Lecture 1

Introduction to
APL705 Finite Element Method

http://web.iitd.ac.in/~hegde/fem

APL705 FINITE ELEMENT METHOD

- An open elective course with 3-0-2=4 credits
- Gives a general introduction to the theory and practice of FE Method
- The associated lab sessions are meant to give a practical approach to the method by using popular commercial FEM software packages and programming tools
- The assignments in the course are designed to give task based opportunities to apply the method

Evaluation

Examinations and Weightage

Theory (70%)

- Minor I 15%
- Minor II 15%
- Major 30%
- Quiz 10%

Practical (30%)

- Lab Quiz 20%
- Programming Assignments (10%)
  - Matlab / Fortran programming assignments 5%
  - ANSYS modeling and simulation exercises 5%

Overview of Contents

1. Matrix algebra - Gaussian elimination, Congjugate Gradient (CG) Method
2. Fundamental concepts, Potential energy and equilibrium – The Raleigh Ritz Method, Galerkin Method
3. One dimensional problems
   Discretization
   Coordinates and shape functions
   Element stiffness matrix
   Assembly of global stiffness matrix and load vector
   Governing equations and treatment of boundary conditions
   Temperature effects
4. Trusses
   Plane trusses
   Three dimensional trusses
   Banded matrix – skyline solutions
5. Two-Dimensional problems using constant strain triangles
6. Axisymmetric solids subject to axisymmetric loading
7. Two-dimensional isoparametric elements and numerical integration
8. Beams and frames
9. Three dimensional problems in stress analysis
10. Scalar Field problems

What is FEM?

- Finite Element Method or FEM is a computational approach to solve engineering problems originally in solid mechanics and later adopted to other areas of structural problems and scalar field problems
- Also known as Finite Element Analysis or FEA
- Today it is a (numerical) method to solve multiphysics problems, both structural and field problems
- Helpful to solve problems with complex geometries, loadings, and material properties where analytical solutions may not be possible.
- FEM gives an approximate solution to boundary value problems

Applications of FEM

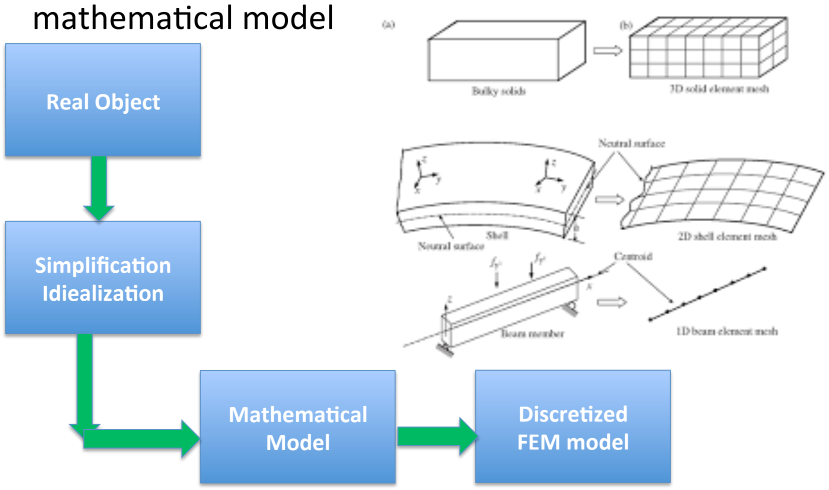
- In which areas and for what type of problems?
- Historical origin in aerospace
- Traditionally, for structural analysis: beams, truss, bridges
- Later applied to wide variety of problems both in structural and mechanical sciences
- Became popular with the advent of computer hardware and software
- Useful where there exists no analytical, closed form solution

Some FEM Application Areas

- Mechanical, Aerospace, structural engineering
- Structural design and Stress Analysis
- Static Vs Dynamic
- Linear Vs Nonlinear
- Fluid Flow
- Heat Transfer
- Electromagnetic Fields
- Soil Mechanics
- Soft materials and Biomechanics

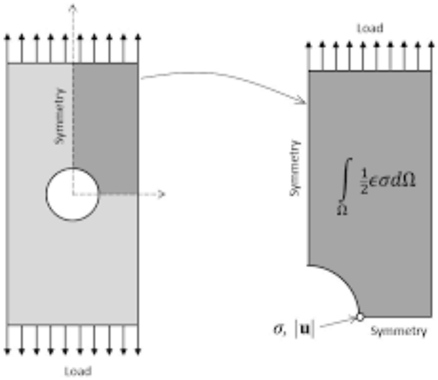
Basic Principles of FEM

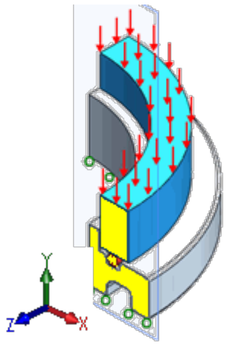
- A continuum is converted to an equivalent (discretized) mathematical model



Basic Principles of FEM

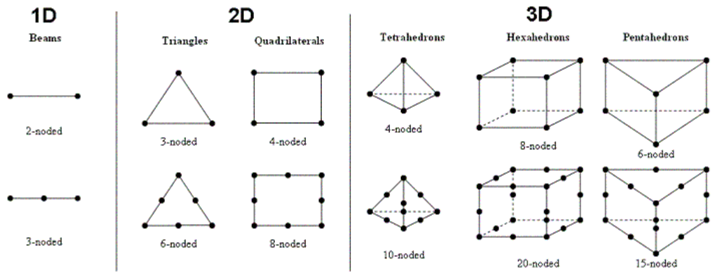
- While simplifying, the problem is reduced by exploiting the symmetry, homogeneity, constant sectional properties in geometry, material and loading





Basic Principles of FEM

- Appropriate discretization tries to represent the essence of the physical object with associated physical properties and application of loads
- The **nodes** and **elements** are the main features of a discretized model.



