I_n(\alpha t),
\end{aligned} \quad (55)$$

where $I_n(\cdot) = I_n(\alpha(t-y))$. The above holds for $n = -1, -2, -3, \dots$. For $n = -1, -2, -3, \dots$, the coefficient of z^{-n} on both sides yield

$$\begin{aligned}
0 = & \mu_1 \int_0^t P_{0,M}(y) [\beta^{-n} I_{-n}(\cdot) - \beta^{-n+1} I_{-n+1}(\cdot)] e^{-(\lambda+\mu_1+\alpha_P+\alpha_L)(t-y)} dy \\
& + \beta_L \int_0^t [\beta^{-n} I_{-n}(\cdot) P_{0,L}(y) + \beta^{-n-1} I_{-n-1}(\cdot) P_{1,L}(y) + \dots + \beta^{-n-n} I_{-2n}(\cdot) P_{n,L}(y) \\
& + \beta^{-2n-1} I_{-2n-1}(\cdot) P_{n+1}(y) + \beta^{-2n-2} I_{-2n-2}(\cdot) P_{n+2}(y) + \dots] \\
& e^{-(\lambda+\mu_1+\alpha_P+\alpha_L)(t-y)} dy + \beta_P \int_0^t \sum_{r=0}^n \beta^{-n-r} I_{-n-r}(\cdot) P_{r,L}(y) e^{-(\lambda+\mu_1+\alpha_P+\alpha_L)(t-y)} dy \\
& + e^{-(\lambda+\mu_1+\alpha_P+\alpha_L)t} I_{-n}(\alpha t).
\end{aligned} \quad (56)$$

After multiplying β^{2n} with equation (56) and then subtracting from (55), and using $I_{-n}(\cdot) = I_n(\cdot)$, we obtain the time-dependent system size probabilities on main HAP $P_{n,M}(t)$ is given in equation (11).

8.3 Evaluation of $P_{n,M}(t)$ in terms of $P_{1,M}(t)$

From equation (1), we obtain

$$\begin{aligned}
f_{0,M}(s) = & \frac{1}{s + \lambda + \alpha_P + \alpha_L} + \frac{\beta_L f_{0,L}(s)}{s + \lambda + \alpha_P + \alpha_L} + \frac{\beta_P f_{0,P}(s)}{s + \lambda + \alpha_P + \alpha_L} \\
& + \frac{\mu_1 f_{1,M}(s)}{s + \lambda + \alpha_P + \alpha_L}.
\end{aligned} \quad (57)$$

Using equations (5) and (7) along with equation (57) and after some simple algebra, one can obtain the following expression

$$f_{0,M}(s) = \frac{\mu_1 f_{1,M}(s)}{s + \lambda + \alpha_P + \alpha_L} \sum_{n=0}^{\infty} [A(s)]^n, \quad (58)$$

where

$$A(s) = \frac{1}{s + \lambda + \alpha_P + \alpha_L} + \frac{\beta_L \alpha_L}{(s + \lambda + \alpha_P + \alpha_L)(s + \lambda + \beta_L)} + \frac{\beta_P \alpha_P}{(s + \lambda + \alpha_P + \alpha_L)(s + \lambda + \beta_P + \gamma)}. \quad (59)$$

Using equation (7), we get

$$\sum_{n=1}^{\infty} f_{n,L}(s) = \frac{\lambda \alpha_L f_{0,M}(s)}{(s + \lambda + \beta_L)(s + \beta_L)} + \frac{\alpha_L}{s + \beta_L} \sum_{n=1}^{\infty} f_{n,M}(s). \quad (60)$$

From equation (8), we obtain

$$\sum_{n=1}^{\infty} f_{n,P}(s) = \frac{\lambda \alpha_P}{(s + \beta_P + \gamma)(s + \lambda + \gamma + \beta_P)} f_{0,M}(s) + \frac{\alpha_P}{s + \beta_P + \gamma} \sum_{n=1}^{\infty} f_{n,M}(s). \quad (61)$$

From equation (2), we obtain

$$\begin{aligned} \sum_{n=1}^{\infty} f_{n,M}(s) &= \frac{\beta_L}{s + \alpha_P + \alpha_L} \sum_{n=1}^{\infty} f_{n,L}(s) + \frac{\beta_P}{s + \alpha_P + \alpha_L} \sum_{n=1}^{\infty} f_{n,P}(s) \\ &\quad + \frac{\lambda}{s + \alpha_P + \alpha_L} f_{0,M}(s) - \frac{\mu_1}{s + \alpha_P + \alpha_L} f_{1,M}(s). \end{aligned} \quad (62)$$

