Lecture 04
Finding Roots of equations
Bracketing Methods

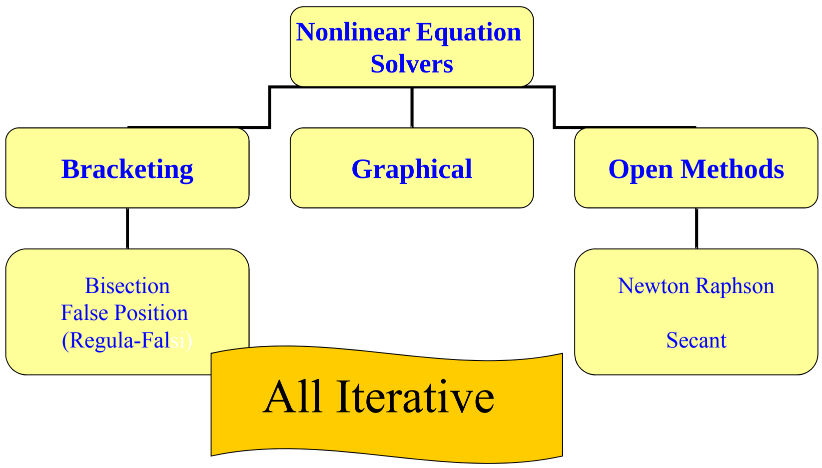
Bracketing Methods

- Root finding methods can be classified into

(a) Bracketing methods and (b) Open methods

- Estimating the errors in computation roots of equations
- The methods we are going to study are:
  - Graphical method
  - Bisection Method
  - False position method
- Compare these methods and their error estimation schemes

Non-linear Equation solving



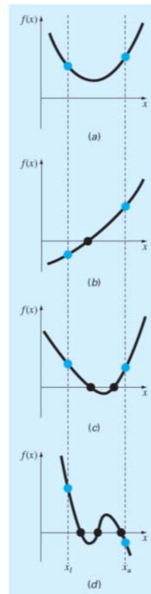
Graphical Method

The real number $x=x_0$ is a root of the polynomial $f(x)$ if and only if $f(x)=0$

At least one root exists between two bounds x_u (upper) and x_l (lower) if the function is real, continuous, and changes sign.

Graphical Method - Roots

- Graphical methods not very precise.
- Serve as rough estimates of roots.
- Helpful to understanding the behavior of the functions
- Upper and lower bounds f(u) and f(l)
- (a) no sign change – no roots
- (b) sign change – 1 root
- (c) no sign change – 2 roots
- (d) sign change – 3 roots

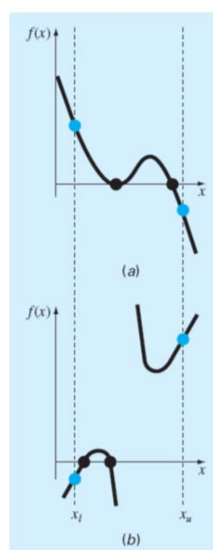


Graphical Method - Exceptions

- There are some exceptions to the general observations made previously.
- (a) The curve is tangential to x-axis

$$f(x) = (x-2)(x-2)(x-4)$$

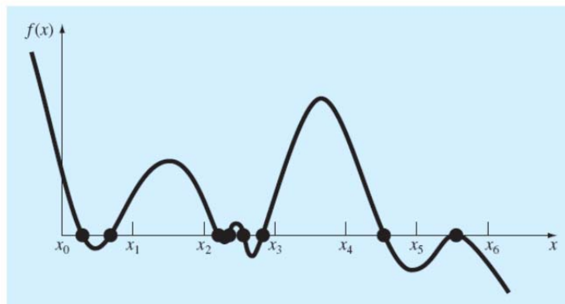
- (b) Discontinuous functions violate the general principles applicable otherwise



Incremental Search

This method is based on the observation that when a real continuous function *f(x)* changes sign there exists a root between them, i.e $f(x_l)f(x_u)<0$

Problem: The choice of the increment length. If the length is too small, the search can be very time consuming. On the other hand, if the length is too great, there is a possibility that closely spaced roots might be missed