Basic Principles of FEM

- From equilibrium conditions of the given problem under the applied load on the model, a set of equations is obtained. The boundary conditions are necessary to solve these equations.
- By solving these equations, we obtain a primary unknown, **displacement**. Later, the secondary variables appropriate to the given problem are computed by assuming suitable laws of mechanics. These secondary quintiles may be the **stresses**, **strains**, etc.

Revision of Matrix Algebra

Consider the following 2 x 2 matrices

$$A = a_{rc} = \begin{array}{c|cc} & c_1 & c_2 \\ \hline r_1 & a_{11} & a_{12} \\ r_2 & a_{21} & a_{22} \end{array} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}; B = b_{rc} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

Matrix Addition

$$A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{bmatrix}$$

Matrix Operations

Multiplication by a scalar quantity

$$kA = \begin{bmatrix} ka_{11} & ka_{12} \\ ka_{21} & ka_{22} \end{bmatrix}$$

Matrix multiplication

Defined for matrices of r x c, c x r dimensions
This operation is comparable to vector dot product

$$C = AB = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$$

$$C_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

Matrix Operations

Consider the following system of equations

$$a_{11}x_1 + a_{12}x_2 + \dots\dots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \dots\dots + a_{2n}x_n = b_2$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$a_{n1}x_1 + a_{n2}x_2 + \dots\dots + a_{nn}x_n = b_n$$

This can be written using matrix equation as follows

$$[A]\{x\} = \{b\} \ OR \ \mathbf{Ax} = \mathbf{b}$$

Here **A** is a square matrix of size (nxn) and **x** and **b** are vectors of size (nx1)

Matrix Operations

Transpose of a matrix – Reflection along leading diagonal

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \Rightarrow A' = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}$$

Identity matrix

Every matrix has a transpose. If $A^t = A^{-1}$, such a matrix is called orthonormal

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$AA^{-1} = A^{-1}A = I$$

Matrix Operations

Transpose of a matrix – Reflection along leading diagonal

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \Rightarrow A' = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}$$

Transpose of a product is given as the product of the transpose in reverse order

$$(ABC)^T = C^T B^T A^T$$

Matrix Operations

A diagonal matrix

A diagonal matrix is a square matrix with non-zero elements only along the principal diagonal

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Symmetric Matrix

$$a_{ij} = a_{ji}$$

$$or \quad A = A^T$$

Matrix Operations

Upper triangular matrix

In an upper triangular matrix all elements below the principal diagonal are zero

$$A = \begin{bmatrix} 3 & 4 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & -1 \end{bmatrix}$$

Determinant of a Matrix

$$\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

Matrix Operations

Inverse of a matrix

Consider a square matrix **A.** if det **A ≠ 0** then **A** has an inverse, denoted by $\mathbf{A^{-1}}$

$$AA^{-1} = A^{-1}A = I$$

If det **A ≠ 0** then we say **A** is nonsingular. If det **A = 0** then we say that **A** is **singular** for which inverse is not defined.

Adjoint of a Matrix

The minor M_{ij} of a matrix A is the determinant of the (n-1)(n-1) matrix obtained by eliminating ith row and jth column of **A**. The cofactor matrix **C** with C_{ij} of matrix **A** is given by

$$C_{ij} = (-1)^{i+j} M_{ij}$$

Now the adjoint of matrix **A** is defined as

$$Adj\ A = C^T$$

Using this we now find the inverse of **A** as

$$A^{-1} = \frac{Adj\ A}{\det A}$$

Matrix Operations

Eigenvalues and Eigenvectors

Consider a square matrix **A of size (nxn)**

$$Ay = \lambda y$$

To find a non-trivial solution to this equation we can write it as

$$[A = \lambda I]y = 0$$

The expression $[A - \lambda I]$ has to be zero or singular, i.e

$$\det[A - \lambda I] = 0$$

This gives the characteristic polynomial equation in λ. The roots of this equation are the eigenvalues. By substituting a particular eigenvalue we get the corresponding eigenvector.

Matrix Operations

- **Positive Definitive matrix**

For a square matrix **A.** if all its eigenvalues λ_i are positive then we say **A** is positive definitive matrix.

For a symmetric matrix **A** of size (nxn) is positive definite for any non zero vector x=[x_1 x_2 .....x_n]^T if

$$x^T A x > 0$$

Matrix Operations

- **Cholesky Decomposition**

A positive definite symmetric matrix **A** can be decomposed to the form

$$A = LL^T$$

where L is a lower triangular and its transpose L^T is upper triangular. The decomposition is performed by the following algorithm:

$$l_{kj} = \frac{\left(a_{kj} - \sum_{i=1}^{j-1} l_{ki} l_{ji}\right)}{l_{jj}} \qquad j = 1 \text{ to } k-1$$

$$l_{kk} = \sqrt{a_{kk} - \sum_{i=1}^{j-1} l_{ki}^2}$$