Then

$$\begin{aligned} \sum_{n=1}^{\infty} f_{n,M}(s) &= \frac{\beta_L}{s + \alpha_P + \alpha_L} \left[\frac{\lambda \alpha_L}{(s + \beta_L)(s + \lambda + \beta_L)} f_{0,M}(s) + \frac{\alpha_L}{s + \beta_L} \sum_{n=1}^{\infty} f_{n,M}(s) \right] \\ &\quad + \frac{\beta_P}{s + \alpha_P + \alpha_L} \left[\frac{\lambda \alpha_P}{(s + \gamma + \beta_P)(s + \lambda + \gamma + \beta_P)} f_{0,M}(s) \right. \\ &\quad \left. + \frac{\alpha_P}{s + \gamma + \beta_P} \sum_{n=1}^{\infty} f_{n,M}(s) \right] + \frac{\lambda}{s + \alpha_P + \alpha_L} f_{0,M}(s) \\ &= \frac{\mu_1}{s + \alpha_P + \alpha_L} f_{1,M}(s) \\ &\quad \left[1 - \frac{\alpha_L \beta_L}{(s + \alpha_P + \alpha_L)(s + \beta_L)} - \frac{\alpha_P \beta_P}{(s + \alpha_P + \alpha_L)(s + \gamma + \beta_P)} \right] \sum_{n=1}^{\infty} f_{n,M}(s) = \\ &\quad \left[\frac{\lambda \alpha_L \beta_L}{(s + \alpha_P + \alpha_L)(s + \lambda + \beta_L)(s + \beta_L)} + \frac{\lambda \alpha_P \beta_P}{(s + \alpha_P + \alpha_L)(s + \gamma + \beta_P)(s + \lambda + \gamma + \beta_P)} \right. \\ &\quad \left. + \frac{\lambda}{s + \alpha_P + \alpha_L} \right] f_{0,M}(s) - \frac{\mu_1}{s + \alpha_P + \alpha_L} f_{1,M}(s) \end{aligned}$$

$$[1 - B(s)]f_{1,M}(s) + [1 - B(s)] \sum_{n=2}^{\infty} f_{n,M}(s) = C(s) \frac{\mu_1}{s + \lambda + \alpha_P + \alpha_L} f_{1,M}(s) \\ \sum_{n=1}^{\infty} [A(s)]^n - \frac{\mu_1}{s + \alpha_P + \alpha_L} f_{1,M}(s),$$

where

$$B(s) = \frac{\alpha_L \beta_L}{(s + \alpha_P + \alpha_L)(s + \beta_L)} + \frac{\alpha_P \beta_P}{(s + \alpha_P + \alpha_L)(s + \beta_P + \gamma)} \\ C(s) = \frac{\lambda \alpha_L \beta_L}{(s + \alpha_P + \alpha_L)(s + \lambda + \beta_L)(s + \beta_L)} \\ + \frac{\lambda \alpha_P \beta_P}{(s + \alpha_P + \alpha_L)(s + \gamma + \beta_P)(s + \lambda + \gamma + \beta_P)} + \frac{\lambda}{s + \alpha_P + \alpha_L}.$$

Therefore,

$$\sum_{n=2}^{\infty} f_{n,M}(s) = \sum_{n=0}^{\infty} [B(s)]^n \left[\frac{\mu_1}{s + \lambda + \alpha_P + \alpha_L} C(s) \sum_{n=1}^{\infty} [A(s)]^n \right. \\ \left. - \frac{\mu_1}{s + \alpha_P + \alpha_L} \right] f_{1,M}(s) - f_{1,M}(s).$$

On inversion, we obtain

$$\sum_{n=2}^{\infty} P_{n,M}(t) = \sum_{n=0}^{\infty} [B(t)]^{*(n)} [\mu_1 e^{-(\lambda + \alpha_P + \alpha_L)t} * C(t) \\ * \sum_{n=1}^{\infty} [A(t)]^{*(n)} - \mu_1 e^{-(\alpha_P + \alpha_L)t}] * P_{1,M}(t) - P_{1,M}(t),$$

where $A(t)$, $B(t)$, $C(t)$ can be obtained by taking inverse Laplace transform of $A(s)$, $B(s)$, $C(s)$.

