Lecture 2: One-dimensional Problems

APL705 Finite Element Method

Steps in FEM Problems

1. System idealization – the given actual problem is broken down into idealized elements
2. Element equilibrium – the equilibrium requirements are established in terms of displacements or the state variables (or primary unknowns)
3. Assembly of elements – the element interconnections are used to develop a set of simultaneous equations in the unknown state variables
4. Calculation of response – by solving the system of equations, the response of each element and thereby that of the entire system under consideration is determined

Direct Stiffness Method

- **The direct aprroach**

The process of obtaining the total or system stiffness matrix by summation of individual element stiffness matrices is called direct stiffness method.

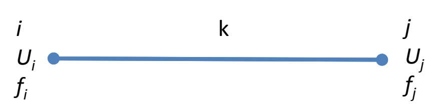
$$K = \sum_{i=1}^{n} k^{(j)}$$

Here $k^{(j)}$ are element stiffness matrices. Through the examples discussed later, we will see that this approach is general and can be applied to other non-structural problems also. This method is simple and gives a basic idea about obtaining the behaviour of a finite element of a continuum.

Some Examples

- **A single element model**

Here we can consider a rod, spring, truss member, beam, pipe or any such simple structural element for analysis. Consider the following single linear spring model as an example.



- Here i and j are the two nodes where the displacements and forces are present and k is the stiffness (spring constant) of the linear spring element under consideration

Simple Examples

- **Force – displacement relation**

Let us consider the equilibrium of forces at nodes i and j:

$$f_i = k(u_i - u_j) = ku_i - ku_j$$

$$f_j = k(u_j - u_i) = ku_j - ku_i$$

Expressing these in a matrix form:

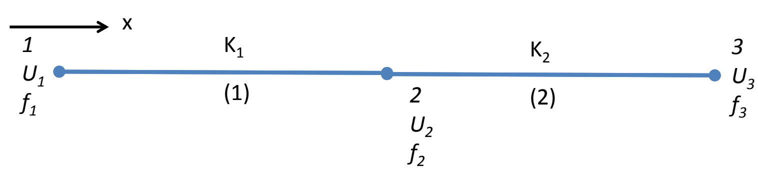
$$\begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_i \\ u_j \end{Bmatrix} = \begin{Bmatrix} f_i \\ f_j \end{Bmatrix} \Rightarrow ku = f$$

- Here [k] – element stiffness matrix, {u}- nodal displacement vector and {f}- element force vector

Direct Approach

- **Two-element model**

Now let us consider the following linear elastic springs connected end to end as follows:



- Here again the nomenclature is similar at nodes and elements. Now we have 3 nodes and two elements with their stiffness as k_1 and k_2 respectively.

Direct Approach

- **Simple Two-element Example**

Let us extend the idea of to a two element system now:

For Element 1

$$\begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} f_1^{(1)} \\ f_2^{(1)} \end{Bmatrix}$$

For Element 2

$$\begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} f_1^{(2)} \\ f_2^{(2)} \end{Bmatrix}$$

- For assembling the system or total stiffness matrix, there are two approaches: (i) Equilibrium of forces (ii) Superposition of element matrices

Assembling Stiffness Matrix

- **(i) Considering Equilibrium of forces**

$$Node\ 1\ \ f_1^{(1)} = R_1 \Rightarrow k_1 u_1 - k_1 u_2$$

$$Node\ 2\ \ f_2^{(1)} + f_1^{(2)} = R_2 \Rightarrow k_1 u_1 + (k_1 + k_2) u_2 - k_2 u_3$$

$$Node\ 3\ \ f_2^{(2)} = R_3 \Rightarrow k_2 u_2 + k_2 u_3$$

$$\begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} R_1 \\ R_2 \\ R_3 \end{Bmatrix}$$

- (ii) Superposition of element matrices

Assembling Stiffness Matrix

- (ii) Superposition of element matrices

Here we take each element matrix of 3x3 size and then just add them together to obtain the system stiffness matrix

$$\begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} f_1^{(1)} \\ f_2^{(1)} \\ 0 \end{Bmatrix} \textit{element } 1$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ f_1^{(2)} \\ f_2^{(2)} \end{Bmatrix} \textit{element } 2$$

- Superposition

$$\begin{bmatrix} k_1 & -k_1 & 0 \\ -k_2 & (k_1+k_2) & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} f_1^{(1)} \\ f_2^{(1)} + f_1^{(2)} \\ f_2^{(2)} \end{Bmatrix} E1+E2$$

Boundary Conditions and Loading

- Treatment of B.Cs and Loading

Let the displacement u_1=0 and loads R_2=R_3=P

$$\begin{bmatrix} k_1 & -k_1 & 0 \\ -k_2 & (k_1+k_2) & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} 0 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} R_1 \\ P \\ P \end{Bmatrix}$$

- Now we can obtain the reaction and displacements as

$$R_1 = -k_1 u_2 \qquad \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 2P/k_1 \\ 2P/k_1 + P/k_2 \end{Bmatrix}$$

- Therefore the reaction is

$$R_1 = -2P$$