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Bisection Method

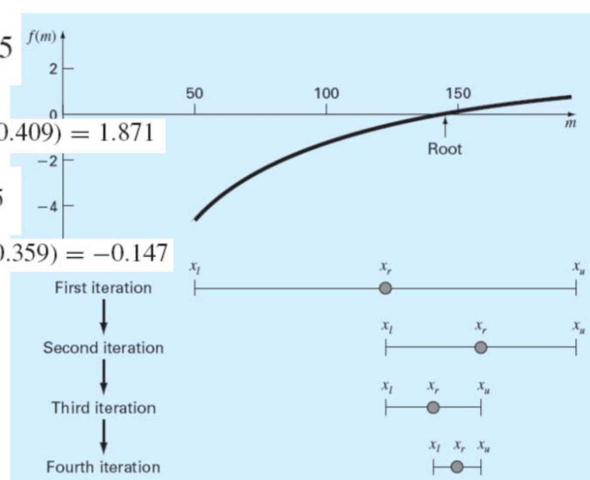
In this method we successively halve the intervals to search for the roots $x_2=(x_0+x_1)/2$

$$x_r = \frac{50+200}{2} = 125$$

$$f(50)f(125) = -4.579(-0.409) = 1.871$$

$$x_r = \frac{125+200}{2} = 162.5$$

$$f(125)f(162.5) = -0.409(0.359) = -0.147$$



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Bisection Method

The Algorithm:

1. For the arbitrary equation of one variable, f(x)=0
2. Pick xl and xu such that they bound the root of interest, check if f(xl).f(xu) <0.
3. Estimate the root by evaluating f[(xl+xu)/2].
4. Find the pair If f(xl). f[(xl+xu)/2]<0, root lies in the lower interval, then xu=(xl+xu)/2 and go to step 2.

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Bisection Method (contd.)

5. If $f(x_l)$. $f[(x_l+x_u)/2]>0$, root lies in the upper interval, then $x_l=[(x_l+x_u)/2$, go to step 2.

6. If $f(x_l)$. $f[(x_l+x_u)/2]=0$, then root is $(x_l+x_u)/2$ and terminate.

Compare ε_s with ε_a

If $\varepsilon_a < \varepsilon_s$, stop. Otherwise repeat the process.

$$\frac{\left(x_l - \frac{x_l+x_u}{2}\right)}{\left(\frac{x_l+x_u}{2}\right)} \times 100$$

or

$$\frac{\left(x_u - \frac{x_l+x_u}{2}\right)}{\left(\frac{x_l+x_u}{2}\right)} \times 100$$

Graphical Veiew of Bisection

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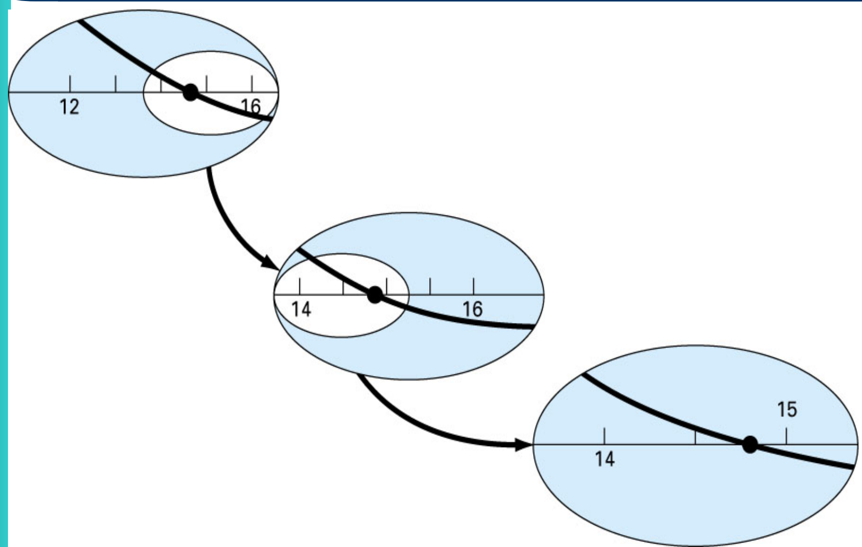

Error Estimation

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- To estimate the relative error, we can base it on the true value of root. If our guess is in doubt the error estimate may not be appropriate.
- Therefore, we require an error estimate that is not contingent on prior knowledge of the root. One way to do this is by estimating an approximate percent relative error as in

$$|\varepsilon_a| = \left| \frac{x_r^{\text{new}} - x_r^{\text{old}}}{x_r^{\text{new}}} \right| 100\%$$

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Example Error Estimation

- A typical error computation is shown in the table below. Here x_r are the roots, ε_r and ε_t are approximate relative error and true relative error based on true value.