$$\begin{aligned}
A(t) &= \left[1 + \frac{\alpha_L \beta_L}{\beta_L - \alpha_L - \alpha_P} + \frac{\alpha_P \beta_P}{\gamma + \beta_P - \alpha_L - \alpha_P} \right] e^{-(\lambda + \alpha_L + \alpha_P)t} \\
&\quad + \frac{\alpha_L \beta_L}{\alpha_L + \alpha_P - \beta_L} e^{-(\lambda + \beta_L)t} + \frac{\alpha_P \beta_P}{\alpha_L + \alpha_P - \gamma - \beta_P} e^{-(\lambda + \gamma + \beta_P)t} \\
B(t) &= \left[\frac{\alpha_L \beta_L}{\beta_L - \alpha_L - \alpha_P} + \frac{\alpha_P \beta_P}{\gamma + \beta_P - \alpha_L - \alpha_P} \right] e^{-(\alpha_L + \alpha_P)t} + \frac{\alpha_L \beta_L}{\alpha_L + \alpha_P - \beta_L} e^{-\beta_L t} \\
&\quad + \frac{\alpha_P \beta_P}{\alpha_L + \alpha_P - \gamma - \beta_P} e^{-(\gamma + \beta_P)t} \\
C(t) &= \left[\lambda + \frac{\lambda \alpha_L \beta_L}{(\lambda + \beta_L - \alpha_P - \alpha_L)(\beta_L - \alpha_P - \alpha_L)} \right. \\
&\quad \left. + \frac{\lambda \alpha_P \beta_P}{(\gamma + \beta_P - \alpha_P - \alpha_L)(\lambda + \gamma + \beta_P - \alpha_P - \alpha_L)} \right] e^{-(\alpha_P + \alpha_L)t} \\
&\quad - \frac{\alpha_L \beta_L}{\alpha_P + \alpha_L - \lambda - \beta_L} e^{-(\lambda + \beta_L)t} + \frac{\alpha_L \beta_L}{\alpha_P + \alpha_L - \beta_L} e^{-\beta_L t} \\
&\quad + \frac{\alpha_P \beta_P}{\alpha_P + \alpha_L - \gamma - \beta_P} e^{-(\gamma + \beta_P)t} - \frac{\alpha_P \beta_P}{\alpha_P + \alpha_L - \lambda - \gamma - \beta_P} e^{-(\lambda + \gamma + \beta_P)t}
\end{aligned}$$

and $*$ denotes convolution, and $[a(t)]^{*(n)}$ denotes n -fold convolution of $a(t)$.

The probabilities $P_{n,L}(t)$, $P_{n,P}(t)$, and $P_{n,B}(t)$ are expressed in terms of $P_{n,M}(t)$. Also, $P_{n,M}(t)$ is expressed in terms of $P_{1,M}(t)$. Using normalization principle, one can obtain the probability $P_{1,M}(t)$ in explicit form. In principle, we obtain all the time-dependent system probabilities in explicit form.

8.4 Derivation of Equation (26):

We are interested in evaluating the integral

$$\int_0^t (1 - F_{02}(x)) dF_{01}(x)$$

where,

- $F_{01}(x)$ is the CDF of a Gamma distribution with shape β and rate λ_{01}
- $F_{02}(x)$ is the CDF of an exponential distribution with rate λ_{02} .

We know

$$1 - F_{02}(x) = e^{-\lambda_{02}x}, \quad dF_{01}(x) = f_{01}(x)dx = \frac{\lambda_{01}^\beta x^{\beta-1} e^{-\lambda_{01}x}}{(\beta-1)!} dx.$$

Thus, the integral becomes

$$\begin{aligned}\int_0^t (1 - F_{02}(x)) dF_{01}(x) &= \int_0^t e^{-\lambda_{02}x} \cdot \frac{\lambda_{01}^\beta x^{\beta-1} e^{-\lambda_{01}x}}{(\beta-1)!} dx \\ &= \frac{\lambda_{01}^\beta}{(\beta-1)!} \int_0^t x^{\beta-1} e^{-(\lambda_{01}+\lambda_{02})x} dx.\end{aligned}$$

This integral does not have a closed-form in elementary functions, and is usually expressed using the lower incomplete gamma function.

We approximate the given expression

$$x^{\beta-1} e^{-ax} \approx t^{\beta-1} e^{-ax}, \quad (\text{holding } x^{\beta-1} \text{ constant at } x = t).$$

Then,

$$\int_0^t x^{\beta-1} e^{-ax} dx \approx t^{\beta-1} \int_0^t e^{-ax} dx = t^{\beta-1} \cdot \frac{1 - e^{-at}}{a}.$$

Hence,

$$\int_0^t e^{-\lambda_{02}x} dF_{01}(x) \approx \frac{\lambda_{01}^\beta}{(\beta-1)!} \cdot t^{\beta-1} \cdot \frac{1 - e^{-(\lambda_{01}+\lambda_{02})t}}{\lambda_{01} + \lambda_{02}}.$$

The above approximation can be improved using moment matching. We consider the integral

$$I(t) = \int_0^t x^{\beta-1} e^{-ax} dx, \quad \text{where } a = \lambda_{01} + \lambda_{02}.$$

We have,

$$1 - F_{02}(x) = e^{-\lambda_{02}x}, \quad dF_{01}(x) = f_{01}(x)dx = \frac{\lambda_{01}^\beta x^{\beta-1} e^{-\lambda_{01}x}}{(\beta-1)!} dx.$$

Thus, the integral simplifies to

$$\int_0^t e^{-\lambda_{02}x} f_{01}(x) dx = \frac{\lambda_{01}^\beta}{(\beta-1)!} \int_0^t x^{\beta-1} e^{-(\lambda_{01}+\lambda_{02})x} dx = \frac{\lambda_{01}^\beta}{(\beta-1)!} \cdot I(t).$$

A basic approximation is obtained by replacing the slowly-varying function $x^{\beta-1}$ with its value at the upper limit $x = t$

$$I(t) \approx t^{\beta-1} \int_0^t e^{-ax} dx = t^{\beta-1} \cdot \frac{1 - e^{-at}}{a}.$$

While this approximation is simple, it does not accurately capture the behavior of the integrand over the full interval $[0, t]$, especially when $x^{\beta-1}$ changes significantly. To improve accuracy, we apply moment matching.

We define

$$A(t) = \frac{1 - e^{-at}}{a}, \quad B(t) = \frac{1 - e^{-\lambda_{01}t}}{\lambda_{01}}.$$

These are the integrals of e^{-ax} and $e^{-\lambda_{01}x}$ over $[0, t]$, respectively. The idea is to approximate $I(t)$ as

$$I(t) \approx t^{\beta-1} \cdot A(t) + \text{correction term}.$$

We observe that

- $A(t)$ approximates the exponential decay with total rate $a = \lambda_{01} + \lambda_{02}$.
- $B(t)$ approximates the decay using only the gamma rate λ_{01} .

The difference $B(t) - A(t)$ reflects the discrepancy in decay rates between the Gamma and exponential assumptions.

To capture the effect of the changing polynomial term $a^{\beta-1}$, we introduce a correction function

$$R(t) = \lambda_{01}t^{\beta-1} - \lambda_{02}\lambda_{01}(a^{\beta-1} - t^{\beta-1}).$$

The corrected approximation is given by

$$I(t) \approx t^{\beta-1} \cdot \frac{1 - e^{-at}}{a} - (B(t) - A(t)) \cdot R(t).$$

Substituting explicitly

$$I(t) \approx t^{\beta-1} \cdot \frac{1 - e^{-at}}{a} - \left(\frac{1 - e^{-\lambda_{01}t}}{\lambda_{01}} - \frac{1 - e^{-at}}{a} \right) \cdot [\lambda_{01}t^{\beta-1} - \lambda_{02}\lambda_{01}(a^{\beta-1} - t^{\beta-1})].$$

Let,

$$\int_0^t a^{\beta-1} e^{-ax} dx \approx \left[\frac{1 - e^{-at}}{a} \right] \cdot t^{\beta-1} - \left[\frac{1 - e^{-\lambda_{01}t}}{\lambda_{01}} - \frac{1 - e^{-at}}{a} \right] \cdot R(t),$$

where $R(t)$ is a correction function chosen to capture the difference between gamma and exponential behavior. Let,

$$R(t) = \lambda_{01}t^{\beta-1} - \lambda_{02}\lambda_{01}(a^{\beta-1} - t^{\beta-1}).$$

Combining all terms, the approximation becomes

$$\int_0^t (1 - F_{02}(x)) dF_{01}(x) \approx \frac{\lambda_{01}}{(\beta - 1)!} \cdot \frac{1 - e^{-\lambda_{01}t}}{\lambda_{01} + \lambda_{02}} - \left(\frac{1 - e^{-\lambda_{01}t}}{\lambda_{01}} - \frac{1 - e^{-(\lambda_{01} + \lambda_{02})t}}{\lambda_{01} + \lambda_{02}} \right) \cdot [\lambda_{01}t^{\beta-1} - \lambda_{02}\lambda_{01}(a^{\beta-1} - t^{\beta-1})].$$