Iteration	x_l	x_u	x_r	$\lvert\varepsilon_a\rvert$ (%)	$\lvert\varepsilon_t\rvert$ (%)
1	50	200	125		12.43
2	125	200	162.5	23.08	13.85
3	125	162.5	143.75	13.04	0.71
4	125	143.75	134.375	6.98	5.86
5	134.375	143.75	139.0625	3.37	2.58
6	139.0625	143.75	141.4063	1.66	0.93
7	141.4063	143.75	142.5781	0.82	0.11
8	142.5781	143.75	143.1641	0.41	0.30

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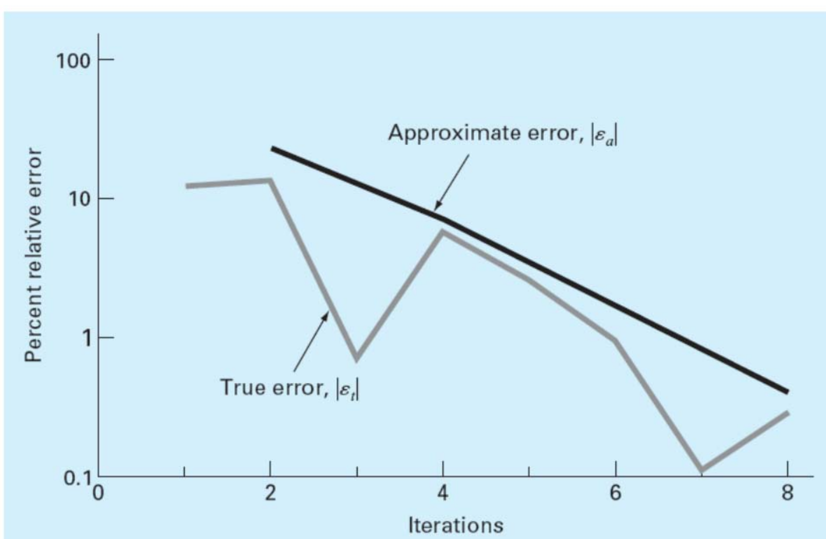
Approximate Error vs True Error



Estimation of Iteration

Length of the first Interval $L_o = b - a$

After 1 iteration $L_1 = L_o/2$

After 2 iterations $L_2 = L_o/4$

After k iterations $L_k = L_o/2^k$

- When ε_a becomes less than a prespecified stopping criterion ε_s , the computation is terminated.

$$\varepsilon_a \leq \frac{L_k}{x} \times 100\% \qquad \varepsilon_a \leq \varepsilon_s$$

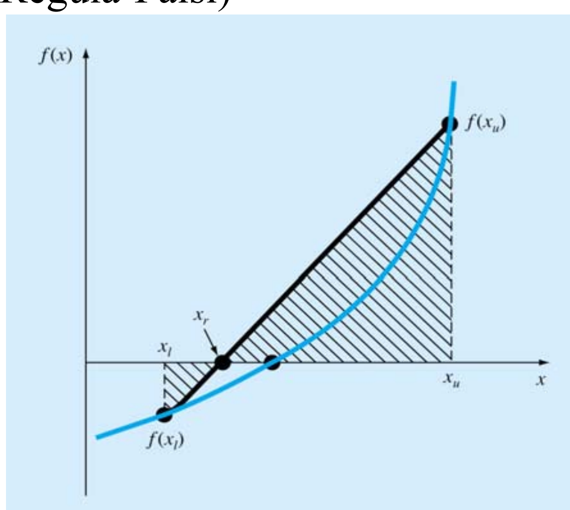
If the absolute magnitude of the error is E and L_o=2, how many iterations will you have to do to get the required accuracy in the solution?