Derivation of mean sojourn time h_i values:

We want to evaluate the integral

$$I(t) = \int_0^t (1 - F_{01}(x))(1 - F_{02}(x)) dx,$$

where

- $F_{01}(x) = 1 - e^{-\lambda_{01}x}$ is the CDF of an exponential distribution with rate λ_{01} .
- $F_{02}(x)$ is the CDF of a gamma distribution with shape parameter $\beta \in \mathbb{N}$ and rate λ_{02} .

Then,

$$1 - F_{01}(x) = e^{-\lambda_{01}x}, \quad 1 - F_{02}(x) = \sum_{k=0}^{\beta-1} \frac{(\lambda_{02}x)^k}{k!} e^{-\lambda_{02}x}.$$

Thus, the integral becomes

$$I(t) = \int_0^t e^{-\lambda_{01}x} \left(\sum_{k=0}^{\beta-1} \frac{(\lambda_{02}x)^k}{k!} e^{-\lambda_{02}x} \right) dx = \sum_{k=0}^{\beta-1} \frac{\lambda_{02}^k}{k!} \int_0^t x^k e^{-(\lambda_{01} + \lambda_{02})x} dx.$$

Now, for the last term in the sum (i.e., $k = \beta - 1$), we use the identity

$$\int_0^t x^n e^{-\alpha x} dx = \frac{n!}{\alpha^{n+1}} - \frac{t^n}{\alpha} e^{-\alpha t} + (\text{higher order terms}),$$

for a compact closed-form, define

$$I_1(t) = \int_0^t (1 - F_{01}(x)) dx = \int_0^t e^{-\lambda_{01}x} dx = \frac{1 - e^{-\lambda_{01}t}}{\lambda_{01}}.$$

Next, we note that

$$F_{02}(t) = \frac{\gamma(\beta, \lambda_{02}t)}{(\beta - 1)!}, \quad (1 - F_{02}(t)) = \frac{1}{(\beta - 1)!} \int_{\lambda_{02}t}^{\infty} u^{\beta-1} e^{-u} du.$$

Now substitute back into the integral

$$I(t) = \int_0^t e^{-\lambda_{01}x} \cdot \frac{1}{(\beta - 1)!} \int_{\lambda_{02}x}^{\infty} u^{\beta-1} e^{-u} du dx.$$

Switch the order of integration

$$I(t) = \frac{1}{(\beta-1)!} \int_0^\infty u^{\beta-1} e^{-u} \left(\int_0^{\min(x, u/\lambda_{02})} e^{-\lambda_{01}x} dx \right) du.$$

Now evaluate the inner integral

$$\int_0^{\min(x, u/\lambda_{02})} e^{-\lambda_{01}t} dt = \begin{cases} \frac{1-e^{-\lambda_{01} \cdot (u/\lambda_{02})}}{\lambda_{01}}, & \text{if } u \leq \lambda_{02}x, \\ \frac{1-e^{-\lambda_{01}x}}{\lambda_{01}}, & \text{if } u > \lambda_{02}x. \end{cases}$$

Now split the integral at $u = \lambda_{02}x$, and combine the two parts

$$I(t) = \frac{1}{(\beta-1)!} \left[\frac{1}{\lambda_{01}} \int_0^\infty u^{\beta-1} e^{-u} du - \left(\frac{1}{\lambda_{01}} - \frac{1}{\lambda_{01} + \lambda_{02}} \right) a^{\beta-1} \right].$$

Using $\int_0^\infty u^{\beta-1} e^{-u} du = (\beta-1)!$, the expression simplifies to

$$I(t) = \frac{1}{(\beta-1)!} \left[\frac{(\beta-1)!}{\lambda_{01}} - a^{\beta-1} \left(\frac{1}{\lambda_{01}} - \frac{1}{\lambda_{01} + \lambda_{02}} \right) \right].$$

Final result,

$$\int_0^t (1 - F_{01}(x))(1 - F_{02}(x)) dx = \frac{1}{(\beta-1)!} \left[\frac{(\beta-1)!}{\lambda_{01}} - a^{\beta-1} \left(\frac{1}{\lambda_{01}} - \frac{1}{\lambda_{01} + \lambda_{02}} \right) \right].$$

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Declarations

Conflict of interest The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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