E= $10^{-4} = \dfrac{2}{2^k} \quad \Rightarrow 2^k = 2 \times 10^4 \quad \Rightarrow k \cong 14.3 = 15$

False Position Method

(Regula-Falsi)

- If a real root is bounded by x_l and x_u of f(x)=0, then approximate solution is a linear interpolation between the points $[x_l, f(x_l)]$ and $[x_u, f(x_u)]$ to find the x_r value such that $l(x_r)=0$, l(x) is the linear approximation of f(x).
- The method is also known as Regula-Falsi

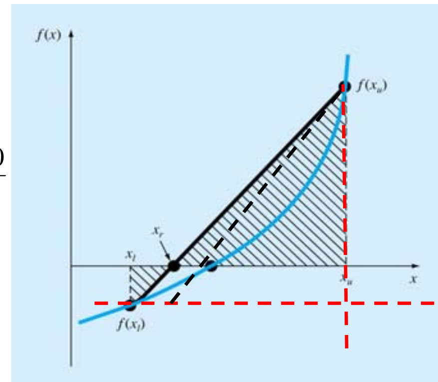


False Position Method

- To determine x_r
- Consider the slope of the chord of f(x)=0

$$\frac{f(x_u)-f(x_l)}{x_u-x_l}=\frac{f(x_u)-f(x_r)}{x_u-x_r}=\frac{f(x_l)-0}{x_u-x_r}$$

$$x_u-x_r=\frac{f(x_u)(x_u-x_l)}{f(x_u)-f(x_l)}$$



$$x_r=x_u-\frac{f(x_u)(x_l-x_u)}{f(x_l)-f(x_u)}=\frac{x_l f(x_u)-x_u f(x_l)}{f(x_l)-f(x_u)}$$

The Regula-falsi Algorithm

1. Find a pair of values of x, x_l and x_u such that $f_l=f(x_l)<0$ and $f_u=f(x_u)>0$.
2. Estimate the value of the root from the following formula and evaluate $f(x_r)$.
3. $x_r = x_u - \frac{f(x_u)(x_l-x_u)}{f(x_l)-f(x_u)} \Rightarrow x_r=\frac{x_l f_u - x_u f_l}{f_u-f_l}$

Use the new point to replace one of the original points, keeping the two points on opposite sides of the x axis.

If $f(x_r)<0$ then $x_l=x_r$ ==> $f_l=f(x_r)$

If $f(x_r)>0$ then $x_u=x_r$ ==> $f_u=f(x_r)$

If $f(x_r)=0$ then you have found the root and need go no further!

Regula-falsi Algorithm (contd.)

4. See if the new x_l and x_u are close enough for convergence to be declared. If they are not go back to step 2.

Why this method?

Faster (than bisection)

Always converges for a single root.

Note: Always check by substituting estimated root in the original equation to determine whether $f(x_r) \approx 0$.

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Bisection and False Position

Consider the solution of $f(x) = x^{10} - 1$ between 0 and 1.3 first by bisection and then by false position

Iteration	x_l	x_u	x_r	ε_a (%)	ε_t (%)
1	0	1.3	0.65	100.0	35
2	0.65	1.3	0.975	33.3	2.5
3	0.975	1.3	1.1375	14.3	13.8
4	0.975	1.1375	1.05625	7.7	5.6
5	0.975	1.05625	1.015625	4.0	1.6

Iteration	x_l	x_u	x_r	ε_a (%)	ε_t (%)
1	0	1.3	0.09430		90.6
2	0.09430	1.3	0.18176	48.1	81.8
3	0.18176	1.3	0.26287	30.9	73.7
4	0.26287	1.3	0.33811	22.3	66.2
5	0.33811	1.3	0.40788	17.1	59.2

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Modified False Position – Illinois Method

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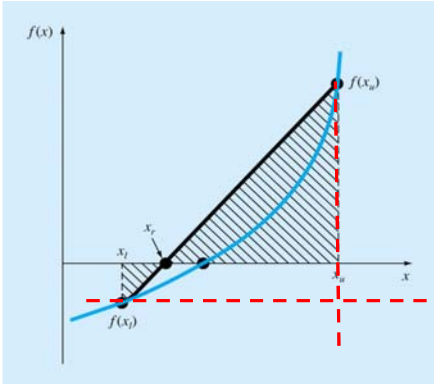
- The improved false position method is obtained by modifying the root finding expression as given below
- This is known as Illinois method. Here we multiply by ½ the first term in the numerator and in the denominator.



$$x_r = \frac{\frac{1}{2} x_l f(x_u) - x_u f(x_l)}{\frac{1}{2} f(x_u) - f(x_l)}$$

- Or $x_r = \dfrac{x_l f(x_u) - \frac{1}{2} x_u f(x_l)}{f(x_u) - \frac{1}{2} f(x_l)}$